# A NOTE ON DIOPHANTINE APPROXIMATION

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We prove the existence of a dense subset $\Delta$ of $[0,4]$ such that for all $\alpha \in \Delta$ there exists a subgroup $X_\alpha$ of infinite rank of $\mathbb{Z}[z]$ such that $X_\alpha$ is a discrete subgroup of $\mathbf{C}[0,\beta]$ for all $\beta \geq \alpha$ but it is not a discrete subgroup of $\mathbf{C}[0,\beta]$ for any $\beta \in (0,\alpha)$.

Given a set of nonnegative real numbers $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$, a $\Lambda$-polynomial (or Müntz polynomial) is a function of the form $p(x) = \sum_{i=0}^{n} a_i z^{\lambda_i}$ $(n \in \mathbb{N})$. We denote by $\Pi(\Lambda)$ the space of $\Lambda$-polynomials and by $\Pi^{\mathbb{Z}}(\Lambda) := \{p(x) = \sum_{i=0}^{n} a_i z^{\lambda_i} \in \Pi(\lambda) : a_i \in \mathbb{Z} \text{ for all } i \geq 0\}$ the set of integral $\Lambda$-polynomials. Clearly, the sets $\Pi^{\mathbb{Z}}(\Lambda)$ are subgroups of infinite rank of $\mathbb{Z}[x]$ whenever $\Lambda \subset \mathbb{N}$, $\#\Lambda = \infty$ (by infinite rank, we mean that the real vector space spanned by $X$ does not have finite dimension. In all what follows we are uniquely interested in groups of infinite rank). Now, it is well known that the problem of approximation of functions on intervals $[a,b]$ by polynomials with integral coefficients is solvable only for intervals $[a,b]$ of length smaller than four and functions $f$ which are interpolable by polynomials of $\mathbb{Z}[x]$ on a certain set (which we call the algebraic kernel of the interval $[a,b]$) $\mathcal{J}(a,b)$. Concretely, it is well known that $\mathbb{Z}[x]$ is a discrete subgroup of $\mathbf{C}[a,b]$ whenever $b-a \geq 4$ and 4 is the smallest number with this property (for these and other interesting results about approximation by polynomials with integral coefficients, see [1, 3] and the references therein. See also the other references at the end of this note). This motivates the following concept.

*Definition 1.* Given $X$ a subgroup of infinite rank of $\mathbb{Z}[x]$, set

$$\alpha_0(X) = \inf\{\alpha > 0 : X \text{ is a discrete subgroup of } \mathbf{C}[0,\alpha]\}. \tag{1}$$

In [1], Ferguson proved that $\alpha_0(X_n) = 4^{1/n}$, where $X_n = \Pi^{\mathbb{Z}}(n\mathbb{N})$ $(n \in \mathbb{N})$, and $\alpha_0(\Pi^{\mathbb{Z}}(\mathbb{P})) = 1$, where $\mathbb{P}$ denotes the set of prime numbers. Moreover, he observed that $1 \leq \alpha_0(\Pi^{\mathbb{Z}}(\Lambda)) \leq 4$ for all $\Lambda \subset \mathbb{N}$ such that $\sum_{i=0}^{\infty}(1/\lambda_i) = \infty$. We would like to observe that the relation $1 \leq \alpha_0(\Pi^{\mathbb{Z}}(\Lambda)) \leq 4$, holds for any infinite set $\Lambda \subset \mathbb{N}$. This clearly holds for the upper bound since $\Pi^{\mathbb{Z}}(\Lambda) \subset \mathbb{Z}[x]$. To prove the lower bound it is enough to observe that $\|x^{\lambda_n}\|_{[0,\alpha]} = \alpha^{\lambda_n} \to 0$ when $n \to \infty$ for all $\alpha \in (0,1)$, where $\Lambda = \{\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots\} \subset \mathbb{N}$.


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In general, the numbers $\alpha_0(X)$ are not easy to compute. Moreover, they do not have nice properties in connection with the elementary operations of sets. For example, the knowledge of $\alpha_0(X_i)$ $(i = 1,2)$ does not help for computing the numbers $\alpha_0(X_1 + X_2)$ and $\alpha_0(X_1 \cap X_2)$.

Anyway, we have the following elementary properties.

Proposition 2. *Assume that $X$ is a discrete subgroup of $\mathbf{C}[0,\alpha]$. Then $X$ is also a discrete subgroup of $\mathbf{C}[0,\beta]$ for all $\beta > \alpha$.*

*Proof.* By definition, $X$ is a discrete subgroup of $\mathbf{C}[0,\alpha]$ if and only if there exists a positive number $\varepsilon > 0$ such that

$$\|p\|_{[0,\alpha]} \geq \varepsilon \quad \forall p \in X \setminus \{0\}. \tag{2}$$

Hence, $\beta \geq \alpha$ implies that $X$ is a discrete subgroup of $\mathbf{C}[0,\beta]$, since

$$\|p\|_{[0,\beta]} \geq \|p\|_{[0,\alpha]} \geq \varepsilon \quad \forall p \in X \setminus \{0\}. \tag{3}$$

□

Corollary 3. *Assume that $X_1 \subset X_2$ are two subgroups of $\mathbb{Z}[x]$ of infinite rank. Then $\alpha_0(X_1) \leq \alpha_0(X_2)$.*

*Proof.* We assume that $\alpha_0(X_2) < \alpha_0(X_1)$ and take $\alpha \in (\alpha_0(X_2), \alpha_0(X_1))$. Then $X_2$ is a discrete subgroup of $\mathbf{C}[0,\alpha]$ (since $\alpha > \alpha_0(X_2)$), so that

$$\|p\|_{[0,\alpha]} \geq \varepsilon \quad \forall p \in X_2 \setminus \{0\}. \tag{4}$$

Hence

$$\|p\|_{[0,\alpha]} \geq \varepsilon \quad \forall p \in X_1 \setminus \{0\}, \tag{5}$$

since $X_1 \subset X_2$. This implies that $\alpha \geq \alpha_0(X_1)$, a contradiction. □

Corollary 4. *With the same notation as in the corollary above, it holds that $\alpha_0(X_1 + X_2) \geq \max\{\alpha_0(X_1), \alpha_0(X_2)\}$ and $\alpha_0(X_1 \cap X_2) \leq \min\{\alpha_0(X_1), \alpha_0(X_2)\}$. Moreover, there are examples where the inequalities are strict.*

*Proof.* We only prove that the inequalities are strict, in general (since the other claim is a trivial consequence of the proposition above). To do this, we give the following examples: We set $X_1 = \Pi^{\mathbb{Z}}(2\mathbb{N})$, $X_2 = \Pi^{\mathbb{Z}}(\mathbb{N} \setminus 2\mathbb{N})$ and $Y_1 = \Pi^{\mathbb{Z}}(2\mathbb{N})$, $Y_2 = \Pi^{\mathbb{Z}}(3\mathbb{N})$. Then

$$\mathbb{Z}[x] = \Pi^{\mathbb{Z}}(2\mathbb{N}) + \Pi^{\mathbb{Z}}(\mathbb{N} \setminus 2\mathbb{N}), \tag{6}$$

$$4 = \alpha_0(\mathbb{Z}[x]) > 2 = \alpha_0(\Pi^{\mathbb{Z}}(2\mathbb{N})) = \alpha_0(\Pi^{\mathbb{Z}}(\mathbb{N} \setminus 2\mathbb{N})). \tag{7}$$

Finally, $Y_1 \cap Y_2 = \Pi^{\mathbb{Z}}(6\mathbb{N})$, so that

$$\alpha_0(Y_1 \cap Y_2) = \alpha_0(\Pi^{\mathbb{Z}}(6\mathbb{N})) = 4^{1/6} < 4^{1/3} = \min\{\alpha_0(Y_1), \alpha_0(Y_2)\}. \tag{8}$$

□

Now we prove the main result of this note.

THEOREM 5. *Set*

$$\Delta = \{\alpha_0(X) : X \textit{ is a subgroup of infinite rank of } \mathbb{Z}[x]\}. \tag{9}$$

*Then* $\overline{\Delta}^{\mathbb{R}} = [0,4]$.

*Proof.* Let $\alpha \in (0,4)$ be a transcendent number and we denote by $J_\alpha = J(0,\alpha)$ the algebraic kernel of the interval $[0,\alpha]$. We know that $J_\alpha$ is a finite subset of $[0,\alpha]$ and $\alpha \notin J_\alpha$. Thus $\rho(\alpha) = \max J_\alpha < \alpha$. For each $\varepsilon \in (0,(\alpha - \rho(\alpha))/2)$, we take a function $\varphi_{\alpha,\varepsilon}$ such that $\varphi_{\alpha,\varepsilon} \in \mathbf{C}^1[0,\alpha]$, $\varphi_{\alpha,\varepsilon}(x) = x$ for all $x \in [0,\rho(\alpha)]$, $\varphi_{\alpha,\varepsilon}(0) = 0$, $\varphi_{\alpha,\varepsilon}(\alpha - \varepsilon) = 4 - \varepsilon$, $\varphi_{\alpha,\varepsilon}(\alpha) = 4 + \varepsilon$, and $\varphi'_{\alpha,\varepsilon}(x) > 0$ for all $x \in [0,\alpha]$. This function satisfies $(\varphi^{(k)}_{\alpha,\varepsilon})_{|J_\alpha} = p^{(k)}_{|J_\alpha}$, $k = 0,1$, for $p(x) = x$. Thus, $\varphi_{\alpha,\varepsilon}$ belongs to the closure of $\mathbb{Z}[x]$ in $\mathbf{C}^1[0,\alpha]$ (see [2, Theorem 12.1]), so that there exists a polynomial $q_{\alpha,\varepsilon} \in \mathbb{Z}[x]$ such that

$$||\varphi_{\alpha,\varepsilon} - q_{\alpha,\varepsilon}||_{\mathbf{C}^1[0,\alpha]} < \frac{\varepsilon}{2}. \tag{10}$$

This implies that $q_{\alpha,\varepsilon}$ is an strictly increasing function on $[0,\alpha]$, $q_{\alpha,\varepsilon}(0) = 0$ and there exists a certain number $\eta_{\alpha,\varepsilon} \in (\alpha - \varepsilon, \alpha)$ such that $q_{\alpha,\varepsilon}(\eta_{\alpha,\varepsilon}) = 4$. Hence, $q_{\alpha,\varepsilon} : [0,\eta_{\alpha,\varepsilon}] \to [0,4]$ is an homeomorphism. We denote by $\mathbb{Z}[q_{\alpha,\varepsilon}]$ the ring of polynomials with integral coefficients in the variable $q_{\alpha,\varepsilon}(x)$. Then $\mathbb{Z}[q_{\alpha,\varepsilon}]$ is a subgroup of $\mathbb{Z}[x]$ of infinite rank. We claim that

$$\alpha_0(\mathbb{Z}[q_{\alpha,\varepsilon}]) = \eta_{\alpha,\varepsilon}. \tag{11}$$

This is clear since, for all $\beta \in (0,4]$ the map $\Phi^\beta_{\alpha,\varepsilon} : \mathbf{C}[0,\beta] \to \mathbf{C}[0,q^{-1}_{\alpha,\varepsilon}(\beta)]$ given by $\Phi^\beta_{\alpha,\varepsilon}(u) = u(q_{\alpha,\varepsilon})$ is an isometry of normed vector spaces which maps $\mathbb{Z}[x]$ onto $\mathbb{Z}[q_{\alpha,\varepsilon}]$ and the fact that $\alpha_0(\mathbb{Z}[x]) = 4$. □

If $X$ is a subring of $\mathbb{Z}[x]$, then $X$ is a discrete subset of $\mathbf{C}[0,\alpha]$ if and only if $\|p\|_{[0,\alpha]} \geq 1$ for all $p \in X \setminus \{0\}$ (since $\|p\|_{[0,\alpha]} < 1$ implies that $\lim_{n\to\infty} \|p^n\|_{[0,\alpha]} = 0$). This means that, for all ring $X \subset \mathbb{Z}[x]$ we have that $\{\alpha \geq 0 : X$ is a discrete subset of $\mathbf{C}[0,\alpha]\}$ is a closed set, so that

$$\alpha_0(X) = \min\{\alpha > 0 : X \text{ is a discrete subgroup of } \mathbf{C}[0,\alpha]\}. \tag{12}$$

In particular, $X$ is a discrete subset of $\mathbf{C}[0,\alpha_0(X)]$. Now, we note that the sets $\mathbb{Z}[q_{\alpha,\varepsilon}]$ we used for the proof of the theorem above were subrings of $\mathbb{Z}[x]$, so that we have proved the existence of a dense subset $\Delta$ of $[0,4]$ such that for all $\alpha \in \Delta$ there exists a subgroup $X_\alpha$ of infinite rank of $\mathbb{Z}[z]$ such that $X_\alpha$ is a discrete subgroup of $\mathbf{C}[0,\beta]$ for all $\beta \geq \alpha$ but it is not a discrete subgroup of $\mathbf{C}[0,\beta]$ for any $\beta \in (0,\alpha)$.

We end this note with two open problems:

(P1) is $\Delta = [0,4]$?

(P2) is $\widetilde{\Delta} = \{\alpha_0(\Pi^{\mathbb{Z}}(\Lambda)) : \Lambda \subset \mathbb{N}, \#\Lambda = \infty\}$ a dense subset of $[0,4]$? In particular, is 4 an accumulation point of $\widetilde{\Delta}$?

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