A FORMULA FOR THE INNER SPECTRAL RADIUS

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This note presents an asymptotic formula for the minimum of the moduli of the elements in the spectrum of a bounded linear operator acting on Banach space $X$. This minimum moduli is called the inner spectral radius, and the formula established herein is an analogue of Gelfand’s spectral radius formula.

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1. Introduction. Let $X$ be a Banach space and let $B(X)$ denote the Banach algebra of all bounded linear operators on $X$, and let $T$ be an element of $B(X)$. We denote by $m(T)$ the “minimum moduli” of $T$ and define it by

$$ m(T) = \inf \left\{ \|Tx\|, \ x \in X, \ |x| = 1 \right\}. $$

(1.1)

In what follows, $i(T)$ and $i_{ap}(T)$ will denote, respectively, inner spectral radius and inner approximate spectral radius of $T$. We define $i(T)$ and $i_{ap}(T)$ by

$$ i(T) = \inf \left\{ |\lambda| : \lambda \in \sigma(T) \right\}, $$

$$ i_{ap}(T) = \inf \left\{ |\lambda| : \lambda \in \sigma_{ap}(T) \right\}. $$

(1.2)

Makai and Zemanek [3] proved that

$$ i_{ap}(T) = \lim_{n \to \infty} \left[ m(T^n) \right]^{1/n}. $$

(1.3)

In this note, we prove the same formula for $i(T)$. The main results established herein are the following theorem and corollary.

**Theorem 1.1.** Let $T \in B(X)$. Then, $r_i(T) = i(T)$ if and only if $r_i(T) \leq r_i(T^*)$.

**Corollary 1.2.** It is not necessary that $r_i(T) = r_i(T^*)$ for any $T \in B(X)$.

2. Basic concepts. Throughout, $X$ will denote a Banach space, $B(X)$ is the Banach algebra of all bounded linear operators on $X$. $T$ will denote an element of $B(X)$. We denote $T^*$ as the transpose of $T$ ($T^*$ is an element of $B(X^*)$, where $X^*$ is dual space of $X$) and define

$$ (T^*g)(x) = g(T(x)), \ x \in X, \ g \in X^*. $$

(2.1)
If $X$ is a Hilbert space, then $T^*$ is the adjoint of $T$ and $T^* \in B(X)$. We denote by $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_p(T)$, and $\sigma_c(T)$, respectively, spectrum, approximate point spectrum, point spectrum, and compression spectrum of $T$ and define

$$
\sigma(T) = \{ \lambda : (T - \lambda I) \text{ is not invertible}, \lambda \in \mathbb{C} \},
\sigma_{ap}(T) = \{ \lambda : (T - \lambda I) \text{ is not bounded below}, \lambda \in \mathbb{C} \},
\sigma_p(T) = \{ \lambda : \ker(T - \lambda I) \neq 0, \lambda \in \mathbb{C} \},
\sigma_c(T) = \{ \lambda : \text{ran}(T - \lambda I) \text{ is not dense in } X, \lambda \in \mathbb{C} \}.
$$

The spectral radius of $T$ is denoted by $r(T)$ and defined by

$$
r(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \}.
$$

We recall the following statements. One can see their proof in [1].

1. $\|T\| = \|T^*\|$.
2. $r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$ (Gelfand’s formula).
3. $r(T) = r(T^*)$.
4. $\sigma(T) = \sigma(T^*)$, if $X$ is a Hilbert space, then $\sigma(T^*) = \overline{\sigma(T)}$, where $\overline{\sigma(T)} = \{ \overline{\lambda}, \lambda \in \sigma(T) \}$.

For operator $T \in B(X)$, define

$$
m(T) = \inf \{ \|Tx\|, x \in X, \|x\| = 1 \}.
$$

$m(T)$ is called the minimum moduli of $T$. Note that by definition of $m(T)$, we have $\|Tx\| \geq m(T)\|x\|$. It is clear that; if $T$ is an invertible element in $B(X)$, then $m(T) = \|T^{-1}\|^{-1}$.

**Definition 2.1.** The inner spectral radius and inner approximate spectral radius of $T$ are denoted, respectively, by $i(T)$ and $i_{ap}(T)$ and defined by (1.2).

**Proposition 2.2.** If $|\lambda| < m(T)$, then $(T - \lambda I)$ is bounded below.

**Proof.** We have

$$
\|(T - \lambda I)x\| \geq \|Tx\| - \|\lambda x\| \geq (m(T) - |\lambda|)\|x\|.
$$

The assumption implies that $m(T) - |\lambda| > 0$ and hence $(T - \lambda I)$ is bounded below by the definition.

**Proposition 2.3.** For every operator $T \in B(X)$,

$$
\lim_{n \to \infty} \left[ m(T^n) \right]^{1/n} = \sup \left[ m(T^n) \right]^{1/n}.
$$
**Proof.** For every operator $T$ and $S$ in $B(X)$, we have

$$m(TS) \geq m(T)m(S),$$

by definition of the minimum moduli. Therefore, for every positive integers $i$ and $j$,

$$m(T^{i+j}) \geq m(T^i)m(T^j).$$

(2.8)

This is the crucial inequality. Let $k$ be fixed. For every integer number $n$, we have $n = kq + r$, $0 \leq r < k$, where $q = q(n)$ and $r = r(n)$ are functions of $n$. Note that $\lim_{n \to \infty} q(n)/n = 1/k$. Thus, by (2.8) we have

$$m(T^n) \geq m(T^k)^q m(T)^r, \text{ for each positive integer } n.$$  

(2.9)

Hence,

$$\lim_{n \to \infty} \inf \left[ m(T^n) \right]^n \geq m(T^k)^{1/k}. $$

(2.10)

Since this equation holds for all $k$, we have

$$\lim_{n \to \infty} \inf \left[ m(T^n) \right]^n \geq \sup m(T^n)^{1/n} \geq \lim \sup \left[ m(T^n) \right]^n$$

(2.11)

and the result follows.

Assume that $r_1(T) = \lim_{n \to \infty} [m(T^n)]^{1/n}$. By Gelfand's formula, it is clear that if $T$ is invertible, then $r_1(T) = [r(T^{-1})]^{-1}$.

**Corollary 2.4.** Let $T \in B(X)$. Then, $0 \in \sigma_{ap}(T)$ if and only if $m(T) = 0$.

**Proof.** The result follows from the facts that $0 \in \sigma_{ap}(T)$ if and only if $T$ is not bounded below and $\|Tx\| \geq m(T)\|x\|$ for each $x \in X$.

**Proposition 2.5.** Let $T \in B(X)$. If $\lambda \in \sigma_{ap}(T)$, then $|\lambda| \geq r_1(T)$.

**Proof.** Suppose $\lambda \in \sigma_{ap}(T)$. Assume, contrary to what we wish to prove, that $|\lambda| < r_1(T)$. Thus, $|\lambda|^n < m(T^n)$ for some integer $n$ by the definition of $r_1(T)$. By Proposition 2.2, $(T^n - \lambda^n I)$ is bounded below. We have

$$T^n - \lambda^n = (T^{n-1} + T^{n-2}\lambda + \cdots + \lambda^n)(T - \lambda).$$

(2.12)

Hence, $(T - \lambda)$ is bounded below and so $\lambda \notin \sigma_{ap}(T)$, which is contradictory to our assumption.
**Corollary 2.6.** For each $T \in B(X)$,

$$
\sigma_{ap}(T) \subseteq \{ \lambda : r_i(T) \leq |\lambda| \leq r(T) \}.
$$

(2.13)

Makai and Zemanek in [3] proved that $i_{ap}(T) = r_i(T)$ for every $T \in B(X)$. In the next section, we will prove that $i(T) = r_i(T)$ if and only if $r_i(T) \leq r_i(T^*)$.

3. **Inner spectral radius.** The purpose of this section is to prove the main result.

We know that $\partial \sigma_{ap}(T) \subseteq \sigma(T)$ and $r_i(T) = i_{ap}(T)$ and so $r_i(T) \in \sigma(T)$. Therefore, for every $T \in B(X)$, we have

$$
i(T) \leq r_i(T).
$$

(3.1)

**Fact 3.1.** If $X$ is a finite-dimensional space, then $\sigma_{ap}(T) = \sigma(T)$ for each $T \in B(X)$ and hence $r_i(T) = i(T)$.

**Fact 3.2.** If $T$ is a compact operator acting on Banach space $X$, then $r_i(T) = i(T)$.

We begin with some general lemmas that we need in the proof of the main theorem.

**Lemma 3.3.** Let $T \in B(X)$. Then, $\sigma_c(T) = \sigma_p(T^*)$. (If $X$ is a Hilbert space, then $\sigma_c(T) = \sigma_p(T^*)$).

**Proof.** First, we show that $\sigma_c(T) \subseteq \sigma_p(T^*)$. Suppose $\lambda$ is an element in $\sigma_c(T)$. Consider $M$ the closure of $\text{ran}(T - \lambda I)$. By definition of $\sigma_c(T)$, $M \neq X$. If $x_0$ is a nonzero element in $X - M$, then by the Hahn-Banach theorem there is $f_0 \in X^*$ such that $f_0(M) = 0$ and $f_0(x_0) = 1$. We have $((T^* - \lambda I)f_0)(x) = f_0(((T - \lambda I)x) = 0$ for every $x \in X$ and hence $f_0 \in \ker(T^* - \lambda I)$, that is $\lambda \in \sigma_p(T^*)$.

Now, we prove $\sigma_p(T^*) \subseteq \sigma_c(T)$. Suppose $\lambda \in \sigma_p(T^*)$, thus, there is a nonzero functional $g$ in $X^*$ such that $(T^* - \lambda I)g = 0$ and so, $g((T - \lambda I)x) = 0$ for each $x \in X$ by (2.1). Hence, $g(t) = 0$ for any $t$ in closure $\text{ran}(T - \lambda I)$.

But $g \neq 0$ on $X$, and hence there is $x_0 \in X - M$ such that $g(x_0) \neq 0$. Therefore, $M \neq X$, that is, $\lambda \in \sigma_c(T)$.

If $X$ is a Hilbert space, then we know that $\ker(T) = (\text{ran}T^*)^\perp$ and closure(ran$T^*) = (\ker(T))^\perp$ in [1, Theorem II.2.19]. Thus, by the definition of $\sigma_p(T)$ and $\sigma_c(T)$, we get the following result.

**Lemma 3.4.** Let $T \in B(X)$. Then, $\sigma(T) = \sigma_{ap}(T) \cup \sigma_c(T)$.

**Proof.** It follows from [1, Proposition VII.6.4] and the definition of $\sigma_{ap}(T)$ and $\sigma_c(T)$.

**Lemma 3.5.** Let $T \in B(X)$. If $\sigma(T) \subseteq \{ \lambda : r_i(T) \leq |\lambda| \leq r(T) \}$, then $r_i(T) = i(T)$.

**Proof.** By assumption, we have $r_i(T) \leq i(T)$ and the result follows the fact that $i(T) \leq r_i(T)$.

**Theorem 3.6.** Let $T \in B(X)$. Then, $r_i(T) = i(T)$ if and only if $r_i(T) \leq r_i(T^*)$. 


**Proof.** First, suppose that $r_i(T) \leq r_i(T^*)$. By Lemmas 3.3 and 3.4, $\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T^*)$ (if $X$ is a Hilbert space, then $\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T^*)$). We have

$$\sigma(T) \subseteq \{ \lambda : r_i(T) \leq |\lambda| \leq r(T) \}. \quad (3.2)$$

Hence, by Lemma 3.5, $r_i(T) = i(T)$.

Conversely, suppose that $r_i(T) = i(T)$. We have $\sigma(T) = \sigma(T^*)$ (if $X$ is a Hilbert space, then $\sigma(T^*) = \sigma(T)$). Thus, $i(T) = i(T^*)$ by definition of $i(T)$, and, therefore,

$$r_i(T) = i(T) = i(T^*) \leq r_i(T^*). \quad (3.3)$$

**Example 3.7.** Let $X$ be a Hilbert space and $N \in B(X)$ a normal operator. Then,

$$i(N) = i(N^*) = r_i(N) = r_i(N^*). \quad (3.4)$$

Since $N$ is normal, $\|Nx\| = \|N^*x\|$ for every $x$ in $X$, and, therefore, $m(N) = m(N^*)$. Similarly, we have $m(N^n) = m(N^*n)$ for each $n$, and so, $i(N) = i(N^*) = r_i(N) = r_i(N^*)$.

If $X$ is a Hilbert space and $N \in B(X)$ is a normal operator, then $r(N) = \|N\|$. In the next proposition, we prove that $r_i(N) = m(N)$ for the normal operator $N$ in $B(X)$.

Recall that for each operator $T \in B(X)$ the numerical range of $T$ is defined and denoted as follows:

$$W(T) = \{ \lambda \in \mathbb{C} : \lambda = \langle Tx, x \rangle, \ x \in X \text{ with } \|x\| = 1 \}. \quad (3.5)$$

The following interesting theorem was proved in [2, Theorem 27.9].

**Theorem 3.8.** If $T$ is a selfadjoint operator in $B(X)$, $M_1$ and $M_2$ denote, respectively, the infimum and the supremum of the numerical range of $T$, then $M_1$ and $M_2$ are approximate eigenvalues of $T$, and the spectrum of $T$ is contained in the interval $[M_1, M_2]$.

By this theorem, for each positive operator $T \in B(X)$ we have

$$i(T) = r_i(T) = \inf \{ \langle Tx, x \rangle, \ x \in X \text{ with } \|x\| = 1 \}. \quad (3.6)$$

**Proposition 3.9.** If $N$ is normal operator acting on Hilbert space $X$, then $i(N) = r_i(N) = m(N)$.

**Proof.** As shown in Example 3.7, we have $m(N) = m(N^*)$. Now, we prove that $m(NN^*) = m(N)^2$. Since $NN^*$ is positive, by (3.6) and Proposition 2.3, we have

$$m(NN^*) \leq r_i(NN^*) = \inf \{ \langle NN^*x, x \rangle, \ x \in X \text{ with } \|x\| = 1 \} \leq \inf \{ \|Nx\|^2, \ x \in X \text{ with } \|x\| = 1 \} = m(N)^2. \quad (3.7)$$

By (2.8), we get

$$m(NN^*) \geq m(N)m(N^*) = m(N)^2. \quad (3.8)$$

Hence,

$$m(NN^*) = m(N)^2. \quad (3.9)$$
By induction, we show that if \( j = 2^n, n = 0,1,2,\ldots \), then \( m(N^j) = m(N)^j \). This is clearly true for \( n = 0 \). Assume it to be true for some \( n \), then for all \( x \in \mathcal{H} \), we have

\[
\|N^{2^{n+1}}(x)\| = \|(N^{2^{n}}N(x))\| = \|(N^{2^{n}})^*(N^{2^{n}}(x))\|,
\]

because \( N^{2^n} \) is normal. This shows that \( m(N^{2^{n+1}}) = m((N^{2^{n}})^*N^{2^{n}}) \), which is equal to \( m(N^{2^n})^2 \). Thus, \( m(N^{2^{n+1}}) = (m(N)^{2^n})^2 = m(N)^{2^{n+1}} \). Therefore,

\[
\ri(N) = \lim_{n \to \infty} \left[ m(N^n) \right]^{1/n} = \lim_{n \to \infty} \left[ m(N^{2^n}) \right]^{1/2^n} = m(N). \tag{3.11}
\]

**Example 3.10.** Suppose \( U \) is a unilateral weighted shift with weights \((1,2,1,\ldots)\) acting on separable Hilbert space \( \mathcal{H} \). William Ridge [4] proved that \( \sigma_{ap}(U) = \{ \lambda : |\lambda| = \sqrt{2} \} \), \( \sigma(U) = \{ \lambda : |\lambda| \leq \sqrt{2} \} \), and \( \sigma_{ap}(U^*) = \sigma(U^*) = \sigma(U) \). Hence, \( \ri(U) = r(U) = \sqrt{2} \), \( \ri(U^*) = i(U^*) = 0 \), and \( i(U) = 0 \). Therefore, we have \( i(U) \neq \ri(U) \) and \( \ri(U^*) < \ri(U) \).

We know that \( r(T) = r(T^*) \) for any \( T \in B(\mathcal{X}) \). But in the above example \( \ri(U^*) < \ri(U) \) so, we can write the next corollary.

**Corollary 3.11.** It is not necessary that \( \ri(T) = \ri(T^*) \) for any \( T \in B(\mathcal{X}) \).

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