OSCILLATION PROPERTIES OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF \( n \)TH ORDER

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We consider the nonlinear neutral functional differential equation

\[
[r(t)[x(t) + \int_a^b p(t, \mu)x(\tau(t, \mu))d\mu]^{(n-1)}}]’ + \delta \int_c^d q(t, \xi)f(x(\sigma(t, \xi)))d\xi = 0
\]

with continuous arguments. We will develop oscillatory and asymptotic properties of the solutions.

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1. Introduction. Recently, several authors \([2, 3, 4, 5, 6, 7, 12, 13, 14]\) have studied the oscillation theory of second-order and higher-order neutral functional differential equations, in which the highest-order derivative of the unknown function is evaluated both at the present state and at one or more past or future states. For some related results, refer to \([1, 8, 10, 11]\).

In this paper, we extend these results to \( n \)th-order nonlinear neutral equations with continuous arguments

\[
\left[ r(t)\left[ x(t) + \int_a^b p(t, \mu)x(\tau(t, \mu))d\mu \right]^{(n-1)}} \right]’ + \delta \int_c^d q(t, \xi)f(x(\sigma(t, \xi)))d\xi = 0,
\]

where \( \delta = \pm 1 \), \( t \geq 0 \), and establish some new oscillatory criteria. Suppose that the following conditions hold:

(a) \( r(t) \in C([t_0, \infty), \mathbb{R}) \), \( r(t) \in C^1 \), \( r(t) > 0 \), and \( \int_{t_0}^{\infty} (dt/r(t)) = \infty \);

(b) \( p(t, \mu) \in C([t_0, \infty) \times [a, b], \mathbb{R}) \), \( 0 \leq p(t, \mu) \);

(c) \( \tau(t, \mu) \in C([t_0, \infty) \times [a, b], \mathbb{R}) \), \( \tau(t, \mu) \leq t \) and \( \lim_{t \to \infty} \min_{\mu \in [a, b]} \tau(t, \mu) = \infty \);

(d) \( q(t, \xi) \in C([t_0, \infty) \times [c, d], \mathbb{R}) \) and \( q(t, \xi) > 0 \);

(e) \( f(x) \in C(\mathbb{R}, \mathbb{R}) \) and \( xf(x) > 0 \) for \( x \neq 0 \);

(f) \( \sigma(t, \xi) \in C([t_0, \infty) \times [c, d], \mathbb{R}) \), and

\[
\lim_{t \to \infty} \min_{\xi \in [c, d]} \sigma(t, \xi) = \infty.
\]

A solution \( x(t) \in C[t_0, \infty) \) of (1.1) is called oscillatory if \( x(t) \) has arbitrarily large zeros in \([t_0, \infty)\), \( t_0 > 0 \). Otherwise, \( x(t) \) is called nonoscillatory.
2. Main results. We will prove the following lemma to be used in Theorem 2.2.

**Lemma 2.1.** Let \( x(t) \) be a nonoscillatory solution of (1.1) and let \( z(t) = x(t) + \int_a^b p(t, \mu) x(\tau(t, \mu)) d\mu \). Then, the following results hold:

(i) there exists a \( T > 0 \) such that for \( \delta = 1 \),
\[
z(t)z^{(n-1)}(t) > 0, \quad t \geq T,
\tag{2.1}
\]
and for \( \delta = -1 \) either
\[
z(t)z^{(n-1)}(t) < 0, \quad t \geq T, \quad \text{or} \quad \lim_{t \to \infty} z^{(n-2)}(t) = \infty,
\tag{2.2}
\]
(ii) if \( r'(t) \geq 0 \), then there exists an integer \( l, l \in \{0, 1, \ldots, n\} \) with \((-1)^{n-l-1} \delta = 1\) such that
\[
z^{(i)}(t) > 0 \quad \text{on} \quad [T, \infty) \quad \text{for} \quad i = 0, 1, 2, \ldots, l,
\tag{2.3}
\]
\[
(-1)^{i-l}z^{(i)}(t) > 0 \quad \text{on} \quad [T, \infty) \quad \text{for} \quad i = l, l+1, \ldots, n
\]
for some \( t \geq T \).

**Proof.** Let \( x(t) \) be an eventually positive solution of (1.1), say \( x(t) > 0 \) for \( t \geq t_0 \). Then, there exits a \( t_1 \geq t_0 \) such that \( x(\tau(t, \mu)) \) and \( x(\sigma(t, \xi)) \) are also eventually positive for \( t \geq t_1, \xi \in [c, d] \), and \( \mu \in [a, b] \). Since \( x(t) \) is eventually positive and \( p(t, \mu) \) is nonnegative, we have
\[
z(t) = x(t) + \int_a^b p(t, \mu) x(\tau(t, \mu)) d\mu > 0 \quad \text{for} \quad t \geq t_1.
\tag{2.4}
\]
(i) From (1.1), we have
\[
\delta [r(t)z^{(n-1)}(t)]' = -\int_c^d q(t, \xi)f(x(\sigma(t, \xi)))d\xi.
\tag{2.5}
\]
Since \( q(t, \xi) > 0 \) and \( f \) is positive for \( t \geq t_1 \), we have \( \delta [r(t)z^{(n-1)}(t)]' < 0 \). For \( \delta = 1 \), \( r(t)z^{(n-1)}(t) \) is a decreasing function for \( t \geq t_1 \). Hence, we can have either
\[
r(t)z^{(n-1)}(t) > 0 \quad \text{for} \quad t \geq t_1
\tag{2.6}
\]
or
\[
r(t)z^{(n-1)}(t) < 0 \quad \text{for} \quad t \geq t_2 \geq t_1.
\tag{2.7}
\]
We claim that (2.6) is satisfied for \( \delta = 1 \). Suppose this is not the case, then we have (2.7). Since \( r(t)z^{(n-1)}(t) \) is decreasing,
\[
r(t)z^{(n-1)}(t) \leq r(t_2)z^{(n-1)}(t_2) < 0 \quad \text{for} \quad t \geq t_2.
\tag{2.8}
\]
Divide both sides of the last inequality by \( r(t) \) and integrate from \( t_2 \) to \( t \), respectively, then we obtain
\[
z^{(n-2)}(t) - z^{(n-2)}(t_2) \leq r(t_2)z^{(n-1)}(t_2) \int_{t_2}^t \frac{dt}{r(t)} < 0 \quad \text{for} \quad t \geq t_2.
\tag{2.9}
\]
Now, taking condition (a) into account we can see that \(z^{(n-2)}(t) - z^{(n-2)}(t_2) \to -\infty\) as \(t \to \infty\). That implies \(z(t) \to -\infty\), but this is a contradiction to \(z(t) > 0\). Therefore, for \(\delta = 1\),

\[
r(t)z^{(n-1)}(t) > 0 \quad \text{for } t \geq t_1.
\]

Since both \(z(t)\) and \(r(t)\) are positive, we conclude that

\[
z(t)z^{(n-1)}(t) > 0.
\]

For \(\delta = -1\), \(r(t)z^{(n-1)}(t)\) is increasing. Hence, either

\[
r(t)z^{(n-1)}(t) < 0 \quad \text{for } t \geq t_1,
\]

or

\[
r(t)z^{(n-1)}(t) > 0 \quad \text{for } t \geq t_2 \geq t_1.
\]

If (2.12) holds, we replace \(z(t)\) for \(r(t)\) to get

\[
z(t)z^{(n-1)}(t) < 0.
\]

If (2.13) holds, using the increasing nature of \(r(t)z^{(n-1)}(t)\), we obtain

\[
r(t)z^{(n-1)}(t) \geq r(t_2)z^{(n-1)}(t_2) > 0 \quad \text{for } t \geq t_2.
\]

Divide both sides of (2.15) by \(r(t)\) and integrate from \(t_2\) to \(t\), then we get

\[
z^{(n-2)}(t) - z^{(n-2)}(t_2) \geq r(t_2)z^{(n-1)}(t_2) \frac{dt}{r(t)} > 0 \quad \text{for } t \geq t_2.
\]

Taking condition (a) into account, it is not difficult to see that \(z^{(n-2)}(t) \to \infty\) as \(t \to \infty\). Hence, for \(\delta = -1\), either (2.14) holds or \(\lim_{t \to \infty} z^{(n-2)}(t) = \infty\).

(ii) From (1.1), we can see that

\[
\delta[r'(t)z^{(n-1)}(t) + r(t)z^{(n)}(t)] = -\int_c^d q(t, \xi) f(x(\sigma(t, \xi))) d\xi,
\]

and then

\[
\delta z^{(n)}(t) = -\frac{\delta r'(t)z^{(n-1)}(t)}{r(t)} - \int_c^d \frac{q(t, \xi) f(x(\sigma(t, \xi)))}{r(t)} d\xi.
\]

Using (i) and (2.18), we obtain

\[
\delta z^{(n)}(t) < 0.
\]

Suppose that \(\lim_{t \to \infty} z^{(n-2)}(t) \neq \infty\) when \(\delta = -1\). Thus, because of the positive nature of \(z(t)\) and (2.19), there exists an integer \(l\), \(l \in \{0, 1, \ldots, n\}\) with \((-1)^{n-l-1}\delta = 1\) by
Kiguradze’s lemma [9] such that
\[ z(i)(t) > 0 \text{ on } [T, \infty) \quad \text{for } i = 0, 1, 2, \ldots, l, \]
\[ (-1)^{i-l} z(i)(t) > 0 \text{ on } [T, \infty) \quad \text{for } i = l, l+1, \ldots, n \]
for some \( t \geq T. \)

If \( \lim_{t \to \infty} z^{(n-2)}(t) = \infty \) and \( \delta = -1, \) \( z^{(n-1)}(t) \) is eventually positive. Moreover, \( z^{(n)}(t) \) is also eventually positive by (2.19). But, this is the case \( l = n \) in (2.20). Thus, the proof is complete. \( \square \)

**Theorem 2.2.** Let \( P(t) = \int_a^b p(t, \mu) d\mu < 1. \) Suppose that \( f \) is increasing and for all constant \( k > 0, \)
\[ \int_{\infty}^{c} \int_{d}^{e} q(s, \xi) f ((1-P(\sigma(s, \xi))) k) d\xi ds = \infty. \] (2.21)

(i) If \( \delta = 1, \) then every solution \( x(t) \) of (1.1) is oscillatory when \( n \) is even, and every solution \( x(t) \) of (1.1) is either oscillatory or satisfies
\[ \liminf_{t \to \infty} |x(t)| = 0 \] (2.22)
when \( n \) is odd.

(ii) If \( \delta = -1, \) then every solution \( x(t) \) of (1.1) is either oscillatory or else
\[ \lim_{t \to \infty} |x(t)| = \infty \quad \text{or} \quad \liminf_{t \to \infty} |x(t)| = 0 \] (2.23)
when \( n \) is even, and every solution \( x(t) \) of (1.1) is either oscillatory or else
\[ \lim_{t \to \infty} |x(t)| = \infty \] (2.24)
when \( n \) is odd.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of (1.1), say \( x(t) > 0 \) for \( t \geq t_0. \) Let \( z(t) \) be a function defined by
\[ z(t) = x(t) + \int_a^b p(t, \mu) x(\tau(t, \mu)) d\mu. \] (2.25)

Recall from Lemma 2.1, if \( \delta = 1, \) then (2.1) holds and if \( \delta = -1, \) either \( z(t) z^{(n-1)}(t) < 0 \) for \( t \geq T \) or \( \lim_{t \to \infty} z^{(n-2)}(t) = \infty. \)

Suppose that \( \lim_{t \to \infty} z^{(n-2)}(t) \neq \infty \) for \( \delta = -1. \) Thus, there exist a \( t_1 \geq T \) and an integer \( l \in \{0, 1, \ldots, n-1\} \) with \((-1)^{n-1} \delta = 1\) such that
\[ z(i)(t) > 0, \quad i = 0, 1, 2, \ldots, l, \]
\[ (-1)^{i-l} z(i)(t) > 0, \quad i = l, l+1, \ldots, n, \quad t \geq t_1, \] (2.26)
by Kiguradze’s lemma [9].
Let \( n \) be even and \( \delta = 1 \), or \( n \) be odd and \( \delta = -1 \). Since \((-1)^{n-l-1}\delta = (-1)^{-l-1} = 1\), then \( l \) is odd. Now, \( z(t) \) is increasing by (2.26). Therefore, we have
\[
z(t) = x(t) + \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu \leq x(t) + \int_a^b p(t,\mu)z(\tau(t,\mu))d\mu,
\]
(2.27)
since \( x(t) \leq z(t) \). Since \( z(t) \) is increasing and \( \tau(t,\mu) < t \), this will imply that
\[
z(t) \leq x(t) + P(t)z(t).
\]
(2.28)
Thus, we have
\[
(1 - P(t))z(t) \leq x(t).
\]
(2.29)
On the other hand, we have \( z(t) \) positive and increasing with \( \lim_{t \to \infty} \min_{\xi \in [a,b]} \sigma(t,\xi) = \infty \). These imply that there exist a \( k > 0 \) and a \( t_2 \geq t_1 \) such that
\[
z(\sigma(t,\xi)) \geq k \quad \text{for} \quad t \geq t_2.
\]
(2.30)
Integrating (1.1) from \( t_2 \) to \( t \), then we have
\[
\delta r(t)z^{(n-1)}(t) - \delta r(t_2)z^{(n-1)}(t_2) + \int_{t_2}^t \int_c^d q(s,\xi)f(x(\sigma(s,\xi)))d\xi ds = 0.
\]
(2.31)
By (2.29), (2.30), and increasing nature of \( f \), we obtain
\[
f(x(\sigma(t,\xi))) \geq f((1 - P(\sigma(t,\xi)))k) \quad \text{for} \quad t \geq t_2.
\]
(2.32)
Substituting (2.32) into (2.31), we get
\[
\delta r(t)z^{(n-1)}(t) - \delta r(t_2)z^{(n-1)}(t_2) + \int_{t_2}^t \int_c^d q(s,\xi)f((1 - P(\sigma(s,\xi)))k)d\xi ds \leq 0.
\]
(2.33)
From (2.21) and (2.33), we can conclude that \( \delta r(t)z^{(n-1)}(t) \to -\infty \) as \( t \to \infty \). This contradicts the following:
\[
z^{(n-1)}(t) > 0 \quad \text{for} \quad \delta = 1,
\]
\[
z^{(n-1)}(t) < 0 \quad \text{for} \quad \delta = -1.
\]
(2.34)
Thus, this proves that \( x(t) \) is oscillatory when \( \delta = 1 \) and \( n \) is even, or \( x(t) \) is either oscillatory or \( \lim_{t \to \infty} z^{(n-2)}(t) = \infty \) when \( \delta = -1 \) and \( n \) is odd. Obviously, if \( \lim_{t \to \infty} z^{(n-2)}(t) = \infty \), then \( \lim_{t \to \infty} x(t) = \infty \).

Let \( n \) be odd and \( \delta = 1 \), or \( n \) be even and \( \delta = -1 \). If the integer \( l > 0 \), then we can find the same conclusion as above. Let \( l = 0 \). Since
\[
\int_0^\infty \int_c^d q(s,\xi)d\xi ds = \infty,
\]
(2.35)
and by using these two in (2.31), then it is easy to see that
\[
\lim_{t \to \infty} \inf f(x(t)) = 0 \quad \text{or} \quad \lim_{t \to \infty} x(t) = 0.
\] (2.36)
This completes the proof. □

**Example 2.3.** Consider the following functional differential equation:
\[
\left[ e^{-t/2} \left[ x(t) + \int_1^2 (1 - e^{-t-\mu}) x(t-\mu) d\mu \right] \right]'' - \int_3^5 \frac{(e^2 + e - 1)(e^t + t\xi)^3}{4e^{7/2}(e-1)} x(t+\xi) d\xi = 0
\] (2.37)
so that \( \delta = -1 \), \( n = 3 \), \( r(t) = e^{-t/2} \), \( p(t,\mu) = 1 - e^{-t-\mu} \), \( q(t,\xi) = (e^2 + e - 1)(e^t + t\xi)^3 / 4e^{7/2}(e-1) \), \( f(x) = x \), \( \sigma(t,\xi) = (t+\xi)/6 \) in (1.1).

We can easily see that the conditions of Theorem 2.2 are satisfied. Then, all solutions of this problem are either oscillatory or tends to infinity as \( t \) goes to infinity. It is easy to verify that \( x(t) = e^t \) is a solution of this problem.

**Theorem 2.4.** Let \( P(t) = \int_a^b p(t,\mu) d\mu < 1 \), and let \( f \) be increasing and \( r(t) = 1 \). Suppose that
\[
\int_\infty^\infty \int_c^d s^{n-1} q(s,\xi) f(\{(1 - P(\sigma(s,\xi)))k\}) d\xi ds = \infty
\] (2.38)
for every constant \( k > 0 \). Then, every bounded solution \( x(t) \) of (1.1) is oscillatory when \((-1)^n\delta = 1\).

**Proof.** Let \( x(t) \) be a nonoscillatory solution of (1.1). We may assume that \( x(t) > 0 \) for \( t \geq t_0 \). Then, obviously there exists a \( t_1 \geq t_0 \) such that \( x(t), x(\tau(t,\mu)) \), and \( x(\sigma(t,\xi)) \) are positive for \( t \geq t_1 \), \( \mu \in [a,b] \), and \( \xi \in [c,d] \). Let \( z(t) = x(t) + \int_a^b p(t,\mu) x(\tau(t,\mu)) d\mu \), then from (1.1), \( \delta z^{(n)}(t) < 0 \) for \( t \geq t_1 \). Hence, for \( \delta = 1 \), \( z^{(n-1)}(t) \) is decreasing and for \( \delta = -1 \), \( z^{(n)}(t) \) is increasing.

Since \( z^{(n)}(t) < 0 \) for \( \delta = 1 \), by Kiguradze’s lemma [9] there exists an integer \( l \), \( 0 \leq l \leq n-1 \) with \( n-l \) is odd and for \( t \geq t_1 \) such that
\[
\begin{align*}
  z^{(i)}(t) &> 0, \quad i = 0,1,\ldots,l, \\
(-1)^{n-l}z^{(i)}(t) &> 0, \quad i = l+1,\ldots,n-1.
\end{align*}
\] (2.39)
For \( \delta = -1 \), \( z^{(n)}(t) > 0 \), by Kiguradze’s lemma [9] either
\[
\begin{align*}
  z^{(i)}(t) &> 0, \quad i = 0,1,\ldots,n-1, \\
(-1)^{n-l}z^{(i)}(t) &> 0, \quad i = l+1,\ldots,n-1.
\end{align*}
\] (2.40)
or there exists an integer \( l \), \( 0 \leq l \leq n-2 \) with \( n-l \) is even and for \( t \geq t_1 \) such that
\[
\begin{align*}
  z^{(i)}(t) &> 0, \quad i = 0,1,\ldots,l, \\
(-1)^{n-l}z^{(i)}(t) &> 0, \quad i = l+1,\ldots,n-1.
\end{align*}
\] (2.41)
Since \( z(t) \) is bounded, \( l \) cannot be 2 for both cases. Then for \((-1)^n\delta = 1\), we have
\[
\begin{align*}
(-1)^{l-1}z^{(i)}(t) &> 0, \quad i = 1,2,\ldots,n-1.
\end{align*}
\] (2.42)
This shows that
\[
\lim_{t \to \infty} z^{(i)}(t) = 0 \quad \text{for } i = 1, 2, \ldots, n - 1.
\] (2.43)

Using (2.43) and integrating (1.1) n times from t to \(\infty\) to find
\[
(-1)^n \delta [z(\infty) - z(t)] = \frac{1}{(n-1)!} \int_t^\infty \int_c d(s-t)^{n-1} \frac{q(s,\xi)}{s} f(x(\sigma(s,\xi))) d\xi ds,
\] (2.44)

where \(z(\infty) = \lim_{t \to \infty} z(t)\). On the other hand, from (2.42), \(z(t)\) is increasing for large \(t\) and \(z(t)\) is positive, so we have
\[
f(x(\sigma(t,\xi))) \geq f((1-P(\sigma(t,\xi)))k) \quad \text{for } t \geq t_1, k > 0
\] (2.45)
as in the proof of Theorem 2.2. Thus, from (2.44) and (2.45), we have
\[
z(\infty) - z(t_1) \geq \frac{1}{(n-1)!} \int_{t_1}^\infty \int_c (s-t)^{n-1} q(s,\xi) f((1-P(\sigma(s,\xi)))k) d\xi ds.
\] (2.46)

By (2.38), the right-hand side of the above inequality is \(\infty\), therefore \(z(\infty) = \infty\) and this contradicts the boundedness of \(z(t)\). Thus, every bounded solution \(x(t)\) of (1.1) is oscillatory when \((-1)^n \delta = 1\).

**Example 2.5.** Consider the following functional differential equation:
\[
\left[ x(t) + \int_0^{2\pi} \frac{(1-e^{-t})}{4} x(t-\mu) d\mu \right]'' + \int_0^{5\pi/2} \left( \frac{1}{2} - e^{-t} \right) x(t+\xi) d\xi = 0, \quad t > -\ln \left( \frac{1}{2} \right)
\] (2.47)

so that \(\delta = 1\), \(n = 2\), \(r(t) = 1\), \(p(t,\mu) = (1-e^{-t})/4\), \(\tau(t,\mu) = t - \mu/2\), \(q(t,\xi) = 1/2 - e^{-t}\), \(f(x) = x\), \(\sigma(t,\xi) = (t + \xi)\) in (1.1).

We can easily see that the conditions of Theorem 2.4 are satisfied. Then, all bounded solutions of this problem are oscillatory. It is easy to verify that \(x(t) = \sin t\) is a solution of this problem.

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