BILINEAR MULTIPLIERS AND TRANSFERENCE

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We give de Leeuw-type transference theorems for bilinear multipliers. In particular, it is shown that bilinear multipliers arising from regulated functions $m(\xi, \eta)$ in $\mathbb{R} \times \mathbb{R}$ can be transferred to bilinear multipliers acting on $\mathbb{T} \times \mathbb{T}$ and $\mathbb{Z} \times \mathbb{Z}$. The results follow from the description of bilinear multipliers on the discrete real line acting on L^p -spaces.

1. Introduction

Let (p_1, p_2, p_3) be such that $0 < p_1, p_2, p_3 \leq \infty$, $1/p_1 + 1/p_2 = 1/p_3$ and let $m(\xi, \eta)$ be a bounded measurable function on $\mathbb{R}^2$. It is said to be a bilinear (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$ if

$$\mathcal{C}_m(f, g)(x) = \int_{\mathbb{R}^2} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) e^{2\pi i x(\xi + \eta)} d\xi d\eta \quad (1.1)$$

(defined for Schwarz test functions f and g in $\mathcal{S}$) extends to a bounded bilinear operator from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$.

The theory of these multipliers has been tremendously developed after the results proved by Lacey and Thiele (see [16, 18, 17]) which establish that $m(\xi, \nu) = \text{sign}(\xi + \alpha\nu)$ is a (p_1, p_2) -multiplier for each triple (p_1, p_2, p_3) such that $1 < p_1, p_2 \leq \infty$, $p_3 > 2/3$, and each $\alpha \in \mathbb{R} \setminus \{0, 1\}$.

The study of such multipliers was started by Coifman and Meyer (see [3, 4, 19]) for smooth symbols and new results for nonsmooth symbols, extending the ones given by the bilinear Hilbert transform, have been achieved by Gilbert and Nahmod (see [8, 9, 10]) and also by Muscalu et al. (see [20]).

We refer the reader also to [7, 12, 11, 15] for new results on bilinear multipliers and related topics.

In a recent paper (see [7]), Fan and Sato have shown certain de Leeuw-type theorems for transferring multilinear operators on Lebesgue and Hardy spaces from $\mathbb{R}^n$ to $\mathbb{T}^n$. Here we will consider bilinear multipliers on Lebesgue spaces $L^p(\mathbb{R})$ and get a characterization which allows us to transfer not only to the bilinear multipliers on $\mathbb{T}$ but also on $\mathbb{Z}$. Our approach will follow closely the ideas in the original paper by de Leeuw (see [6]) and will

provide an alternative proof of some results in [7], whose proof follows, in the multilinear case, the approach used by Stein and Weiss (see [21, page 260]).

We start by setting up natural analogous versions of bilinear multipliers in the periodic and discrete cases. Let $m = (m_{k,k'})$ be a bounded sequence and let $\tilde{m}$ be a periodic function on $\mathbb{T} \times \mathbb{T}$. Define for $\theta \in [-1/2, 1/2]$,

$$\mathcal{P}_m(f, g)(\theta) = \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} \hat{f}(k) \hat{g}(k') m_{k,k'} e^{2\pi i \theta(k+k')} \quad (1.2)$$

for functions f, g defined on $\mathbb{T}$, and for $k \in \mathbb{Z}$,

$$\mathcal{D}_{\tilde{m}}(a, b)(k) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} P(t) Q(s) \tilde{m}(t, s) e^{2\pi i k(t+s)} dt ds \quad (1.3)$$

for sequences $a = (a(n))_{n \in \mathbb{Z}}$ and $b = (b(n))_{n \in \mathbb{Z}}$, where $P(t) = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i n t}$ and $Q(t) = \sum_{n \in \mathbb{Z}} b(n) e^{2\pi i n t}$.

Now we say that m (resp., $\tilde{m}$) is a bilinear (p_1, p_2) -multiplier on $\mathbb{Z} \times \mathbb{Z}$ (resp., $\mathbb{T} \times \mathbb{T}$) if $\mathcal{P}_m$ (resp., $\mathcal{D}_{\tilde{m}}$) defines a bounded bilinear operator from $L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T})$ into $L^{p_3}(\mathbb{T})$ (resp., $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z})$ into $\ell^{p_3}(\mathbb{Z})$), where $1/p_1 + 1/p_2 = 1/p_3$.

Of course we can see these three cases as instances of the general bilinear multiplier acting on different groups. Let G be a locally compact abelian group and $\hat{G}$ its dual group with Haar measure μ . Let $1 \leq p_1, p_2 \leq \infty$ and let m be a bounded measurable function on $\hat{G} \times \hat{G}$. We say that m is a (p_1, p_2) -multiplier on $\hat{G} \times \hat{G}$ if the operator

$$T_m(f, g)(x) = \int_{\hat{G}} \int_{\hat{G}} \mathcal{F}f(\gamma_1) \mathcal{F}g(\gamma_2) m(\gamma_1, \gamma_2) \gamma_1(-x) \gamma_2(-x) d\mu(\gamma_1) d\mu(\gamma_2) \quad (1.4)$$

(defined for simple functions f and g) extends to a bounded bilinear operator from $L^{p_1}(G) \times L^{p_2}(G)$ to $L^{p_3}(G)$, where $1/p_1 + 1/p_2 = 1/p_3$. The reader is referred to [14] for the general theory in the linear case.

The first transference results on linear multipliers were given by de Leeuw (see [6]). He showed, among other things, that if m is regulated (all its points are Lebesgue points) and m is a p -multiplier on $\mathbb{R}$, then $(m(\varepsilon k))_k$ is a uniformly bounded p -multiplier for all $\varepsilon > 0$ on $\mathbb{Z}$ (see [21, page 264] for the converse of this result for continuous multipliers). Transference results of similar nature are presented in [1].

A general transference method was considered by [5] (see also the generalization given by [13]), but we will not consider these approaches in our bilinear generalization in the paper.

In [7], the multilinear version of the continuous result was shown, namely that for any continuous function $m(\xi, \eta)$, one has that m is a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$ if and only if $m(\varepsilon k, \varepsilon k')_{k,k'}$ is a uniformly bounded multiplier on $\mathbb{Z} \times \mathbb{Z}$ for $\varepsilon > 0$. An extension of the result to Lorentz spaces was achieved in [2].

We will first characterize the boundedness of bilinear multipliers on $\mathbb{R} \times \mathbb{R}$ by the existence of a constant K such that

$$\left| \sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t, s) \mu(\{t\}) \nu(\{s\}) \lambda(\{t + s\}) \right| \leq K \|\hat{\mu}\|_{B_{p_1}} \|\hat{\nu}\|_{B_{p_2}} \|\hat{\lambda}\|_{B_{p_3'}} \quad (1.5)$$

for all measures μ, ν, λ of finite supports.

This allows us to transfer from the continuous $\mathcal{C}_m$ to the discrete case $\mathcal{D}_{\tilde{m}}$ recovering some of the Fan-Sato results in [7].

We also obtain the transference from the continuous case $\mathcal{C}_m$ to the periodic case $\mathcal{P}_m$. Our main result establishes that m is a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$ if and only if $D_\varepsilon m = m_{\varepsilon \cdot, \varepsilon \cdot} \chi_{[-1/2, 1/2] \times [-1/2, 1/2]}$ (extended by periodicity) are uniformly bounded (p_1, p_2) -multipliers on $\mathbb{T} \times \mathbb{T}$.

The reader should be aware that the results of the paper can be stated for multilinear multipliers, with the condition $1/p = \sum_{i=1}^n (1/p_i)$, by considering the corresponding multilinear notions, for instance, for $m(\xi_1, \dots, \xi_n)$, one has

$$\mathcal{C}_m(f_1, \dots, f_n)(x) = \int_{\mathbb{R}^n} \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) m(\xi_1, \dots, \xi_n) e^{2\pi i x(\xi_1 + \cdots + \xi_n)} d\xi_1 \cdots d\xi_n, \quad (1.6)$$

and similar modifications for $\mathcal{P}_m$ and $\mathcal{D}_{\tilde{m}}$. We simply do the bilinear case for the sake of simplicity.

Throughout the paper, $1 \leq p_1, p_2, p_3 \leq \infty$ and $1/p_3 = 1/p_1 + 1/p_2$. For a given finite Borel measure μ on $\mathbb{R}$, we write $\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi t} d\mu(t)$ and, for an almost periodic function g , we denote $\|g\|_{B_p} = \lim_{T \rightarrow \infty} ((1/2T) \int_{-T}^T |g(t)|^p dt)^{1/p}$. We will use the notations $D_\varepsilon m(x, y) = m(\varepsilon x, \varepsilon y)$ and $\phi_\varepsilon(x) = (1/\varepsilon)\phi(x/\varepsilon)$.

2. Bilinear multipliers on $\mathbb{R} \times \mathbb{R}$

We start by reformulating the condition of (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$ using duality. The proof is straightforward and is left to the reader.

LEMMA 2.1. *Let $m(\xi, \eta)$ be a bounded measurable function on $\mathbb{R} \times \mathbb{R}$. Then m is a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$ if and only if there exists a constant K so that*

$$\left| \int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq K \|\hat{\phi}\|_{p_1} \|\hat{\psi}\|_{p_2} \|\hat{\nu}\|_{p_3'} \quad (2.1)$$

for all $\phi, \psi, \nu \in \mathcal{S}$.

Now we present some behavior of multipliers on $\mathbb{R} \times \mathbb{R}$ with respect to convolution and dilation operators to be used later on.

LEMMA 2.2. *Let $m(\xi, \eta)$ be a bounded measurable function on $\mathbb{R} \times \mathbb{R}$. If $\Phi \in L^1(\mathbb{R}^2)$ and m is a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$, then $\Phi * m$ is a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$ and $\|\mathcal{C}_{\Phi * m}\| \leq \|\Phi\|_1 \|\mathcal{C}_m\|$, where $\|\mathcal{C}_m\|$ stands for the norm of the corresponding bilinear map from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$.*

Proof. Let $f_s(x) = f(x+s)$ for any $s \in \mathbb{R}$ and function f . Then for any $s, t \in \mathbb{R}$ and $\phi, \psi, \nu \in \mathcal{S}$ with $\|\hat{\phi}\|_{p_1} = \|\hat{\psi}\|_{p_2} = \|\hat{\nu}\|_{p'_3} = 1$, we have

$$\left| \int_{\mathbb{R}^2} \phi_s(\xi) \psi_t(\eta) \nu_{t+s}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq K. \quad (2.2)$$

Now

$$\begin{aligned} & \int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) \Phi * m(\xi, \eta) d\xi d\eta \\ &= \int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) \left(\int_{\mathbb{R}^2} m(\xi - s, \eta - t) \Phi(s, t) ds dt \right) d\xi d\eta \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(\xi + s) \psi(\eta + t) \nu(\xi + \eta + s + t) m(\xi, \eta) \Phi(s, t) d\xi d\eta ds dt. \end{aligned} \quad (2.3)$$

And the result follows by Lemma 2.1. $\square$

LEMMA 2.3. Let $\varepsilon > 0$ and $m(\xi, \eta)$ be a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$. Then $m(\varepsilon\xi, \varepsilon\eta)$ is also a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$ and $\|\mathcal{C}_{m(\varepsilon, \cdot, \cdot)}\| \leq \|\mathcal{C}_m\|$.

Proof. For $\phi, \psi, \nu \in \mathcal{S}$ and $\|\hat{\phi}\|_{p_1} = \|\hat{\psi}\|_{p_2} = \|\hat{\nu}\|_{p'_3} = 1$, we have

$$\begin{aligned} & \int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) m(\varepsilon\xi, \varepsilon\eta) d\xi d\eta \\ &= \int_{\mathbb{R}^2} \frac{1}{\varepsilon^{1/p'_1}} \phi\left(\frac{\xi}{\varepsilon}\right) \frac{1}{\varepsilon^{1/p'_2}} \psi\left(\frac{\eta}{\varepsilon}\right) \frac{1}{\varepsilon^{1/p'_3}} \nu\left(\frac{\xi + \eta}{\varepsilon}\right) m(\xi, \eta) d\xi d\eta. \end{aligned} \quad (2.4)$$

The proof is finished invoking Lemma 2.1 again. $\square$

THEOREM 2.4. Let $m(\xi, \eta)$ be a bounded continuous function on $\mathbb{R} \times \mathbb{R}$. The following are equivalent:

- (i) m is a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$;
- (ii) there exists a constant K so that

$$\left| \sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t, s) \mu(\{t\}) \nu(\{s\}) \lambda(\{t + s\}) \right| \leq K \|\hat{\mu}\|_{B_{p_1}} \|\hat{\nu}\|_{B_{p_2}} \|\hat{\lambda}\|_{B_{p'_3}} \quad (2.5)$$

for all measures μ, ν, λ supported on a finite number of points.

Proof. (i) $\Rightarrow$ (ii). Assume that m is a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$. Denote by ϕ the Gaussian function $\phi(x) = e^{-x^2/2}$. Then for any $\alpha > 0$ and $a \in \mathbb{R}$,

$$\left(\frac{1}{\varepsilon}\right)^\alpha \phi^\alpha\left(\frac{\xi - a}{\varepsilon}\right) = \delta_a * (\phi_\varepsilon)^\alpha(\xi). \quad (2.6)$$

Now choose $0 < \alpha, \beta, \gamma$ such that $\alpha + \beta + \gamma = 2$, and $\mu = \delta_a$, $\nu = \delta_b$, and $\lambda = \delta_c$ for $a, b, c \in \mathbb{R}$. It is easily checked that

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{1}{\varepsilon^2} \phi^\alpha\left(\frac{\xi-a}{\varepsilon}\right) \phi^\beta\left(\frac{\eta-b}{\varepsilon}\right) \phi^\gamma\left(\frac{\xi+\eta-c}{\varepsilon}\right) m(\xi, \eta) d\xi d\eta \\ &= \int_{\mathbb{R}^2} \phi^\alpha(\xi) \phi^\beta(\eta) \phi^\gamma\left(\xi + \eta + \frac{a+b-c}{\varepsilon}\right) m(a + \varepsilon\xi, b + \varepsilon\eta) d\xi d\eta \\ &= \int_{\mathbb{R}^2} \mu * (\phi_\varepsilon)^\alpha(\xi) \nu * (\phi_\varepsilon)^\beta(\eta) \lambda * (\phi_\varepsilon)^\gamma(\xi + \eta) m(\xi, \eta) d\xi d\eta. \end{aligned} \quad (2.7)$$

Since

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \phi^\alpha(\xi) \phi^\beta(\eta) \phi^\gamma\left(\xi + \eta + \frac{a+b-c}{\varepsilon}\right) m(a + \varepsilon\xi, b + \varepsilon\eta) \\ &= \delta_c(a+b) \phi^\alpha(\xi) \phi^\beta(\eta) \phi^\gamma(\xi + \eta) m(a, b), \end{aligned} \quad (2.8)$$

the Lebesgue convergence theorem implies that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{1}{\varepsilon^2} \phi^\alpha\left(\frac{\xi-a}{\varepsilon}\right) \phi^\beta\left(\frac{\eta-b}{\varepsilon}\right) \phi^\gamma\left(\frac{\xi+\eta-c}{\varepsilon}\right) m(\xi, \eta) d\xi d\eta \\ &= C m(a, b) \delta_c(a+b) = C m(a, b) \mu(\{a\}) \nu(\{b\}) \lambda(\{a+b\}), \end{aligned} \quad (2.9)$$

where $C = \int_{\mathbb{R}^2} \phi^\alpha(\xi) \phi^\beta(\eta) \phi^\gamma(\xi + \eta) d\xi d\eta$.

Therefore we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \mu * (\phi_\varepsilon)^\alpha(\xi) \nu * (\phi_\varepsilon)^\beta(\eta) \lambda * (\phi_\varepsilon)^\gamma(\xi + \eta) m(\xi, \eta) d\xi d\eta \\ &= C \sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t, s) \mu(\{t\}) \nu(\{s\}) \lambda(\{(t+s)\}) \end{aligned} \quad (2.10)$$

for all measures μ, ν, λ having their supports on finite sets of points.

On the other hand, from (i) and Lemma 2.1, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \mu * (\phi_\varepsilon)^\alpha(\xi) \nu * (\phi_\varepsilon)^\beta(\eta) \lambda * (\phi_\varepsilon)^\gamma(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \\ &\leq K \|\widehat{\mu}(\phi_\varepsilon)^\alpha\|_{p_1} \|\widehat{\nu}(\phi_\varepsilon)^\beta\|_{p_2} \|\widehat{\lambda}(\phi_\varepsilon)^\gamma\|_{p_3}. \end{aligned} \quad (2.11)$$

We now choose $\alpha = 1/p'_1$, $\beta = 1/p'_2$, and $\gamma = 1/p_3$. Since $(\phi_\varepsilon)^\alpha = \varepsilon^{1-\alpha}/\alpha^{1/2} \phi_{\varepsilon\alpha^{-1/2}}$, we get $\widehat{(\phi_\varepsilon)^\alpha}(\xi) = C_\alpha \varepsilon^{1/p_1} e^{-\varepsilon^2 \xi^2/2\alpha}$, $\widehat{(\phi_\varepsilon)^\beta}(\xi) = C_\beta \varepsilon^{1/p_2} e^{-\varepsilon^2 \xi^2/2\beta}$, and $\widehat{(\phi_\varepsilon)^\gamma}(\xi) = C_\gamma \varepsilon^{1/p_3} e^{-\varepsilon^2 \xi^2/2\gamma}$ for some constants C_α , C_β , and C_γ .

Now taking into account that $\int_{\mathbb{R}} e^{-\varepsilon^2 p_1 \xi^2/2\alpha} d\xi = C'_\alpha \varepsilon^{-1}$, we have

$$\|\widehat{\mu}(\phi_\varepsilon)^\alpha\|_{p_1} = C \left(\frac{1}{A(\varepsilon)} \int_{\mathbb{R}} |\widehat{\mu}(\xi)|^{p_1} \varepsilon^{-p_1 \varepsilon^2 \xi^2/2\alpha} d\xi \right)^{1/p_1}, \quad (2.12)$$

for $A(\varepsilon) = \int_{\mathbb{R}} e^{-\varepsilon^2 p_1 \xi^2/2\alpha} d\xi$. Hence $C \|\widehat{\mu}\|_{B_{p_1}} = \lim_{\varepsilon \rightarrow 0} \|\widehat{\mu} \phi_\varepsilon^\alpha\|_{p_1}$.

Applying a similar procedure for ν and λ , we finish this implication.

(ii) $\Rightarrow$ (i). From (ii) we can get that the inequality holds for all finite measures μ, ν, λ , with countable supports. We take ϕ, ψ , and ρ such that $\hat{\phi}, \hat{\psi}$, and $\hat{\rho}$ have compact support contained in $[-N/2, N/2]$ for N big enough. Now consider μ_N, ν_N , and λ_N the measures with support in $(1/N)\mathbb{Z}$ whose Fourier transform coincides with the periodic extensions of $\hat{\phi}, \hat{\psi}$, and $\hat{\rho}$. In particular, we have

$$\mu_N\left(\left\{\frac{n}{N}\right\}\right) = \frac{1}{N}\phi\left(\frac{n}{N}\right), \quad \nu_N\left(\left\{\frac{n}{N}\right\}\right) = \frac{1}{N}\psi\left(\frac{n}{N}\right), \quad \lambda_N\left(\left\{\frac{n}{N}\right\}\right) = \frac{1}{N}\rho\left(\frac{n}{N}\right). \quad (2.13)$$

Therefore we have

$$\begin{aligned} \lim_{N \rightarrow \infty} N \sum_{(t,s) \in \mathbb{R} \times \mathbb{R}} m(t,s) \mu_N(\{t\}) \nu_N(\{s\}) \lambda_N(\{t+s\}) \\ = \lim_{N \rightarrow \infty} \sum_{(n,m) \in \mathbb{Z} \times \mathbb{Z}} m\left(\frac{n}{N}, \frac{m}{N}\right) \phi\left(\frac{n}{N}\right) \psi\left(\frac{m}{N}\right) \rho\left(\frac{n+m}{N}\right) \frac{1}{N^2} \\ = \int_{\mathbb{R}^2} m(\xi, \nu) \phi(\xi) \psi(\eta) \rho(\xi + \eta) d\xi d\eta. \end{aligned} \quad (2.14)$$

Now observe that $\|\hat{\mu}_N\|_{B_{p_1}} = ((1/2N) \int_{-N}^N |\hat{\phi}(\xi)|^{p_1} d\xi)^{1/p_1} = (1/2N)^{1/p_1} \|\hat{\phi}\|_{p_1}$ and the same for the others.

Using that $\|\hat{\mu}_N\|_{B_{p_1}} \cdot \|\hat{\nu}_N\|_{B_{p_2}} \|\hat{\lambda}_N\|_{B_{p'_3}} = 1/2N$ and passing to the limit, we get the result. $\square$

Remark 2.5. We point out that condition (ii) in Theorem 2.4 is simply a way to say that m defines a multiplier on $\mathbb{D} \times \mathbb{D}$ where $\mathbb{D}$ is the group $\mathbb{R}$ with the discrete topology (see [6]).

Recall that a function m is called regulated if

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon^2} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} m(x-s, y-t) ds dt = m(x, y) \quad (2.15)$$

for all $(x, y) \in \mathbb{R}^2$.

THEOREM 2.6. *Let $m(\xi, \eta)$ be a bounded regulated function on $\mathbb{R} \times \mathbb{R}$. Then m is a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$ if and only if there exists a constant K so that*

$$\left| \sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t,s) \mu(\{t\}) \nu(\{s\}) \lambda(\{t+s\}) \right| \leq K \|\hat{\mu}\|_{B_{p_1}} \|\hat{\nu}\|_{B_{p_2}} \|\hat{\lambda}\|_{B_{p'_3}} \quad (2.16)$$

for all measures μ, ν, λ having their supports on finite sets of points.

Proof. Assume that m is a (p_1, p_2) -multiplier. Let $\Phi(s, t) = (1/4)\chi_{[-1,1]}(s)\chi_{[-1,1]}(t)$ and $\Phi_\varepsilon(\xi, \eta) = (1/\varepsilon^2)\Phi(\xi/\varepsilon, \eta/\varepsilon)$ for $\varepsilon > 0$. Now Lemma 2.2, Theorem 2.4, and the fact that $m(x, y) = \lim_{\varepsilon \rightarrow 0} m * \Phi_\varepsilon(x, y)$ give the direct implication.

Conversely, assume (2.16) for μ, ν, λ having finite supports. Then

$$\begin{aligned} & \sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} (m * \Phi_\varepsilon)(t, s) \mu(\{t\}) \nu(\{s\}) \lambda(\{t+s\}) \\ &= \int_{\mathbb{R}^2} \left(\sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t-u, s-v) \mu(\{t\}) \nu(\{s\}) \lambda(\{t+s\}) \right) \Phi_\varepsilon(u, v) du dv \\ &= \int_{\mathbb{R}^2} \left(\sum_{t \in \mathbb{R}} \sum_{s \in \mathbb{R}} m(t, s) \mu(\{t+u\}) \nu(\{s+v\}) \lambda(\{t+s+u+v\}) \right) \Phi_\varepsilon(u, v) du dv. \end{aligned} \quad (2.17)$$

This shows that $m * \Phi_\varepsilon$ verifies (2.16) with a uniform constant for all $\varepsilon > 0$. Now apply Theorem 2.4 to get that $m * \Phi_\varepsilon$ are (p_1, p_2) -multipliers with uniform norm.

Finally we have that for $\phi, \psi, \nu \in \mathcal{S}$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \\ &= \left| \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \phi(\xi) \psi(\eta) \nu(\xi + \eta) (m * \Phi_\varepsilon)(\xi, \eta) d\xi d\eta \right| \\ &\leq C \|\hat{\phi}\|_{p_1} \|\hat{\psi}\|_{p_2} \|\hat{\nu}\|_{p'_3}. \end{aligned} \quad (2.18)$$

The result now follows from Lemma 2.1. □

3. Transference theorems

We mention the formulations for (p_1, p_2) -multipliers on the groups $\mathbb{T}$ and $\mathbb{Z}$ which follow directly from duality.

LEMMA 3.1. *Let $\tilde{m}(t, s)$ be a bounded measurable function on $\mathbb{T} \times \mathbb{T}$. Then m is a (p_1, p_2) -multiplier on $\mathbb{T} \times \mathbb{T}$ if and only if there exists a constant K so that*

$$\left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} P_a(t) P_b(s) P_c(t+s) \tilde{m}(t, s) dt ds \right| \leq K \|a\|_{p_1} \|b\|_{p_2} \|c\|_{p'_3} \quad (3.1)$$

for all finite sequences $(a(n))_n, (b(n))_n, (c(n))_n$, where $P_a(t) = \sum_n a(n) e^{2\pi i n t}$.

LEMMA 3.2. *Let $(m_{k,k'})$ be a bounded sequence on $\mathbb{Z} \times \mathbb{Z}$. Then m is a (p_1, p_2) -multiplier on $\mathbb{Z} \times \mathbb{Z}$ if and only if there exists a constant K so that*

$$\left| \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} m_{k,k'} \hat{P}(k) \hat{Q}(k') \hat{R}(k+k') \right| \leq K \|P\|_{p_1} \|Q\|_{p_2} \|R\|_{p'_3} \quad (3.2)$$

for all trigonometric polynomials P, Q , and R .

THEOREM 3.3 (see [7, Theorem 1]). *Let $m(\xi, \eta)$ be a regulated bounded function on $\mathbb{R} \times \mathbb{R}$. If $m(\xi, \eta)$ is a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$, then $(m(k, k'))_{k, k'}$ is a (p_1, p_2) -multiplier on $\mathbb{Z} \times \mathbb{Z}$.*

Proof. According to Lemma 3.2, we have to show that there exists a constant K so that

$$\left| \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} m(k, k') \hat{P}(k) \hat{Q}(k') \hat{R}(k + k') \right| \leq K \|P\|_{p_1} \|Q\|_{p_2} \|R\|_{p'_3} \quad (3.3)$$

for all trigonometric polynomials P , Q , and R .

This follows by selecting the measures μ , ν , λ in Theorem 2.6 such that $\hat{\mu} = P$, $\hat{\nu} = Q$, and $\hat{\lambda} = R$. $\square$

THEOREM 3.4. *Let $m(\xi, \eta)$ be a bounded regulated function on $\mathbb{R} \times \mathbb{R}$. The following are equivalent:*

- (i) $m(\xi, \eta)$ is a (p_1, p_2) -multiplier on $\mathbb{R} \times \mathbb{R}$;
- (ii) $m(\varepsilon \cdot, \varepsilon \cdot) \chi_{[-1/2\varepsilon, 1/2\varepsilon]} \chi_{[-1/2\varepsilon, 1/2\varepsilon]}$ (extended by periodicity) are uniformly bounded (p_1, p_2) -multipliers on $\mathbb{T} \times \mathbb{T}$.

Proof. (i) $\Rightarrow$ (ii). Using Lemma 3.1, it suffices to show that for any finite sequences $(a(n))_n$, $(b(n))_n$, and $(c(n))_n$ with $\|a\|_{p_1} = \|b\|_{p_2} = \|c\|_{p'_3} = 1$, there exists a constant $K > 0$ such that

$$\left| \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} m(\xi, \eta) P_a(\xi) P_b(\eta) P_c(\xi + \eta) d\xi d\eta \right| \leq K, \quad (3.4)$$

where $P_a(\xi) = \sum_n a(n) e^{2\pi i n \xi}$.

Since $P_a(x) \chi_{[-1/2, 1/2]}(x) = \hat{\phi}_a(x)$, where $\phi_a(x) = \sum_n a(n) (\sin(\pi(x - n))/\pi(x - n))$, and $P_c(x) \chi_{[-1, 1]}(x) = \hat{\psi}_c(x)$, where $\psi_c(x) = \sum_n c(n) (\sin(2\pi(x - n))/\pi(x - n))$, we can write

$$\begin{aligned} & \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} m(\xi, \eta) P_a(\xi) P_b(\eta) P_c(\xi + \eta) d\xi d\eta \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} m(\xi, \eta) \hat{\phi}_a(\xi) \hat{\phi}_b(\eta) \hat{\psi}_c(\xi + \eta) d\xi d\eta. \end{aligned} \quad (3.5)$$

Using now the assumption and Shanon's sampling theorem, one gets $\|\psi_a\|_{L^p(\mathbb{R})} \leq C_1 \|\phi_a\|_{L^p(\mathbb{R})} \leq C_2 \|a\|_{\ell_p} \leq C_3 \|\psi_a\|_{L^p(\mathbb{R})}$ for some constants C_i for $i = 1, 2, 3$. Hence the desired inequality follows.

Now we apply Lemma 2.3 to get the result for each ε .

(ii) $\Rightarrow$ (i). We take ϕ and ψ such that $\text{supp } \phi$ and $\text{supp } \psi$ are contained in $[-1/4, 1/4]$. For a fixed $u \in [-1/2, 1/2]$, consider the periodic extensions of the functions $\hat{\phi}(\xi) e^{2\pi i u \xi}$, $\hat{\psi}(\eta) e^{2\pi i u \eta}$ to be denoted $\tilde{P}_u$ and $\tilde{Q}_u$, respectively.

If $a^u(n) = \int_{-1/2}^{1/2} \tilde{P}_u(\xi) e^{-i2\pi n \xi} d\xi$, $b^u(n) = \int_{-1/2}^{1/2} \tilde{Q}_u(\xi) e^{-i2\pi n \xi} d\xi$ for all $n \in \mathbb{Z}$, we have that if $x = k + u$ for some $k \in \mathbb{Z}$ and $u \in [-1/2, 1/2]$,

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} m(\xi, \eta) \hat{\phi}(\xi) \hat{\psi}(\eta) e^{2\pi i x(\xi + \eta)} d\xi d\eta \\ &= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} m(\xi, \eta) \tilde{P}_u(\xi) \tilde{Q}_u(\eta) e^{2\pi i k(\xi + \eta)} d\xi d\eta. \end{aligned} \quad (3.6)$$

Let $\tilde{m}(\xi, \eta) = m(\xi, \eta) \chi_{[-1/2, 1/2]}(\xi) \chi_{[-1/2, 1/2]}(\eta)$. Hence for $x = u + k$,

$$\mathcal{C}_m(\phi, \psi)(x) = \mathcal{D}_{\tilde{m}}(a^u, b^u)(k). \quad (3.7)$$

Now

$$\begin{aligned}
 & \int_{\mathbb{R}} |\mathcal{C}_m(\phi, \psi)(x)|^{p_3} dx \\
 &= \sum_k \int_{-1/2}^{1/2} |\mathcal{C}_m(\phi, \psi)(k+u)|^{p_3} du \\
 &= \int_{-1/2}^{1/2} \sum_k |\mathcal{D}_{\tilde{m}}(a^u, b^u)(k)|^{p_3} du \\
 &\leq \|\mathcal{D}_{\tilde{m}}\|^{p_3} \int_{-1/2}^{1/2} \left(\sum_k |a^u(k)|^{p_1} \right)^{p_3/p_1} \left(\sum_k |b^u(k)|^{p_2} \right)^{p_3/p_2} du \\
 &\leq \|\mathcal{D}_{\tilde{m}}\|^{p_3} \left(\int_{-1/2}^{1/2} \sum_k |a^u(k)|^{p_1} du \right)^{p_3/p_1} \left(\int_{-1/2}^{1/2} \sum_k |b^u(k)|^{p_2} du \right)^{p_3/p_2} \\
 &= \|\mathcal{D}_{\tilde{m}}\|^{p_3} \left(\int_{-1/2}^{1/2} \sum_k |\phi(u+k)|^{p_1} du \right)^{p_3/p_1} \left(\int_{-1/2}^{1/2} \sum_k |\psi(u+k)|^{p_2} du \right)^{p_3/p_2} \\
 &= \|\mathcal{D}_{\tilde{m}}\|^{p_3} \|\phi\|_{p_1}^{p_3} \|\psi\|_{p_2}^{p_3}.
 \end{aligned} \tag{3.8}$$

In the general case if ϕ, ψ are such that $\hat{\phi}, \hat{\psi}$ have compact support, then there exists $\varepsilon > 0$ so that $\hat{\phi}_\varepsilon, \hat{\psi}_\varepsilon$ have their support in $[-1/4, 1/4]$. Now observe that

$$\mathcal{C}_m(\phi, \psi)(x) = \varepsilon^2 C_{m(\varepsilon \cdot, \varepsilon \cdot)}(\phi_\varepsilon, \psi_\varepsilon)(\varepsilon x). \tag{3.9}$$

Applying the previous case and the assumption, we obtain

$$\begin{aligned}
 \|\mathcal{C}_m(\phi, \psi)\|_{p_3} &= \varepsilon^{2-1/p_3} \|C_{m(\varepsilon \cdot, \varepsilon \cdot)}(\phi_\varepsilon, \psi_\varepsilon)\|_{p_3} \\
 &\leq K \varepsilon^{2-1/p_3} \|\phi_\varepsilon\|_{p_1} \|\psi_\varepsilon\|_{p_2} \\
 &= K \varepsilon^{2-1/p_3} \|\phi\|_{p_1} \varepsilon^{-1/p_1'} \|\psi\|_{p_1} \varepsilon^{-1/p_2'} \\
 &= K \|\phi\|_{p_1} \|\psi\|_{p_1}.
 \end{aligned} \tag{3.10}$$

□

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