ANNIHILATORS OF NILPOTENT ELEMENTS

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Let x be a nilpotent element of an infinite ring R (not necessarily with 1). We prove that $A(x)$ —the two-sided annihilator of x —has a large intersection with any infinite ideal I of R in the sense that $\text{card}(A(x) \cap I) = \text{card} I$. In particular, $\text{card} A(x) = \text{card} R$; and this is applied to prove that if N is the set of nilpotent elements of R and $R \neq N$, then $\text{card}(R \setminus N) \geq \text{card} N$.

For an element x of a ring R , let $A_\ell(x)$, $A_r(x)$, and $A(x)$ denote, respectively, the left, right and two-sided annihilator of x in R . For a set X , we denote $\text{card} X$ by $|X|$; and say that a subset Y of X is *large* in X if $|Y| = |X|$. We prove that if x is any nilpotent element and I is any infinite ideal of R , then $A(x) \cap I$ is large in I , and in particular $|A_\ell(x)| = |A_r(x)| = |A(x)| = |R|$. The last result is applied to obtain a generalization of a result of Putcha and Yaquub [2] which shows that an infinite nonnil ring has infinitely many nonnilpotent elements. A short proof of their result is given in [1]. We prove a much stronger result showing that the set of nonnilpotent elements of a nonnil ring is at least as large as is its set of nilpotent elements. The following lemma is simple but crucial.

LEMMA 1. *Let R be an infinite ring, $(S, +)$ an infinite subgroup of $(R, +)$, and x an element of R . Then either $|Sx| = |S|$ or $|A_\ell(x) \cap S| = |S|$, and similarly $|xS| = |S|$ or $|A_r(x) \cap S| = |S|$.*

Proof. Consider the map $y \mapsto yx$ from $(S, +)$ onto $(Sx, +)$. The kernel is $A_\ell(x) \cap S$, so $|S| = |Sx| |A_\ell(x) \cap S|$ and the result follows since S is infinite. $\square$

A subset of a ring R is said to be *root closed* if whenever it contains a power of an element, it also contains the element itself.

THEOREM 2. *Let R be an infinite ring and α an infinite cardinal. Then, the following hold.*

- (i) *For any left (right) ideal I of R , the set $\{x \in R \mid |A_r(x) \cap I| = \alpha\}$ (resp., $\{x \in R \mid |A_\ell(x) \cap I| = \alpha\}$) is root closed. In particular, if I is infinite, $\{x \in R \mid |A_r(x) \cap I| = |I|\}$ (resp., $\{x \in R \mid |A_\ell(x) \cap I| = |I|\}$) is root closed, so it contains the set N of nilpotent elements of R .*

(ii) For any ideal I of R , $\{x \in R \mid |A(x) \cap I| = \alpha\}$ is root closed. In particular, if I is infinite, $\{x \in R \mid |A(x) \cap I| = |I|\}$ is root closed, so it contains N .

Proof. (i) Let $|A_r(x^n) \cap I| = \alpha$ for some $n \geq 2$ and consider $x^{n-1}(A_r(x^n) \cap I)$. By Lemma 1, either $|x^{n-1}(A_r(x^n) \cap I)| = \alpha$ or $|A_r(x^{n-1}) \cap I| = |A_r(x^{n-1}) \cap (A_r(x^n) \cap I)| = \alpha$. Now $x^{n-1}(A_r(x^n) \cap I) \subseteq A_r(x) \cap I \subseteq A_r(x^{n-1}) \cap I \subseteq A_r(x^n) \cap I$, so $|A_r(x^{n-1}) \cap I| = \alpha$ even when $|x^{n-1}(A_r(x^n) \cap I)| = \alpha$. It follows by induction that $|A_r(x) \cap I| = \alpha$.

(ii) Let $|A(x^n) \cap I| = \alpha$ for some $n \geq 2$. Since $A_\ell(x^n) \cap I$ is a left ideal and $|A_r(x^n) \cap (A_\ell(x^n) \cap I)| = |A(x^n) \cap I| = \alpha$, it follows by (i) that $|A_r(x) \cap (A_\ell(x^n) \cap I)| = \alpha = |A_\ell(x^n) \cap (A_r(x) \cap I)|$; and since $A_r(x) \cap I$ is a right ideal, we get, again by (i), that $|A_\ell(x) \cap (A_r(x) \cap I)| = \alpha$, namely $|A(x) \cap I| = \alpha$. $\square$

Applying the previous theorem for $I = R$, we obtain the following corollary.

COROLLARY 3. *Let x be a nilpotent element of an infinite ring R , then $|A_\ell(x)| = |A_r(x)| = |A(x)| = |R|$.*

The previous corollary will be applied in the proof of the above-mentioned generalization of a result of Putcha and Yaqub [2]. We also need the following result.

LEMMA 4. *Let b be a nonnilpotent element of an infinite ring R . If $R \setminus N$ is infinite, then $|A_\ell(b)| \leq |R \setminus N|$ and $|A_r(b)| \leq |R \setminus N|$.*

Proof. Let $x \in A_\ell(b) \cap N$, then $xb = 0$ and $x^n = 0$ for some $n \geq 1$. Let $m \geq n$, then $(b + x)^m = b^m + b^{m-1}x + \dots + bx^{m-1}$. Since $(b^{m-1}x + \dots + bx^{m-1})^2 = 0$ and $b \notin N$, $b^{2m} \neq 0$ and $(b + x)^m \neq 0$, so $b + x \notin N$. Hence, the map $x \mapsto b + x$ is $1 - 1$ from $A_\ell(b) \cap N$ into $R \setminus N$ and therefore $|A_\ell(b) \cap N| \leq |R \setminus N|$. Since $R \setminus N$ is infinite, we get that $|A_\ell(b)| = |A_\ell(b) \setminus N| + |A_\ell(b) \cap N| \leq |R \setminus N| + |R \setminus N| = |R \setminus N|$. $\square$

In a ring with 1, the map $x \mapsto 1 + x$ from N into $R \setminus N$ is $1 - 1$, so $|R \setminus N| \geq |N|$. The next theorem shows that the same result holds in any nonnil ring. In particular, we get the result of Putcha and Yaqub [2] stating that R is finite when $R \setminus N$ is finite and not empty.

THEOREM 5. *Let R be a nonnil ring, then $|R \setminus N| \geq |N|$.*

Proof. We start with R infinite. Suppose $|R \setminus N| < |N|$, then $|N| = |R|$ and $|R \setminus N| < |R|$. By the previous lemma, if $b \in R \setminus N$, $|A_\ell(b)| \leq |R \setminus N|$, so $|A_\ell(b)| < |R|$ and by Lemma 1, $|Rb| = |R|$. Now $|R| = |Rb| \leq |Nb| + |(R \setminus N)b|$ and $|(R \setminus N)b| \leq |R \setminus N| < |R|$, so $|Nb| = |R|$. Therefore, $|\{b + xb \mid x \in N\}| = |R|$, so since $|R \setminus N| < |R|$, there exists $x \in N$ such that $b + xb \notin R \setminus N$, namely $b + xb \in N$. Since $x \in N$, $1 + x$ is formally invertible, so $A_r(b + xb) = A_r(b)$. By Corollary 3, $|A_r(b + xb)| = |R|$ and by Lemma 4, $|A_r(b)| \leq |R \setminus N| < |R|$, a contradiction.

Now let R be finite and let J be its radical. Since J is nilpotent, if $a \in R$, $a + J$ is nilpotent in R/J if and only if a is nilpotent, and if $a \notin N$, $(a + J) \cap N = \emptyset$. Since R/J is a finite semisimple ring, it has 1, so at least half of its elements are nonnilpotent, hence at least half of the distinct cosets $a + J$, $a \in R$, do not intersect N , and therefore at least half of the elements of R are not nilpotent, so $|R \setminus N| \geq |N|$. $\square$

References

- [1] H. E. Bell and A. A. Klein, *On finiteness, commutativity, and periodicity in rings*, Math. J. Okayama Univ. **35** (1993), 181–188 (1995).
- [2] M. S. Putcha and A. Yaqub, *Rings with a finite set of nonnilpotents*, Int. J. Math. Math. Sci. **2** (1979), no. 1, 121–126.

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