We give a general condition which enables one to easily establish fixed point theorems for a pair of maps satisfying a contractive inequality of integral type.

Branciari [1] obtained a fixed point result for a single mapping satisfying an analogue of Banach’s contraction principle for an integral-type inequality. The second author [3] proved two fixed point theorems involving more general contractive conditions. In this paper, we establish a general principle, which makes it possible to prove many fixed point theorems for a pair of maps of integral type.

Define

\[ \Phi = \{ \varphi : \mathbb{R}^+ \to \mathbb{R} \mid \varphi \text{ is nonnegative, Lebesgue integrable, and satisfies } \int_0^{\epsilon} \varphi(t) dt > 0 \text{ for each } \epsilon > 0 \}. \]

Let \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfy that

(i) \( \psi \) is nonnegative and nondecreasing on \( \mathbb{R}^+ \),
(ii) \( \psi(t) < t \) for each \( t > 0 \),
(iii) \( \sum_{n=1}^{\infty} \psi^n(t) < \infty \) for each fixed \( t > 0 \).

Define \( \Psi = \{ \psi : \psi \text{ satisfies (i)-(iii)} \} \).

**Lemma 1.** Let \( S \) and \( T \) be self-maps of a metric space \((X,d)\). Suppose that there exists a sequence \( \{x_n\} \subset X \) with \( x_0 \in X \), \( x_{2n} := Sx_{2n} \), \( x_{2n+1} := Tx_{2n+1} \), such that \( \overline{\{x_n\}} \) is complete and there exists a \( k \in [0,1) \) such that

\[ \int_0^{d(Sx,Ty)} \varphi(t) dt \leq \psi \left( \int_0^{d(x,y)} \varphi(t) dt \right) \]

for each distinct \( x,y \in \overline{\{x_n\}} \) satisfying either \( x = Ty \) or \( y = Sx \), where \( \varphi \in \Phi \), \( \psi \in \Psi \).
Then, either
(a) $S$ or $T$ has a fixed point in $\{x_n\}$ or
(b) $\{x_n\}$ converges to some point $p \in X$ and
\[
\int_0^\infty d(x_n, p) \varphi(t) \, dt \leq \sum_{i=n}^{\infty} \psi^i(d) \quad \text{for } n > 0,
\]
where
\[
d := \int_0^\infty d(x_0, x_1) \varphi(t) \, dt.
\]

Proof. Suppose that $x_{2n+1} = x_{2n}$ for some $n$. Then $x_{2n} = x_{2n+1} = Sx_{2n}$, and $x_{2n}$ is a fixed point of $S$. Similarly, if $x_{2n+2} = x_{2n+1}$ for some $n$, then $x_{2n+1}$ is a fixed point of $T$.

Now assume that $x_n \neq x_{n+1}$ for each $n$. With $x = x_{2n}$, $y = x_{2n+1}$, (2) becomes
\[
\int_0^\infty d(x_{2n}, x_{2n+1}) \varphi(t) \, dt \leq \psi \left( \int_0^\infty d(x_{2n}, x_{2n+1}) \varphi(t) \, dt \right).
\]
Substituting $x = x_{2n}$, $y = x_{2n-1}$, (2) becomes
\[
\int_0^\infty d(x_{2n}, x_{2n-1}) \varphi(t) \, dt \leq \psi \left( \int_0^\infty d(x_{2n}, x_{2n-1}) \varphi(t) \, dt \right).
\]
Therefore, for each $n \geq 0$,
\[
\int_0^\infty d(x_n, x_{n+1}) \varphi(t) \, dt \leq \psi \left( \int_0^\infty d(x_{n-1}, x_n) \varphi(t) \, dt \right) \leq \cdots \leq \psi^n(d).
\]
Let $m, n \in \mathbb{N}$, $m > n$. Then, using the triangular inequality,
\[
d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}).
\]
It can be shown by induction that
\[
\int_0^\infty d(x_n, x_m) \varphi(t) \, dt \leq \sum_{i=n}^{m-1} \int_0^\infty d(x_i, x_{i+1}) \varphi(t) \, dt.
\]
Using (7) and (9),
\[
\int_0^\infty d(x_n, x_m) \varphi(t) \, dt \leq \sum_{i=n}^{\infty} \psi^i(d) \leq \sum_{i=n}^{\infty} \psi^i(d).
\]
Taking the limit of (10) as $m, n \to \infty$ and using condition (iii) for $\psi$, it follows that $\{x_n\}$ is Cauchy, hence convergent, since $X$ is complete. Call the limit $p$. Taking the limit of (10) as $m \to \infty$ yields (3).
Theorem 2. Let \((X,d)\) be a complete metric space, and let \(S, T\) be self-maps of \(X\) such that for each distinct \(x, y \in X\),
\[
\int_0^{d(Sx,Ty)} \varphi(t) dt \leq \psi \left( \int_0^{M(x,y)} \varphi(t) dt \right),
\]
where \(k \in [0,1)\), \(\varphi \in \Phi\), \(\psi \in \Psi\), and
\[
M(x,y) := \max \left\{ d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty) + d(y,Sx)}{2} \right\}.
\]
Then \(S\) and \(T\) have a unique common fixed point.

Proof. We will first show that any fixed point of \(S\) is also a fixed point of \(T\), and conversely. Let \(p = Sp\). Then
\[
M(p,p) = \max \left\{ 0, 0, d(p,Tp), \frac{d(p,Tp)}{2} \right\} = d(p,Tp),
\]
and (11) becomes
\[
\int_0^{d(p,Tp)} \varphi(t) dt \leq \psi \left( \int_0^{d(p,Tp)} \varphi(t) dt \right),
\]
which, from (1), implies that \(p = Tp\).

Similarly, \(p = Tp\) implies that \(p = Sp\).

We will now show that \(S\) and \(T\) satisfy (2).
\[
M(x,Sx) = \max \left\{ d(x,Sx), d(x,Sx), d(Sx,TSx), \frac{d(x,TSx) + 0}{2} \right\}.
\]
From the triangular inequality,
\[
\frac{d(x,TSx)}{2} \leq \frac{d(x,Sx) + d(Sx,TSx)}{2} \leq \max \left\{ d(x,Sx), d(Sx,TSx) \right\}.
\]
Thus, (11) becomes
\[
\int_0^{d(Sx,TSx)} \varphi(t) dt \leq k \int_0^{d(Sx,TSx)} \varphi(t) dt,
\]
a contradiction to (1).

Therefore, for all \(x \in X\), \(M(x,Sx) = d(x,Sx)\), and (2) is satisfied. If condition (a) of Lemma 1 is true, then \(S\) or \(T\) has a fixed point. But it has already been shown that any fixed point of \(S\) is also a fixed point of \(T\), and conversely. Thus \(S\) and \(T\) have a common fixed point.

Suppose that conclusion (b) of Lemma 1 is true. Then, from (3),
\[
\int_0^{d(Sx_{2n},Tp)} \varphi(t) dt \leq \psi \left( \int_0^{d(x_{2n},p)} \varphi(t) dt \right),
\]
which implies, since \(X\) is complete, that \(\lim d(Sx_{2n},Tp) = 0\).
Therefore,
\[ d(p, Tp) \leq d(p, Sx_{2n}) + d(Sx_{2n}, Tp) \to 0, \]  
(19)
and \( p \) is a fixed point of \( T \), hence a fixed point of \( S \). Condition (11) clearly implies uniqueness of the fixed point. \( \square \)

Every contractive condition of integral type automatically includes a corresponding contractive condition not involving integrals, by setting \( \varphi(t) \equiv 1 \) over \( \mathbb{R}^+ \).

There are many contractive conditions of integral type which satisfy (2). Included among these are the analogues of the many contractive conditions involving rational expressions and/or products of distances. We conclude this paper with one such example.

**Corollary 3.** Let \((X, d)\) be a complete metric space, \( S \) and \( T \) self-maps of \( X \) such that, for each distinct \( x, y \in X \),
\[ \int_0^{d(Sx, Ty)} \varphi(t) \, dt \leq k \int_0^{n(x, y)} \varphi(t) \, dt, \]  
(20)
where \( \varphi \in \Phi \), \( k \in [0, 1) \), and
\[ n(x, y) := \max \left\{ \frac{d(y, Ty)[1 + d(x, Sx)]}{1 + d(x, y)}, d(x, y) \right\}. \]  
(21)
Then \( S \) and \( T \) have a unique common fixed point.

**Proof.**
\[ n(x, Sx) = \max \{ d(Sx, TSx), d(x, Sx) \}. \]  
(22)

As in the proof of Theorem 2, it is easy to show that any fixed point of \( S \) is also a fixed point of \( T \), and conversely.

If \( n(x, Sx) = d(Sx, TSx) \), then an argument similar to that of Theorem 2 leads to a contradiction. Therefore \( n(x, Sx) = d(x, Sx) \), and either \( S \) or \( T \) has a common fixed point, or (3) is satisfied. In the latter case, with \( \lim x_n = \rho \), \( n(\rho, \rho) = 0 \), so that, from (20), \( p \) is a fixed point of \( S \), hence of \( T \). Uniqueness of \( p \) is easily established.

Corollary 3 is also a consequence of Lemma 1.

We now provide an example, kindly supplied by one of the referees, to show that Lemma 1 is more general than [2, Theorem 3.1].

**Example 4.** Let \( X := \{1/n : n \in \mathbb{N} \cup \{0\} \} \) with the Euclidean metric and \( S, T \) are self-maps of \( X \) defined by
\[
S\left(\frac{1}{n}\right) = \begin{cases} 
\frac{1}{n+1} & \text{if } n \text{ is odd,} \\
\frac{1}{n+2} & \text{if } n \text{ is even,} \\
0 & \text{if } n = \infty,
\end{cases} \quad T\left(\frac{1}{n}\right) = \begin{cases} 
\frac{1}{n+1} & \text{if } n \text{ is even,} \\
\frac{1}{n+2} & \text{if } n \text{ is odd,} \\
0 & \text{if } n = \infty.
\end{cases}
\]  
(23)
For each \( n \), define \( x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1} \). With \( x_0 = 1 \), let \( O(1) \) denote the orbit of \( x_0 = 1 \); that is, \( O(1) = \{1, 1/2, 1/3, \ldots\} \) and \( \overline{O(1)} = O(1) \cup \{0\} = X \). For \( x, y \in O(1) \), \( y = 1/m, \ m \) even and \( x = 1/n = Ty = 1/(m+1), Sx = 1/(m+2) \), so that

\[
d(Sx, Ty) = \left| \frac{1}{m+2} - \frac{1}{m+1} \right| = \frac{1}{m+2} - \frac{1}{m+1} = \frac{1}{(m+1)(m+2)},
\]

\[
d(x, y) = \left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{1}{m+1} - \frac{1}{n} \right| = \frac{1}{m+1} - \frac{1}{n} = \frac{1}{m(m+1)}.
\]

Thus

\[
\frac{d(Sx, Ty)}{d(x, y)} = \frac{m}{m+2} \leq 1.
\]

Also

\[
\sup_{n \in \mathbb{N}} \frac{d(Sx, Ty)}{d(x, y)} = 1,
\]

so that there is no number \( c \in [0, 1) \) such that \( d(Sx, Ty) \leq cd(x, y) \) for \( x, y \in O(1) \) and \( x = Ty \). Therefore, [2, Theorem 3.1] cannot be used. On the other hand, the hypotheses of Lemma 1 are satisfied. To see this, it will be shown that condition (2) is satisfied for some \( \varphi \in \Phi \).

We will first show that for any \( x = 1/n, y = 1/m \in O(1) \) satisfying either \( x = Ty \) or \( y = Sx \),

\[
d(Sx, Ty) \leq \left| \frac{1}{n+1} - \frac{1}{m+1} \right|.
\]

There are four cases.

**Case 1.** \( y = 1/m, \ m \) even, \( x = 1/n = Ty = 1/(m+1) \), and \( Sx = 1/(m+2) \). Then

\[
d(Sx, Ty) = \left| \frac{1}{m+2} - \frac{1}{m+1} \right| = \left| \frac{1}{n+1} - \frac{1}{m+1} \right|.
\]

**Case 2.** \( y = 1/m, \ m \) odd, \( x = 1/n = Ty = 1/(m+2) \), and \( Sx = 1/(m+3) \). Then

\[
\begin{align*}
d(Sx, Ty) & = \left| \frac{1}{m+3} - \frac{1}{m+2} \right| = \frac{1}{m+2} - \frac{1}{m+3} \\
& \leq \frac{1}{m+1} - \frac{1}{m+3} = \left| \frac{1}{n+1} - \frac{1}{m+1} \right|.
\end{align*}
\]

**Case 3.** \( x = 1/n, \ n \) even, \( y = 1/m = Sx = 1/(n+2) \), and \( Ty = 1/(n+3) \). Then

\[
\begin{align*}
d(Sx, Ty) & = \left| \frac{1}{n+2} - \frac{1}{n+3} \right| = \frac{1}{n+2} - \frac{1}{n+3} \\
& \leq \frac{1}{n+1} - \frac{1}{n+3} = \left| \frac{1}{n+1} - \frac{1}{n+3} \right|.
\end{align*}
\]
Case 4. $x = 1/n$, $n$ odd, $y = 1/m = Sx = 1/(n + 1)$, and $Ty = 1/(n + 2)$. Then

$$d(Sx, Ty) = \left| \frac{1}{n+1} - \frac{1}{n+2} \right| = \left| \frac{1}{n+1} - \frac{1}{m+1} \right|.$$  \hfill (31)

Thus in all cases, (20) is satisfied.

Define $\varphi$ by $\varphi(t) = t^{1/2} - 2[1 - \log t]$ for $t > 0$ and $\varphi(0) = 0$. Then, for any $\tau > 0$,

$$\int_{0}^{\tau} \varphi(t) \, dt = \tau^{1/\tau},$$  \hfill (32)

and $\varphi \in \Phi$.

Using [1, Example 3.6],

$$\int_{0}^{d(Sx, Ty)} \varphi(t) \, dt \leq d(Sx, Ty)^{1/d(Sx, Ty)}$$

$$\leq \left| \frac{1}{n+1} - \frac{1}{m+1} \right|^{1/(1/(n+1) - (1/m+1))}$$

$$\leq \frac{1}{2} \left| \frac{1}{n} - \frac{1}{m} \right|^{1/(1/n - (1/m))} = d(x, y)^{1/d(x, y)}$$

for each $x, y$ as in Lemma 1, and condition (2) is satisfied with $\psi(t) = t/2$. \hfill \square

Acknowledgment

The authors thank each of the referees for careful reading of the manuscript.

References


P. Vijayaraju: Department of Mathematics, Anna University, Chennai-600 025, India
E-mail address: vijay@annauniv.edu

B. E. Rhoades: Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, USA
E-mail address: rhoades@indiana.edu

R. Mohanraj: Department of Mathematics, Anna University, Chennai-600 025, India
E-mail address: vrmraj@yahoo.com
Submit your manuscripts at http://www.hindawi.com