SCHATTEN'S THEOREMS ON FUNCTIONALLY DEFINED SCHUR ALGEBRAS

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For each triple of positive numbers $p,q,r \geq 1$ and each commutative C^*-algebra $\mathcal{B}$ with identity 1 and the set $s(\mathcal{B})$ of states on $\mathcal{B}$, the set $\mathcal{S}^r(\mathcal{B})$ of all matrices $A = [a_{jk}]$ over $\mathcal{B}$ such that $\varphi[A^{[r]}] := [\varphi(|a_{jk}|^r)]$ defines a bounded operator from ℓ^p to ℓ^q for all $\varphi \in s(\mathcal{B})$ is shown to be a Banach algebra under the Schur product operation, and the norm $||A|| = |||A|||_{p,q,r} = \sup\{||\varphi[A^{[r]}]||^{1/r} : \varphi \in s(\mathcal{B})\}$. Schatten's theorems about the dual of the compact operators, the trace-class operators, and the decomposition of the dual of the algebra of all bounded operators on a Hilbert space are extended to the $\mathcal{S}^r(\mathcal{B})$ setting.

1. Introduction

Fix p and q with $1 \leq p,q < \infty$. The space of pth power summable sequences of complex numbers is denoted by ℓ^p, and the space of matrices which define bounded linear transformations from ℓ^p to ℓ^q is denoted by $\mathcal{B}(\ell^p,\ell^q)$. Let $A = [a_{jk}]$, $B = [b_{jk}]$ be infinite matrices, not necessarily in $\mathcal{B}(\ell^p,\ell^q)$. The *Schur product* $A \bullet B$ of A and B is defined by $A \bullet B = [a_{jk}b_{jk}]$. Many areas in mathematics such as matrix theory, function theory, operator theory, and operator algebras have made use of results from the study of Schur product and have injected new problems in return. See [1, 4, 6] for further references to the related literature.

As a generalization of a result of Schur in [9] for $p = 2$, and $q = 2$, Bennett proved in [1, Theorem 2.2] the following theorem.

Theorem 1.1 (Bennett). *If* $\Lambda = [\lambda_{jk}], \Sigma = [\sigma_{jk}] \in \mathcal{B}(\ell^p,\ell^q)$, *then* $\Lambda \bullet \Sigma = [\lambda_{jk}\sigma_{jk}] \in \mathcal{B}(\ell^p,\ell^q)$, *and*

$$||\Lambda \bullet \Sigma||_{p,q} \leq ||\Lambda||_{p,q}||\Sigma||_{p,q}. \tag{1.1}$$

(i.e., $\mathcal{B}(\ell^p,\ell^q)$ *is a commutative Banach algebra under the Schur product operation and operator norm* $||\cdot||_{p,q}$*.)*

Based on these results of Schur and Bennett, in [2] we studied algebras, under the Schur product operation, of matrices over a Banach algebra, which have the matrix of

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the norms of the entries define bounded operators. Here we give a functional version of the generalization. This is another direction of generalization of the numerical Schur r-algebras, $\mathcal{S}^r$ discussed therein.

A matrix $A = [a_{jk}]$, over $\mathbb{C}$, is in the *absolute Schur r-algebra* $\mathcal{S}^r$ if the *Schur rth power* $A^{[r]} := [|a_{jk}|^r]$ of A is in $\mathcal{B}(\ell^p, \ell^q)$. The *$r$-norm* of A is defined by $\|A\| := |||A|||_{p,q,r} := \|A^{[r]}\|^{1/r}$, where $\|A^{[r]}\| = \|A^{[r]}\|_{p,q}$ is the operator norm of $A^{[r]}$ as an element of $\mathcal{B}(\ell^p, \ell^q)$. That $|||\cdot|||_{p,q,r}$ is a norm follows, as expected, from an inequality that is analogous to the Hölder inequality.

Theorem 1.2. *For each $r \geq 1$, $\mathcal{S}^r$ is a Banach algebra under the Schur multiplication and the norm $|||\cdot|||_{p,q,r}$.*

This is a special case of a result in [2] that will be used here. The matrices could be over a Banach algebra, and the norm is defined by the nonnegative matrix of the rth power of norms of the entries. Here we investigate the situation where the matrices are over a commutative C^*-algebra $\mathcal{B}$, the norm is defined by the absolute values of linear functional values of the entries, as the linear functional runs over the set of states on $\mathcal{B}$.

In [7] Schatten's theorems [8] concerning the dual of the compact operators, the trace-class operators, and the decomposition of the dual of the algebra of bounded operators on a Hilbert space have been extended to the setting of matrices of functions acting on function sequence space analogous to the ℓ^2 sequence space. Here we extend these Schatten-type theorems to the algebra $\mathcal{S}^r(\mathcal{B})$ consisting of certain classes of matrices over a commutative C^*-algebra $\mathcal{B}$, in a situation where p and q need not be equal to 2.

Since the p and q will be fixed throughout our discussion, we will sometimes suppress the subscripts p, q in $\|\cdot\|_{p,q}$ and write $\|\cdot\|$ instead, if no confusion can arise. We will occasionally use subscripts if an emphasis for clarity is warranted. We also use $\|\cdot\|$ to denote the norm on a Banach space, and let the context determine which one is intended.

For convenience of reference, we also record the following simple useful fact.

Lemma 1.3. *Let $[\alpha_{jk}]$ and $[\beta_{jk}]$ be matrices over the complex field $\mathbb{C}$ such that $|\alpha_{jk}| \leq \beta_{jk}$ for all j and k. Suppose that $[\beta_{jk}] \in \mathcal{B}(\ell^p, \ell^q)$; that is, the matrix defines a bounded linear transformation from ℓ^p to ℓ^q. Then $[\alpha_{jk}] \in \mathcal{B}(\ell^p, \ell^q)$ and $\|[\alpha_{jk}]\| \leq \|[\beta_{jk}]\|$.*

2. Definitions and preliminary results

We establish some of our results in a more general setting before moving on to the settings on which Schur product makes sense. Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|$, and dual space $\mathcal{X}^\#$. Consider the set $\mathcal{M}(\mathcal{X})$ of all infinite matrices over $\mathcal{X}$. Let $(\mathcal{X}^\#)_1$ denote the set of all bounded linear functionals on $\mathcal{X}$ which have norm 1. Let $s(\mathcal{X}) \subseteq (\mathcal{X}^\#)_1$ be a set of linear functionals on $\mathcal{X}$ such that

(i) $\|x\| = \sup\{|f(x)| : f \in s(\mathcal{X})\}$ for every $x \in \mathcal{X}$;

(ii) the linear span of $s(\mathcal{X})$ is all of $X^\#$.

One such example is $s(\mathcal{X}) = (\mathcal{X}^\#)_1$ by the Hahn-Banach theorem and the fact that all linear functionals in $\mathcal{X}^\#$ are multiples of elements in $(\mathcal{X}^\#)_1$.

For each $f \in \mathcal{X}^\#$, and each matrix $A = [a_{jk}] \in \mathcal{M}(\mathcal{X})$, denote by $f[A] = [f(a_{jk})]$ the complex matrix whose (j,k) entry is $f(a_{jk})$. For fixed p and q with $1 \leq p, q < \infty$, regard

the matrix $f[A]$ as a linear transformation of ℓ^p to ℓ^q, if it is defined. (The closed graph theorem implies that it is bounded if it is everywhere defined.) Let $\mathcal{S}(\mathcal{X}) = \mathcal{S}(\mathcal{X})_{p,q}$ be the set of matrices $A = [a_{jk}] \in \mathcal{M}(\mathcal{X})$ such that $f[A] \in \mathcal{B}(\ell^p, \ell^q)$ for all $f \in s(\mathcal{X})$. The following result provides us with a natural way of defining a norm on $\mathcal{S}(\mathcal{X})$.

Theorem 2.1. *Let $\mathcal{X}$ and $s(\mathcal{X})$ be as above, and $A = [a_{jk}] \in \mathcal{S}(\mathcal{X})$, so that $f[A] \in \mathcal{B}(\ell^p, \ell^q)$ for all $f \in s(\mathcal{X})$. Then*

$$\sup\{\|f[A]\| : f \in s(\mathcal{X})\} < \infty. \tag{2.1}$$

Proof. First we note that since each $f \in \mathcal{X}^\#$ is a linear combination $f = \sum_{j=1}^n \alpha_j g_j$ of elements $g_1, \ldots, g_n$ in $s(\mathcal{X})$, and for the given $A \in \mathcal{S}(\mathcal{X})$, $g_j[A] \in \mathcal{B}(\ell^p, \ell^q)$, $1 \le j \le n$, and hence $f[A] = \alpha_1(g_1[A]) + \cdots + \alpha_n(g_n[A]) \in \mathcal{B}(\ell^p, \ell^q)$. Therefore, the map $\mathcal{T}_A : f \mapsto f[A]$ is a linear transformation from $\mathcal{X}^\#$ to $\mathcal{B}(\ell^p, \ell^q)$. Since both the domain $\mathcal{X}^\#$ and codomain $\mathcal{B}(\ell^p, \ell^q)$ are Banach spaces, the continuity of the map $\mathcal{T}_A$ will follow if we can show that the graph of $\mathcal{T}_A$ is closed. To that end, suppose that $\{(f_n, \mathcal{T}_A(f_n))\}$ is a sequence in the graph $\mathcal{G}(\mathcal{T}_A) \subseteq \mathcal{X}^\# \oplus_\infty \mathcal{B}(\ell^p, \ell^q)$ of $\mathcal{T}_A$ that converges to some $(f, M) \in \mathcal{X}^\# \oplus_\infty \mathcal{B}(\ell^p, \ell^q)$ with $M = [m_{jk}]$. Then $f_n \to f$ in $\mathcal{X}^\#$ and $f_n[A] \to M$ in $\mathcal{B}(\ell^p, \ell^q)$. Thus, for each $j, k = 1, 2, \ldots$, $|f_n(a_{jk}) - m_{jk}| \le \|f_n[A] - M\| \to 0$ as $n \to \infty$. Also the convergence of $\{f_n\}$ to f implies that $|f_n(a_{jk}) - f(a_{jk})| \to 0$ as $n \to \infty$. Therefore, $f(a_{jk}) = m_{jk}$ for all j, k, and hence $f[A] = M$. Thus the graph of $\mathcal{T}_A$ is closed. By the closed graph theorem, $\mathcal{T}_A$ is bounded. Therefore,

$$\sup\{\|f[A]\| : f \in s(\mathcal{X})\} = \sup\{|\mathcal{T}_A(f)| : f \in s(\mathcal{X})\} \le \|\mathcal{T}_A\| < \infty. \tag{2.2}$$

□

Next we prove that $\mathcal{S}(\mathcal{X})$ is a Banach space.

Theorem 2.2. *The set $\mathcal{S}(\mathcal{X})$ is a Banach space under the usual (entrywise) scalar multiplication and addition, and the norm*

$$\|A\| := \sup\{\|f[A]\| : f \in s(\mathcal{X})\}, \quad A \in \mathcal{S}(\mathcal{X}). \tag{2.3}$$

Proof. First we show that the function as defined in Theorem 2.1 is indeed a norm on $\mathcal{S}(\mathcal{X})$. Let $A, B \in \mathcal{S}(\mathcal{X})$ and let $f \in s(\mathcal{X})$. Then $\|f[A+B]\| = \|f[A] + f[B]\| \le \|f[A]\| + \|f[B]\| \le \|A\| + \|B\|$. Therefore, $\|f[A+B]\| \le \|A\| + \|B\|$ for all $f \in s(\mathcal{X})$, and hence $\|A+B\| \le \|A\| + \|B\|$.

To prove that $\mathcal{S}(\mathcal{X})$ is complete, let $\{A_n\}$ be a Cauchy sequence in $\mathcal{S}(\mathcal{X})$. Then for each $f \in s(\mathcal{X})$,

$$\|f[A_n] - f[A_m]\| = \|f[A_n - A_m]\| \le \|A_n - A_m\| \longrightarrow 0 \quad \text{as } n, m \longrightarrow \infty. \tag{2.4}$$

Thus $\{f[A_n]\}$ is a Cauchy sequence in $\mathcal{B}(\ell^p, \ell^q)$. Since $\mathcal{B}(\ell^p, \ell^q)$ is complete, there exists a bounded matrix $\Lambda_f \in \mathcal{B}(\ell^p, \ell^q)$ to which $\{f[A_n]\}$ converges in $\mathcal{B}(\ell^p, \ell^q)$. For each $j, k = 1, 2, \ldots$,

$$\left| f\left(a_{jk}^{(n)}\right) - f\left(a_{jk}^{(m)}\right) \right| \le \|f[A_n] - f[A_m]\| \le \|A_n - A_m\| \longrightarrow 0 \tag{2.5}$$

as $n,m \to \infty$. Since this is true for every $f \in s(\mathscr{X})$, the quantity $\|A_n - A_m\|$ does not depend on f, and $\|x\| = \sup\{|\psi(x)| : \psi \in s(\mathscr{X})\}$ for all $x \in \mathscr{X}$, we have

$$\begin{aligned}\left\|a_{jk}^{(n)} - a_{jk}^{(m)}\right\| &= \sup_{f\in s(\mathscr{X})} \left| f\left(a_{jk}^{(n)} - a_{jk}^{(m)}\right)\right| = \sup_{f\in s(\mathscr{X})} \left| f\left(a_{jk}^{(n)}\right) - f\left(a_{jk}^{(m)}\right)\right| \\ &\le \|A_n - A_m\| \longrightarrow 0 \quad \text{as } n,m \longrightarrow 0.\end{aligned} \tag{2.6}$$

Thus the sequence $\{a_{jk}^{(n)}\}$ is a Cauchy sequence in $\mathscr{X}$. By the completeness of $\mathscr{X}$, there exists an $a_{jk} \in \mathscr{X}$ such that $a_{jk}^{(n)} \to a_{jk}$ in $\mathscr{X}$. Thus there is an $A = [a_{jk}] \in \mathscr{M}(\mathscr{X})$ such that $\{A_n\}$ converges to A entrywise. It is clear from this construction that for each $f \in s(\mathscr{X})$, $f[A] = \Lambda_f$. Since each $\Lambda_f \in \mathscr{B}(\ell^p, \ell^q)$, we have $A \in \mathscr{S}(\mathscr{X})$.

We have to show that $\|A_n - A\| \to 0$ as $n \to \infty$. Let $\epsilon > 0$ be given. Since $\{A_n\}$ is a Cauchy sequence, there exists an N such that $\|A_n - A_m\| < \epsilon/3$, for all $n,m \ge N$. Let $n \ge N$ be arbitrarily fixed. We will show that $\|A_n - A\| < \epsilon$. There exists an $f \in s(\mathscr{X})$ such that

$$\|A_n - A\| < \|f[A_n - A]\| + \frac{\epsilon}{3} = \|f[A_n] - f[A]\| + \frac{\epsilon}{3} = \|f[A_n] - \Lambda_f\| + \frac{\epsilon}{3}. \tag{2.7}$$

Since $f[A_n] \to \Lambda_f$ in the norm of $\mathscr{B}(\ell^p, \ell^q)$, there exists a $\nu > N$ such that $\|f[A_\nu] - \Lambda_f\| < \epsilon/3$. Thus

$$\begin{aligned}\|A_n - A\| &< \|f[A_n] - \Lambda_f\| + \frac{\epsilon}{3} \le \|f[A_n] - f[A_\nu]\| + \|f[A_\nu] - \Lambda_f\| + \frac{\epsilon}{3} \\ &< \|A_n - A_\nu\| + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon.\end{aligned} \tag{2.8}$$

Therefore, $\mathscr{S}(\mathscr{X})$ is complete. □

For a given matrix $A = [a_{jk}]$, denote by $A_{\underline{n}}$ the matrix whose (j,k) entry is a_{jk} for $1 \le j,k \le n$ and 0 otherwise.

Proposition 2.3. *Let $A = [a_{jk}] \in \mathscr{S}(\mathscr{X})$. Then $\|A_{\underline{n}}\| \nearrow \|A\|$ as $n \to \infty$.*

Proof. Let $\xi = \{\xi_k\}_{k=1}^{\infty} \in \ell^p$; $f \in s(\mathscr{X})$; and $n \in \mathbb{N}$. Denote $\xi^{(n)} = \{\xi_1, \xi_2, \dots, \xi_n, 0, \dots\}$. Then

$$\begin{aligned}\left\|f[A_{\underline{n}}]\xi\right\| &= \left(\sum_{j=1}^{n}\left|\sum_{k=1}^{n} f(a_{jk})\xi_k\right|^q\right)^{1/q} \le \left(\sum_{j=1}^{n+1}\left|\sum_{k=1}^{n} f(a_{jk})\xi_k\right|^q\right)^{1/q} \\ &= \left\|f[A_{(n+1)\underline{n}}]\xi^{(n)}\right\| \le \|f[A_{(n+1)\underline{n}}]\|_{p,q}\|\xi^{(n)}\|_p \le \|A_{(n+1)\underline{n}}\|\|\xi\|.\end{aligned} \tag{2.9}$$

So

$$\|f[A_{\underline{n}}]\| \le \|A_{(n+1)\underline{n}}\|. \tag{2.10}$$

Since this is true for every $f \in s(\mathscr{X})$,

$$\|A_{\underline{n}}\| \le \|A_{(n+1)\underline{n}}\|. \tag{2.11}$$

Let $\epsilon > 0$. There is an $f \in s(\mathscr{X})$ such that $\|f[A]\| \geq \|A\| - \epsilon/4$. Since $f[A]$ is in $\mathscr{B}(\ell^p, \ell^q)$, there is a unit vector $\xi = \{\xi_k\}_{k=1}^{\infty} \in \ell^p$ such that

$$\|f[A]\xi\|_q = \left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} f(a_{jk})\xi_k\right|^q\right)^{1/q} \geq \|f[A]\| - \frac{\epsilon}{4}. \tag{2.12}$$

Thus there exists an n_0 such that

$$\left(\sum_{j=1}^{n_0}\left|\sum_{k=1}^{\infty} f(a_{jk})\xi_k\right|^q\right)^{1/q} \geq \|f[A]\| - \frac{\epsilon}{4} - \frac{\epsilon}{12} = \|f[A]\| - \frac{\epsilon}{3}. \tag{2.13}$$

For the finite sum, there is an n_1 such that $n_1 \geq n_0$ and that

$$\begin{aligned}\left\|f[A_{(n_1)}]\xi\right\| &= \left(\sum_{j=1}^{n_1}\left|\sum_{k=1}^{n_1} f(a_{jk})\xi_k\right|^q\right)^{1/q} \geq \left(\sum_{j=1}^{n_0}\left|\sum_{k=1}^{n_1} f(a_{jk})\xi_k\right|^q\right)^{1/q} \\ &\geq \|f[A]\| - \frac{\epsilon}{3} - \frac{\epsilon}{6} = \|f[A]\| - \frac{\epsilon}{2}.\end{aligned} \tag{2.14}$$

So, for $n \geq n_1$,

$$\begin{aligned}\|A_{\underline{n}}\| &\geq \|A_{(n_1)}\| \geq \|f[A_{(n_1)}]\| \geq \|f[A_{(n_1)}]\xi\| \geq \|f[A]\| - \frac{\epsilon}{2} \\ &\geq \|A\| - \frac{\epsilon}{4} - \frac{\epsilon}{2} > \|A\| - \epsilon.\end{aligned} \tag{2.15}$$

Therefore, $\|A_{\underline{n}}\| \nearrow \|A\|$, as asserted. □

With $\mathscr{X}$ and $s(\mathscr{X})$ as above and for $1 \leq r < \infty$, let $\mathscr{S}^r(\mathscr{X})$ denote the set of all matrices $A = [a_{jk}] \in \mathscr{M}(\mathscr{X})$ with the property that $f[A]^{[r]} = [|f(a_{jk})|^r] \in \mathscr{B}(\ell^p, \ell^q)$ for all $f \in s(\mathscr{X})$.

Theorem 2.4. *For each $A \in \mathscr{S}^r(\mathscr{X})$, it holds that*

$$\sup\left\{\left\|f[A]^{[r]}\right\| : f \in s(\mathscr{X})\right\} < \infty. \tag{2.16}$$

Proof. By Theorem 1.2, the set of all matrices $\Lambda = [\lambda_{jk}]$ over $\mathbb{C}$ with bounded absolute Schur rth power ($\Lambda^{[r]} = [|\lambda|^r] \in \mathscr{B}(\ell^p, \ell^q)$) is a Banach algebra. Since $X^{\#}$ is the linear span of $s(\mathscr{X})$, for the fixed $A = [a_{jk}] \in \mathscr{S}^r(\mathscr{X})$, the map $\mathscr{T}_A : f \mapsto f[A]$ is a linear map from $\mathscr{X}^{\#}$ into $\mathscr{S}^r$. To show that $\mathscr{T}_A$ is a closed map, let $\{(f_n, f_n[A])\}$ be a sequence in the graph $\mathscr{G}(\mathscr{T}_A) \subseteq \mathscr{X}^{\#} \oplus_\infty \mathscr{S}^r$ of $\mathscr{T}_A$ converging to some $(f, \Lambda) \in \mathscr{X}^{\#} \oplus_\infty \mathscr{S}^r$. Then $f_n \to f$ in $\mathscr{X}^{\#}$ and $f_n[A] \to \Lambda$ in $\mathscr{S}^r$. Let $\Lambda = [\lambda_{jk}]$. We then have, for each $(j,k) \in \mathbb{N} \times \mathbb{N}$,

$$f_n(a_{jk}) \longrightarrow f(a_{jk}), \quad f_n(a_{jk}) \longrightarrow \lambda_{jk} \quad \text{as } n \longrightarrow \infty. \tag{2.17}$$

Therefore, $f[A] = \Lambda$ and $\mathscr{T}_A$ has a closed graph. Since $\mathscr{X}^{\#}$ and $\mathscr{S}^r$ are Banach spaces, $\mathscr{T}_A$ is bounded by the closed graph theorem. Thus

$$\sup\{\|f[A]\|_{\mathscr{S}^r} : f \in s(\mathscr{X})\} \leq \sup\{\|\mathscr{T}_A(f)\| : \|f\| \leq 1\} = \|\mathscr{T}_A\| < \infty. \tag{2.18}$$

Since $\|f[A]\|_{\mathcal{S}^r} = |||f[A]|||_{p,q,r} = \|f[A]^{[r]}\|^{1/r}$, we have

$$\sup\left\{\left\|f[A]^{[r]}\right\| : f \in s(\mathcal{X})\right\} < \infty. \tag{2.19}$$

□

For each $A \in \mathcal{S}^r(\mathcal{X})$, define $\|A\|$ or $|||A|||_{p,q,r}$ by

$$\begin{aligned}\|A\| := |||A|||_{p,q,r} &:= \sup\{|||f[A]|||_{p,q,r} : f \in s(\mathcal{X})\} \\ &= \sup\left\{\left\|f[A]^{[r]}\right\|^{1/r} : f \in s(\mathcal{X})\right\}.\end{aligned} \tag{2.20}$$

The preceding theorem guarantees that $\|\cdot\|$ is a function defined on $\mathcal{S}^r(\mathcal{X})$. We will prove that it is a norm in Theorem 2.6. Using an argument similar to that in Proposition 2.3, we can show that this function has the same monotone property.

Proposition 2.5. *Let $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{X})$. Then $\|A_{\underline{n}}\| \nearrow \|A\|$ as $n \to \infty$.*

Proof. This is just a routine adaptation of the proof of Proposition 2.3, therefore omitted. □

Theorem 2.6. *For $r \geq 1$, the function $\|\cdot\|$ defined in (2.20) is a norm on the space $\mathcal{S}^r(\mathcal{X})$, and $\mathcal{S}^r(\mathcal{X})$ is a Banach space under this norm and the usual addition and scalar multiplication.*

Proof. To see that $\|\cdot\|$ is indeed a norm, let $A, B \in \mathcal{S}^r(\mathcal{X})$ and $f \in s(\mathcal{X})$. Then by Theorem 1.2,

$$\begin{aligned}\left\|f[A+B]^{[r]}\right\| &= \left\|(f[A]+f[B])^{[r]}\right\| = |||f[A]+f[B]|||^r_{p,q,r} \\ &\leq \left(|||f[A]|||_{p,q,r} + |||f[B]|||_{p,q,r}\right)^r \\ &\qquad \text{(by the triangle inequality for the norm on } \mathcal{S}^r) \\ &\leq \left(\|A\| + \|B\|\right)^r.\end{aligned} \tag{2.21}$$

Since $f \in s(\mathcal{X})$ is arbitrary, we have

$$\|A+B\| \leq \|A\| + \|B\|, \tag{2.22}$$

as asserted.

Let $\{A_n = [a_{jk}^{(n)}]\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{S}^r(\mathcal{X})$. Then for each $(j,k) \in \mathbb{N} \times \mathbb{N}$,

$$\begin{aligned}\left\|a_{jk}^{(n)} - a_{jk}^{(m)}\right\| &= \sup\left\{\left|f\left(a_{jk}^{(n)} - a_{jk}^{(m)}\right)\right| : f \in s(\mathcal{X})\right\} \\ &\leq \sup\{|||f[A_n - A_m]|||_{p,q,r} : f \in s(\mathcal{X})\} \\ &= \|A_n - A_m\| \longrightarrow 0 \quad \text{as } n, m \longrightarrow \infty.\end{aligned} \tag{2.23}$$

Thus each $\{a_{jk}^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{X}$. By the completeness of $\mathcal{X}$, there is an $a_{jk} \in \mathcal{X}$ to which $\{a_{jk}^{(n)}\}_{n=1}^{\infty}$ converges. Let $A = [a_{jk}]$. For each $f \in s(\mathcal{X})$, $\{f[A_n]\}_{n=1}^{\infty} \subseteq \mathcal{S}^r$

is a Cauchy sequence. By the completeness of $\mathcal{S}^r$, there is a $\Lambda_f \in \mathcal{S}^r$ such that $\|f[A_n] - \Lambda_f\| \to 0$ in $\mathcal{S}^r$. Since $f(a_{jk}^{(n)}) \to f(a_{jk})$ for all j, k, as $n \to \infty$, $f[A] = \Lambda_f \in \mathcal{S}^r$. As this is true for all $f \in s(\mathcal{X})$, we have $A \in \mathcal{S}^r(\mathcal{X})$. To see that $\|A_n - A\| \to 0$, we first note that for each $\nu \in \mathbb{N}$, we have, by the finiteness of ν and Lemma 1.3,

$$\|(A_n)_{\underline{\nu}} - A_{\underline{\nu}}\| \le \left\| \left[\|a_{jk}^{(n)} - a_{jk}\|^r \right]_{1\le j,k\le\nu} \right\|_{p,q}^{1/r} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.24}$$

Let $\epsilon > 0$. There is an $N \in \mathbb{N}$ such that

$$\|A_n - A_{n+l}\| < \epsilon \quad \forall\ n \ge N,\ \forall l \ge 1. \tag{2.25}$$

Then, by Proposition 2.5,

$$\left\|(A_n)_{\underline{\nu}} - (A_{n+l})_{\underline{\nu}}\right\| = \left\|(A_n - A_{n+l})_{\underline{\nu}}\right\| \le \|A_n - A_{n+l}\| < \epsilon \quad \forall n \ge N,\ \forall l \ge 1. \tag{2.26}$$

Taking limit as $l \to \infty$ and using (2.24), we have

$$\left\|(A_n)_{\underline{\nu}} - A_{\underline{\nu}}\right\| \le \epsilon \quad \forall n \ge N. \tag{2.27}$$

Since $\nu \in \mathbb{N}$ is arbitrary, we have, by Proposition 2.5,

$$\|A_n - A\| \le \epsilon \quad \forall\ n \ge N; \tag{2.28}$$

thus $\|A_n - A\| \to 0$ as $n \to \infty$. □

3. Schur algebras over commutative C^*-algebras

In this section, we fix a commutative C^*-algebra $\mathcal{B}$ with identity 1 and the set $s(\mathcal{B})$ of states on $\mathcal{B}$, that is, the set of positive linear functionals of norm 1 on $\mathcal{B}$. By the Gelfand-Naimark theorem, $\mathcal{B}$ is isometrically $*$-isomorphic to the function algebra $C(X)$ for some compact Hausdorff space X. We will treat $\mathcal{B}$ as $C(X)$. Since $s(\mathcal{B})$ contains all evaluation linear functionals $\varphi_x(a) = a(x)$ for $x \in X$, and $a \in \mathcal{B}$, we have $\|a\| = \sup\{|\varphi(a)| : \varphi \in s(\mathcal{B})\}$, for all $a \in \mathcal{B}$.

As in the scalar case, define the *Schur product* (also known as *Hadamard product*, or *entrywise product*) of two matrices $A = [a_{jk}]$, $B = [b_{jk}]$ in $\mathcal{M}(\mathcal{B})$ (the set of all matrices over $\mathcal{B}$) by $A \bullet B = [a_{jk}b_{jk}]$. The product $a_{jk}b_{jk}$ is the algebra product defined in $\mathcal{B}$. Schur product of this form with entries in the algebra of bounded linear operators on a Hilbert space was first studied in [6]. For each $\varphi \in s(\mathcal{B})$, and for each $A = [a_{jk}] \in \mathcal{M}(\mathcal{B})$, $\varphi[A] = [\varphi(a_{jk})]$ is the scalar matrix obtained by applying φ to each entry of A. Let $\mathcal{S}(\mathcal{B})$ denote the set of all matrices $A \in \mathcal{M}(\mathcal{B})$ with the property that $\varphi[A]$ defines a bounded linear transformation from ℓ^p to ℓ^q for every $\varphi \in s(\mathcal{B})$. Denote by $\mathcal{B}^\#$ the Banach space dual of $\mathcal{B}$ (the space of bounded linear functionals on $\mathcal{B}$). Since each $f \in \mathcal{B}^\#$ is a linear combination of at most four states (see [5, Corollary 4.3.7, page 260]), it then follows from Theorem 2.1 that

$$\sup\{\|\varphi[A]\| : \varphi \in s(\mathcal{B})\} < \infty \tag{3.1}$$

(where the norm $||\varphi[A]||$ is the operator norm of $\varphi[A]$ as an element in $\mathcal{B}(\ell^p,\ell^q)$). Define the norm of $A \in \mathcal{S}(\mathcal{B})$ by $||A|| = \sup\{||\varphi[A]|| : \varphi \in s(\mathcal{B})\}$.

The following proposition follows immediately from Theorem 2.2.

Proposition 3.1. *For a commutative C^*-algebra $\mathcal{B}$ with state space $s(\mathcal{B})$, $\mathcal{S}(\mathcal{B})$ is a Banach space under the norm $||\cdot||$ and the usual addition and scalar multiplication.*

We do not know, however, whether $\mathcal{S}(\mathcal{B})$ is a Banach algebra under the Schur multiplication operation.

We now turn our attention to functional analog of Schur r-algebras. As before, we fix a commutative C^*-algebra $\mathcal{B}$ with the set $s(\mathcal{B})$ of states. For each real number $r \geq 1$, $A = [a_{jk}] \in \mathcal{M}(\mathcal{B})$, denote by $A^{[r]} = [|a_{jk}|^r]$, where $|x| = \sqrt{x^*x}$ for $x \in \mathcal{B}$. For $\varphi \in s(\mathcal{B})$, denote by $\varphi[A^{[r]}] := [\varphi(|a_{jk}|^r)]$. Let $\mathcal{S}^r(\mathcal{B})$ denote the set of all matrices $A \in \mathcal{M}(\mathcal{B})$ such that $\varphi[A^{[r]}] \in \mathcal{B}(\ell^p,\ell^q)$, that is, the numerical matrix $\varphi[A^{[r]}]$ defines a bounded linear transformation from ℓ^p to ℓ^q for every $\varphi \in s(\mathcal{B})$.

We show that for each $A \in \mathcal{S}^r(\mathcal{B})$,

$$||A|| := |||A|||_{p,q,r} := \sup\left\{\left|\left|\varphi\left[A^{[r]}\right]\right|\right|^{1/r} : \varphi \in s(\mathcal{B})\right\} < \infty, \tag{3.2}$$

and that this is indeed a norm on $\mathcal{S}^r(\mathcal{B})$. Furthermore, we show that $\mathcal{S}^r(\mathcal{B})$ is in fact a Banach algebra under the Schur multiplication and this norm. As a suitable adaptation of the argument used in the proof of Proposition 2.3, we have the following.

Proposition 3.2. *Let $A \in \mathcal{S}^r(\mathcal{B})$. Then $||A_{\underline{n}}|| \nearrow ||A||$ as $n \to \infty$.*

We also need this simple observation.

Lemma 3.3 (Minkowski's inequality for linear functionals). *Let $\varphi \in s(\mathcal{B})$. Then*

$$[\varphi(|a+b|^r)]^{1/r} \leq [\varphi(|a|^r)]^{1/r} + [\varphi(|b|^r)]^{1/r} \quad \forall a,b \in \mathcal{B}. \tag{3.3}$$

Proof. Since $\mathcal{B} = C(X)$ for some compact Hausdorff space X, the given $\varphi \in s(\mathcal{B})$ has an integral representation $\varphi(a) = \int_X a\,d\mu_\varphi$ for all $a \in \mathcal{B}$ and some measure μ_φ on X. The Minkowski inequality for μ_φ is exactly the asserted inequality. □

Theorem 3.4. *The function $||\cdot||$ defined in (3.2) above is a norm on $\mathcal{S}^r(\mathcal{B})$; and $\mathcal{S}^r(\mathcal{B})$ is a Banach algebra under the Schur product operation and this norm.*

Proof. Let $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{B})$. Since each $f \in \mathcal{B}^\#$ is a linear combination of at most four states, the map $f \mapsto f[A^{[r]}]$ is a linear transformation from $\mathcal{B}^\#$ to $\mathcal{B}(\ell^p,\ell^q)$. Using arguments similar to that used in the proof of Theorem 2.1, we have

$$\sup\left\{\left|\left|\varphi[A^{[r]}]\right|\right| : \varphi \in s(\mathcal{B})\right\} < \infty. \tag{3.4}$$

Therefore, the expression (3.2) indeed defines a function on $\mathcal{S}^r(\mathcal{B})$.

To see that (3.2) indeed defines a norm, let $A = [a_{jk}]$, $B = [b_{jk}] \in \mathcal{S}^r(\mathcal{B})$. Then $\varphi[A^{[r]}]^{[1/r]} = [(\varphi(|a_{jk}|^r))^{1/r}] \in \mathcal{S}^r$ for all $\varphi \in s(\mathcal{B})$. Using the norm on $\mathcal{S}^r$, we have

$$\begin{aligned}
\left\|\varphi\left[(A+B)^{[r]}\right]\right\|^{1/r} &= \left\|\left[\left((\varphi(|a_{jk}+b_{jk}|^r))^{1/r}\right)^r\right]\right\|^{1/r} \\
&\le \left\|\left[\left((\varphi(|a_{jk}|^r))^{1/r} + (\varphi(|b_{jk}|^r))^{1/r}\right)^r\right]\right\|^{1/r} \\
&\qquad \text{(by Lemmas 1.3 and 3.3)} \\
&\le \|[\varphi(|a_{jk}|^r)]\|^{1/r} + \|[\varphi(|b_{jk}|^r)]\|^{1/r} \\
&\qquad \text{(by triangle inequality for the norm on } \mathcal{S}^r) \\
&\le \|A\| + \|B\|.
\end{aligned} \tag{3.5}$$

Since this is true for all $\varphi \in s(\mathcal{B})$, we have $\|A+B\| \le \|A\| + \|B\|$.

For the completeness, pick a Cauchy sequence $\{A^{(n)} = [a_{jk}^{(n)}]\}$ in $\mathcal{S}^r(\mathcal{B})$. For each $\varphi \in s(\mathcal{B})$,

$$\varphi\left(\left|a_{jk}^{(n)} - a_{jk}^{(m)}\right|^r\right)^{1/r} \le \left\|\varphi\left[\left(A^{(n)} - A^{(m)}\right)^{[r]}\right]\right\|^{1/r} \le \|A^{(n)} - A^{(m)}\| \longrightarrow 0. \tag{3.6}$$

Since $\|a\| = \sup_{\varphi \in s(\mathcal{B})} |\varphi(a)|$ for all $a \in \mathcal{B}$,

$$\left\|a_{jk}^{(n)} - a_{jk}^{(m)}\right\| \longrightarrow 0 \quad \text{as } n, m \longrightarrow \infty. \tag{3.7}$$

Since $\mathcal{B}$ is complete, there is an $a_{jk} \in \mathcal{B}$ such that $a_{jk}^{(n)} \to a_{jk}$. Let $A = [a_{jk}]$. We show that $A \in \mathcal{S}^r(\mathcal{B})$ and $A^{(n)} \to A$.

For a fixed $\nu \in \mathbb{N}$, since $\varphi(|a_{jk}^{(n)} - a_{jk}|^r) \le \|a_{jk}^{(n)} - a_{jk}\|^r$ for all $\varphi \in s(\mathcal{B})$ and all $(j,k) \in \mathbb{N} \times \mathbb{N}$, we have, by Lemma 1.3,

$$\left\|A_{\underline{\nu}}^{(n)} - A_{\underline{\nu}}\right\| \le \left\|\left[\left\|a_{jk}^{(n)} - a_{jk}\right\|^r\right]_{1 \le j,k \le \nu}\right\|^{1/r} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.8}$$

Let $\epsilon > 0$. There is an N such that $\|A^{(n)} - A^{(m)}\| < \epsilon$ for all $n, m \ge N$. Then, for a fixed $\nu \in \mathbb{N}$, by Proposition 3.2,

$$\left\|A_{\underline{\nu}}^{(n)} - A_{\underline{\nu}}^{(m)}\right\| = \left\|\left(A^{(n)} - A^{(m)}\right)_{\underline{\nu}}\right\| \le \|A^{(n)} - A^{(m)}\| < \epsilon \quad \forall m, n \ge N. \tag{3.9}$$

Taking limit as $m \to \infty$, we have $\|A_{\underline{\nu}}^{(n)} - A_{\underline{\nu}}\| \le \epsilon$ for all $n \ge N$. Since this is true for all $\nu \in \mathbb{N}$, we have, by Proposition 3.2, $\|A^{(n)} - A\| \le \epsilon$ for all $n \ge N$. Thus $\|A\| \le \|A - A^{(N)}\| + \|A^{(N)}\| \le \epsilon + \|A^{(N)}\| < \infty$, and hence $A \in \mathcal{S}^r(\mathcal{B})$, and also $A^{(n)} \to A$.

To see the submultiplicativity of the norm, let $A, B \in \mathcal{S}^r(\mathcal{B})$; and let $\varphi \in s(\mathcal{B})$. Then $\varphi[A^{[r]}], \varphi[B^{[r]}] \in \mathcal{B}(\ell^p, \ell^q)$. For each (j,k),

$$|a_{jk}b_{jk}|^r = |a_{jk}|^r |b_{jk}|^r \le \|a_{jk}\|^r |b_{jk}|^r \le \|A\|^r |b_{jk}|^r. \tag{3.10}$$

Thus

$$\varphi(|a_{jk}b_{jk}|^r) = \varphi(|a_{jk}|^r |b_{jk}|^r) \le \|A\|^r \varphi(|b_{jk}|^r). \tag{3.11}$$

Therefore, by Lemma 1.3,

$$\begin{aligned} \left\|\varphi\left[(A \bullet B)^{[r]}\right]\right\|^{1/r} &= \|[\varphi(|a_{jk}b_{jk}|^r)]\|^{1/r} \le \|[\|A\|^r \varphi(|b_{jk}|^r)]\|_{p,q}^{1/r} \\ &= \|A\| \left\|\varphi\left[B^{[r]}\right]\right\|_{p,q}^{1/r} \le \|A\| \|B\|. \end{aligned} \tag{3.12}$$

Since $\varphi \in s(\mathcal{B})$ is arbitrary, $A \bullet B \in \mathcal{S}^r(\mathcal{B})$ and

$$\|A \bullet B\| \le \|A\| \|B\|. \tag{3.13}$$

This completes the proof. □

The following simple observation will be used to prove a Hölder inequality for the norm $|||\cdot|||_{p,q,r}$.

Lemma 3.5 (Hölder's inequality for positive linear functionals). *Let $a, b \in \mathcal{B}$. Then, for each $\varphi \in s(\mathcal{B})$, $r \in (1,\infty)$, and r^* satisfying $1/r + 1/r^* = 1$,*

$$\varphi(|ab|) \le [\varphi(|a|^r)]^{1/r} \left[\varphi\left(|b|^{r^*}\right)\right]^{1/(r^*)}. \tag{3.14}$$

Proof. As in the proof of Lemma 3.3, φ has an integral representation, and the asserted inequality is just the usual Hölder inequality, written in functional form. □

Theorem 3.6 (Hölder's inequality). *Let $A = [a_{jk}]$ and $B = [b_{jk}]$ be matrices with entries in $\mathcal{B}$. Then, for $r \in (1,\infty)$ and r^* satisfying $1/r + 1/r^* = 1$,*

$$|||A \bullet B|||_{p,q,1} \le |||A|||_{p,q,r} \cdot |||B|||_{p,q,r^*}. \tag{3.15}$$

Note that this inequality should be interpreted with the conventions $0 \cdot \infty = 0$, and $a \cdot \infty = \infty$ for $a \in (0,\infty)$.

Proof. If the right-hand side is ∞, there is nothing to prove. So suppose that both factors on the right are finite and nonzero. Let $\varphi \in s(\mathcal{B})$ and $\xi = \{\xi_k\}_{k=1}^{\infty} \in \ell^p$. Write $|\xi| = \{|\xi_k|\}_{k=1}^{\infty}$. Using Lemma 3.5 and the usual Hölder inequality, we have

$$\begin{aligned}
&\left\|\left(\varphi\left[(A\bullet B)^{[1]}\right]\right)\xi\right\|_q \\
&\quad= \left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}\varphi(|a_{jk}b_{jk}|)\xi_k\right|^q\right)^{1/q} \le \left(\sum_{j=1}^{\infty}\left[\sum_{k=1}^{\infty}\varphi(|a_{jk}b_{jk}|)|\xi_k|\right]^q\right)^{1/q} \\
&\quad\le \left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\left([\varphi(|a_{jk}|^r)]^{1/r}|\xi_k|^{1/r}\right)\left(\left[\varphi\left(|b_{jk}|^{r^*}\right)\right]^{1/(r^*)}|\xi_k|^{1/(r^*)}\right)\right)^q\right)^{1/q} \\
&\qquad\text{(by Lemma 3.5)} \\
&\quad\le \left(\sum_{j=1}^{\infty}\left[\sum_{k=1}^{\infty}[\varphi(|a_{jk}|^r)]|\xi_k|\right]^{q/r}\left[\sum_{k=1}^{\infty}\left[\varphi\left(|b_{jk}|^{r^*}\right)\right]|\xi_k|\right]^{q/(r^*)}\right)^{1/q} \qquad (3.16)\\
&\quad\le \left(\sum_{j=1}^{\infty}\left[\sum_{k=1}^{\infty}\varphi(|a_{jk}|^r)|\xi_k|\right]^q\right)^{1/(qr)}\left(\sum_{j=1}^{\infty}\left[\sum_{k=1}^{\infty}\varphi\left(|b_{jk}|^{r^*}\right)|\xi_k|\right]^q\right)^{1/(qr^*)} \\
&\quad= \left\|\varphi\left[A^{[r]}\right]|\xi|\right\|_q^{1/r}\cdot\left\|\varphi\left[B^{[r^*]}\right]|\xi|\right\|_q^{1/(r^*)} \\
&\quad\le \left\|\varphi\left[A^{[r]}\right]\right\|_{p,q}^{1/r}\cdot\|(|\xi|)\|_p^{1/r}\left\|\varphi\left[B^{[r^*]}\right]\right\|_{p,q}^{1/(r^*)}\cdot\|(|\xi|)\|_p^{1/(r^*)} \\
&\quad\le |||A|||_{p,q,r}|||B|||_{p,q,r^*}\||\xi|\|_p = |||A|||_{p,q,r}|||B|||_{p,q,r^*}\|\xi\|_p.
\end{aligned}$$

Therefore,

$$|||\varphi[A\bullet B]|||_{p,q,1} \le |||A|||_{p,q,r}|||B|||_{p,q,r^*}. \qquad (3.17)$$

Since $\varphi \in s(\mathcal{B})$ is arbitrary, we have, as asserted,

$$|||A\bullet B|||_{p,q,1} \le |||A|||_{p,q,r}|||B|||_{p,q,r^*}. \qquad (3.18)$$

□

Here is an analogue of the relationship between ℓ^p and its dual space.

Theorem 3.7. *Let $B = [b_{jk}]$ be a matrix with entries in $\mathcal{B}$ and $1 < r < \infty$. Then $B \in \mathcal{S}^r(\mathcal{B})$ if and only if $A\bullet B \in \mathcal{S}^1(\mathcal{B})$ for all $A \in \mathcal{S}^{r^*}(\mathcal{B})$, where $1/r + 1/r^* = 1$. Moreover, whenever applied,*

$$|||B|||_{p,q,r} = \sup\{|||A\bullet B|||_{p,q,1} : A \in \mathcal{S}^{r^*}(\mathcal{B}),\ |||A|||_{p,q,r^*} \le 1\}. \qquad (3.19)$$

Proof. Sufficiency follows from the preceding Theorem 3.6. For necessity, define $\Phi : \mathcal{S}^{r^*}(\mathcal{B}) \to \mathcal{S}^1(\mathcal{B})$ by $\Phi(A) = A \bullet B$ for all $A \in \mathcal{S}^{r^*}(\mathcal{B})$. We show that the graph of Φ is closed. Suppose $A_n = [a_{jk}^{(n)}] \to A = [a_{jk}]$ in $\mathcal{S}^{r^*}(\mathcal{B})$ and $A_n \bullet B \to C = [c_{jk}]$ in $\mathcal{S}^1(\mathcal{B})$. Then for all (j,k),

$$a_{jk}^{(n)} \longrightarrow a_{jk}, \quad a_{jk}^{(n)} b_{jk} \longrightarrow c_{jk} \quad \text{as } n \longrightarrow \infty. \tag{3.20}$$

Therefore, $a_{jk}b_{jk} = c_{jk}$. Since this is true for all (j,k), we have the graph of Φ is closed and Φ is a bounded linear transformation, by the closed graph theorem.

By the same argument, we see also that $B^{[1]} = [|b_{jk}|]$ has the same property as B, and $B^{[1]}$ defines a bounded linear transformation $\Phi_{B^{[1]}}$ from $\mathcal{S}^{r^*}(\mathcal{B})$ to $\mathcal{S}^1(\mathcal{B})$. For each $n \in \mathbb{N}$, $(B^{[r^*]})_{\underline{n}} = [|b_{jk}|^{r^*}]_{\underline{n}}$ is in $\mathcal{S}^1(\mathcal{B})$, as it has only finitely many nonzero entries, and $\varphi[(B^{[r^*]})^{[1]}] = \varphi[B^{[r^*]}]$ for all $\varphi \in s(\mathcal{B})$. Thus

$$\left|\left|\left| (B^{[r]})_{\underline{n}} \right|\right|\right|_{p,q,1} \le \left|\left|\left| (B^{[r-1]})_{\underline{n}} \bullet B^{[1]} \right|\right|\right|_{p,q,1} \le \|\Phi_{B^{[1]}}\| \left|\left|\left| (B^{[r-1]})_{\underline{n}} \right|\right|\right|_{p,q,r^*}. \tag{3.21}$$

It is just a matter of writing out the definitions to see that $\|\Phi_{B^{[1]}}\| = \|\Phi_B\|$, and that $|||(B^{[r]})_{\underline{n}}|||_{p,q,1} = |||B_{\underline{n}}|||_{p,q,r}^r$ and $|||(B^{[r-1]})_{\underline{n}}|||_{p,q,r^*} = |||B_{\underline{n}}|||_{p,q,r}^{r/r^*}$. Therefore, the preceding inequality is equivalent to

$$|||B|||_{p,q,r} \le \left|\left|\Phi_{B^{[1]}}\right|\right| = \|\Phi_B\| < \infty. \tag{3.22}$$

Thus $B \in \mathcal{S}^r(\mathcal{B})$. We also have

$$|||B|||_{p,q,r} \le \|\Phi\| = \sup\left\{|||A \bullet B|||_{p,q,1} : A \in \mathcal{S}^{r^*}(\mathcal{B}),\ |||A|||_{p,q,r^*} \le 1\right\} \le |||B|||_{p,q,r}, \tag{3.23}$$

by Hölder's inequality, Theorem 3.6. Therefore, all inequalities reduce to equalities. □

Arguments similar to those used in [2] can be used to proof the following.

Proposition 3.8. *For $1 \le r < r' < \infty$, and $\mathcal{B}$ as above, $\mathcal{S}^r(\mathcal{B}) \subsetneq \mathcal{S}^{r'}(\mathcal{B})$.*

Proof. Let $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{B})$. Then for each (j,k), we have

$$\|a_{jk}\| = \|(|a_{jk}|^r)\|^{1/r} = \left(\sup_{\varphi \in s(\mathcal{B})} \varphi(|a_{jk}|^r)\right)^{1/r} \le |||A|||_{p,q,r}. \tag{3.24}$$

Choose a suitable constant α, with $0 < \alpha \le 1$, such that αA has all entries bounded in norm by 1. Then for $r < r'$ and $\varphi \in s(\mathcal{B})$, we have, for each $(j,k) \in \mathbb{N} \times \mathbb{N}$,

$$\alpha^{r'} \varphi\left(|a_{jk}|^{r'}\right) \le \alpha^r \varphi(|a_{jk}|^r); \tag{3.25}$$

that is, each entry of $\varphi[(\alpha A)^{[r']}]$ is bounded by the corresponding entry of $\varphi[(\alpha A)^{[r]}]$. Thus

$$\alpha|||A|||_{p,q,r'} = |||\alpha A|||_{p,q,r'} \leq |||\alpha A|||_{p,q,r} = \alpha|||A|||_{p,q,r}. \tag{3.26}$$

So that $|||A|||_{p,q,r'} \leq |||A|||_{p,q,r}$, and hence $\mathcal{S}^r(\mathcal{B}) \subseteq \mathcal{S}^{r'}(\mathcal{B})$. To see that the inclusion is proper, we take a sequence $\{\alpha_k\}$ of nonnegative numbers which is in $\ell^{r'}$ but not in ℓ^r (more explicitly, take $\alpha_k = [1/k]^{1/r}$). Then the matrix A with the first column, the sequence $\{(\alpha_k)^{1/q}1\}$ (1 the identity of $\mathcal{B}$), and all other columns 0 is in $\mathcal{S}^{r'}(\mathcal{B})$ but not in $\mathcal{S}^r(\mathcal{B})$. □

4. Dual spaces

We consider in this section an analogue of compact and trace-class operators on a Hilbert space, in the Schur algebras considered in the preceding sections. Analogues of Schatten's trace duality theorems will be proved in this setting. Let $\mathcal{B}$ be a unital commutative C^*-algebra with the set $s(\mathcal{B})$ of states as in the preceding section. Let $\mathcal{M}_0$ be the set of all infinite matrices with entries in $\mathcal{B}$ having only finitely many nonzero entries. Denote by $\mathcal{K}^r(\mathcal{B})$ the closure in $\mathcal{S}^r(\mathcal{B})$ of $\mathcal{M}_0$. That is,

$$\mathcal{K}^r(\mathcal{B}) = \{A \in \mathcal{S}^r(\mathcal{B}) : \forall \epsilon > 0\ \exists A_0 \in \mathcal{M}_0 \text{ such that } \|A_0 - A\| < \epsilon\}, \tag{4.1}$$

where the norm $\|\cdot\|$ is the norm $|||\cdot|||_{p,q,r}$ of $\mathcal{S}^r(\mathcal{B})$. With the norm inherited from $\mathcal{S}^r(\mathcal{B})$, $\mathcal{K}^r(\mathcal{B})$ is a Banach space. We will identify the dual of $\mathcal{K}^r(\mathcal{B})$, in analogy with the fact that the dual of the compact operators on a Hilbert space is the trace-class operators.

Denote by $(\mathcal{AS})$ the space of matrices over the complex field $\mathbb{C}$ that are absolutely summable. This is just the space $\ell^1(\mathbb{N}\times\mathbb{N})$. Therefore, it is a Banach space with the ℓ^1 norm; that is, a matrix $S = [s_{jk}]$ over $\mathbb{C}$ is in $(\mathcal{AS})$ if and only if

$$\|S\|_{(\mathcal{AS})} := \sum_{j,k=1}^{\infty} |s_{jk}| < \infty. \tag{4.2}$$

As an analogue of the trace-class operators on a Hilbert space, we consider the space $\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$ defined as follows:

$$\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS})) := \left\{\Phi = [\varphi_{jk}] : \varphi_{jk} \in \mathcal{B}^{\#},\ \sum_{j,k=1}^{\infty} |\varphi_{jk}(a_{jk})| < \infty\ \forall A = [a_{jk}] \in \mathcal{S}^r(\mathcal{B})\right\}. \tag{4.3}$$

Thus a matrix Φ of functionals is in $\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$ if and only if it "Schur multiplies" each matrix in $\mathcal{S}^r(\mathcal{B})$ to a matrix in $(\mathcal{AS})$. Each $\Phi \in \mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$ defines a bounded linear transformation by the *Schur multiplication* by Φ, that is, by the closed graph theorem, $\Phi \bullet [a_{jk}] = [\varphi_{jk}(a_{jk})]$ from the Banach space $\mathcal{S}^r(\mathcal{B})$ to the Banach space $(\mathcal{AS})$ is a bounded linear transformation. Therefore, it has an operator norm

$$\|\Phi\|_{\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))} := \sup\{\|\Phi \bullet A\|_{(\mathcal{AS})} : A \in \mathcal{S}^r(\mathcal{B}),\ \|A\| \leq 1\}. \tag{4.4}$$

Theorem 4.1. *The space $\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$ equipped with the norm defined above is a Banach space.*

Proof. Since $\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS})) \subseteq \mathcal{B}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$, the space of bounded linear transformations from $\mathcal{S}^r(\mathcal{B})$ to $(\mathcal{AS})$, it suffices to show that it is closed. To that end, suppose that $\{\Psi_n = [\psi_{jk}^{(n)}]\}$ is a sequence in the space $\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$ such that $\Psi_n \to T \in \mathcal{B}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$. Then each $\{\psi_{jk}^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{B}^{\#}$, therefore converges to some $\psi_{jk} \in \mathcal{B}^{\#}$. Let $\Psi = [\psi_{jk}]$. We show that $\Psi = T$. Let $\epsilon > 0$. There is an N such that

$$\|\Psi_n - \Psi_{n+l}\| < \epsilon \quad \forall n \geq N,\ l \geq 1. \tag{4.5}$$

Let $A \in \mathcal{S}^r(\mathcal{B})$; and $m \in \mathbb{N}$. Then

$$\begin{aligned} \left\|[\Psi_n \bullet A - \Psi_{n+l} \bullet A]_{m_\sqcup}\right\|_{(\mathcal{AS})} &= \left\|\Psi_n \bullet A_{m_\sqcup} - \Psi_{n+l} \bullet A_{m_\sqcup}\right\|_{(\mathcal{AS})} \\ &\leq \epsilon \|A_{m_\sqcup}\| \leq \epsilon \|A\| \quad \forall\ n \geq N,\ l \geq 1. \end{aligned} \tag{4.6}$$

Taking limits as $l \to \infty$, we have

$$\left\|[\Psi_n \bullet A - \Psi \bullet A]_{m_\sqcup}\right\|_{(\mathcal{AS})} \leq \epsilon \|A\| \quad \forall n \geq N,\ \forall A \in \mathcal{S}^r(\mathcal{B}). \tag{4.7}$$

Now as $m \to \infty$, we obtain

$$\|\Psi_n \bullet A - \Psi \bullet A\|_{(\mathcal{AS})} \leq \epsilon \|A\| \quad \forall n \geq N,\ \forall A \in \mathcal{S}^r(\mathcal{B}). \tag{4.8}$$

So we see that $\Psi_n \to \Psi$. Therefore, $T(A) = \Psi \bullet A$ for all $A \in \mathcal{S}^r(\mathcal{B})$. □

Let $\Psi \in (\mathcal{S}^r(\mathcal{B}))^{\#}$. For each (j,k), define a linear functional on $\mathcal{B}$ as follows. For $b \in \mathcal{B}$, let $A_{b,j,k}$ be the matrix whose (j,k) entry is b and all others 0. Put

$$\psi_{jk}(b) = \Psi(A_{b,j,k}). \tag{4.9}$$

Then ψ_{jk} is a bounded linear functional on $\mathcal{B}$. Put $B_\Psi = [\psi_{jk}]$.

Each matrix $\Phi = [\varphi_{jk}] \in \mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$ defines a linear functional

$$\widetilde{\Phi}(A) = \sum_{j,k=1}^{\infty} \varphi_{jk}(a_{jk}), \quad A = [a_{jk}] \in \mathcal{S}^r(\mathcal{B}). \tag{4.10}$$

Theorem 4.2. (1) *Let $\Psi \in (\mathcal{S}^r(\mathcal{B}))^{\#}$; and let ψ_{jk} and B_Ψ be as defined above. Then*

$$B_\Psi \in \mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS})), \quad \Psi - \widetilde{B}_\Psi \in \mathcal{K}^r(\mathcal{B})^{\perp}. \tag{4.11}$$

(2) *The map $\Psi \mapsto B_\Psi$ from $(\mathcal{K}^r(\mathcal{B}))^{\#}$ to $\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$ is an isometric isomorphism between $(\mathcal{K}^r)^{\#}$ and $\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$.*

Proof. (1) Let $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{B})$. For $z \in \mathbb{C}$, let $\operatorname{sgn}(z) = z/|z|$ if $z \neq 0$, and $\operatorname{sgn}(0) = 0$. Let $\bar{A} = [\operatorname{sgn}(\psi_{jk}(a_{jk}))a_{jk}]$. Then $\bar{A} \in \mathcal{S}^r(\mathcal{B})$, and for each n,

$$\sum_{j,k=1}^{n} |\psi_{jk}(a_{jk})| = \|\Psi(\bar{A}_{\underline{n}})\|_{(\mathcal{AS})} \leq \|\Psi\|\|\bar{A}_{\underline{n}}\| \leq \|\Psi\|\|\bar{A}\| \leq \|\Psi\|\|A\|. \tag{4.12}$$

So

$$\sum_{j,k=1}^{\infty} |\psi_{jk}(a_{jk})| \leq \|\Psi\|\|A\| < \infty; \tag{4.13}$$

and $B_\Psi \bullet A \in (\mathcal{AS})$ for all $A \in \mathcal{S}^r(\mathcal{B})$; that is, $B_\Psi \in \mathcal{M}(\mathcal{S}^r(\mathcal{B}), (\mathcal{AS}))$.

For $A = [a_{jk}] \in \mathcal{K}^r(\mathcal{B})$, we have for each n,

$$\widetilde{B}_\Psi(A_{\underline{n}}) = \Psi(A_{\underline{n}}) \tag{4.14}$$

by linearity. Since $\|A_{\underline{n}} - A\| \to 0$ as $n \to \infty$, by the continuity of both functionals, we have

$$\widetilde{B}_\Psi(A) = \Psi(A) \quad \forall\ A \in \mathcal{K}^r(\mathcal{B}). \tag{4.15}$$

(2) Let $\Psi \in (\mathcal{K}^r(\mathcal{B}))^{\#}$. For each $A = [a_{jk}] \in \mathcal{K}^r(\mathcal{B})$ such that $\|A\| \leq 1$,

$$|\Psi(A)| = |\widetilde{B}_\Psi(A)| = \left| \sum_{j,k=1}^{\infty} \psi_{jk}(a_{jk}) \right| \leq \|\widetilde{B}_\Psi\|_{\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))}, \tag{4.16}$$

by the definition of the norm on $\mathcal{M}(\mathcal{S}^r(\mathcal{B}), (\mathcal{AS}))$. Therefore,

$$\|\Psi\| \leq \|B_\Psi\|_{\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))}. \tag{4.17}$$

Let $\epsilon > 0$. There is an $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{B})$ such that $\|A\| \leq 1$ and

$$\|B_\Psi\|_{\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))} - \frac{\epsilon}{3} < \sum_{j,k=1}^{\infty} |\psi_{jk}(a_{jk})|. \tag{4.18}$$

By the absolute convergence of the series, there is an N such that

$$\sum_{j,k=1}^{N} |\psi_{jk}(a_{jk})| > \sum_{j,k=1}^{\infty} |\psi_{jk}(a_{jk})| - \frac{\epsilon}{3} > \|B_\Psi\|_{\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))} \frac{-2\epsilon}{3}. \tag{4.19}$$

Let $\hat{A}$ be the matrix whose (j,k) entry is $(\operatorname{sgn}(\psi_{jk}(a_{jk}))a_{jk})$ for $j,k = 1,2,\ldots,N$ and all others 0. Then $\hat{A} \in \mathcal{K}^r(\mathcal{B})$, $\|\hat{)}\|A \leq 1$, and

$$\Psi(\hat{)}A = \sum_{j,k=1}^{N} |\psi_{jk}(a_{jk})| > \|B_\Psi\|_{\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))} \frac{-2\epsilon}{3}. \tag{4.20}$$

Therefore, we have

$$||\Psi|| > ||B_\Psi||_{\mathcal{M}(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))} - \epsilon. \tag{4.21}$$

Since ϵ is arbitrary, we have equality of the norms. □

A linear functional Φ on $\mathcal{S}^r(\mathcal{B})$ is *singular* if $\Phi(A) = 0$ for all $A \in \mathcal{K}^r(\mathcal{B})$. Recall that for each $\Psi \in \mathcal{S}^r(\mathcal{B})^\#$, there corresponds a bounded linear functional $\widetilde{B}_\Psi$ on $\mathcal{S}^r(\mathcal{B})$. Denote

$$\mathcal{S}^r(\mathcal{B})_a^\# := \{\widetilde{B}_\Psi : \Psi \in \mathcal{S}^r(\mathcal{B})^\#\}. \tag{4.22}$$

Theorem 4.3. *The set $\mathcal{S}^r(\mathcal{B})_s^\#$ consisting of all singular linear functionals together with the zero functional on $\mathcal{S}^r(\mathcal{B})$ is a nontrivial closed subspace of the dual $\mathcal{S}^r(\mathcal{B})^\#$ of $\mathcal{S}^r(\mathcal{B})$. Furthermore,*

$$\mathcal{S}^r(\mathcal{B})^\# = \mathcal{S}^r(\mathcal{B})_a^\# \oplus \mathcal{S}^r(\mathcal{B})_s^\#. \tag{4.23}$$

Proof. Since $\mathcal{K}^r(\mathcal{B})$ is a nontrivial closed subspace of $\mathcal{S}^r(\mathcal{B})$, the Hahn-Banach theorem ensures that $\mathcal{S}^r(\mathcal{B})_s^\#$ is a nonempty proper subset of $\mathcal{S}^r(\mathcal{B})^\#$.

The preceding theorem shows that $\mathcal{S}^r(\mathcal{B})^\# = \mathcal{S}^r(\mathcal{B})_a^\# + \mathcal{S}^r(\mathcal{B})_s^\#$. Let $\Psi = [\psi_{jk}] \in \mathcal{S}^r(\mathcal{B})_a^\# \bigcap \mathcal{S}^r(\mathcal{B})_s^\#$. Then $\Psi(K) = 0$ for all $K \in \mathcal{K}^r(\mathcal{B})$. Let $b \in \mathcal{B}$; and let $A_{b,j,k}$ be the matrix whose (j,k) entry is b and all others 0. Then for each (j,k),

$$\psi_{jk}(b) = \Psi(A_{b,j,k}) = 0. \tag{4.24}$$

Therefore, $\Psi = 0$. □

Since $\mathcal{B}$ may not be the dual of any normed space, we cannot expect $\mathcal{S}^r(\mathcal{B})$ to be a dual space. For if it were, then it would not be hard to see that $\mathcal{B}$ must be a dual space as well. We therefore assume, from this point on, that $\mathcal{B}$ is the dual of some Banach space $\mathcal{B}_\#$. We can then consider the space

$$\mathcal{M}_\#^0(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS})) = \left\{B = [b_{jk}] : b_{jk} \in \mathcal{B}_\#, \sum_{j,k=1}^{\infty} |a_{jk}(b_{jk})| < \infty, \ \forall A = [a_{jk}] \in \mathcal{S}^r(\mathcal{B})\right\} \tag{4.25}$$

with the norm

$$||B|| = \sup\left\{\left|\sum_{j,k=1}^{\infty} a_{jk}(b_{jk})\right| : A = [a_{jk}] \in \mathcal{S}^r(\mathcal{B}), |||A|||_{p,q,r} \le 1\right\}, \tag{4.26}$$

for $B = [b_{jk}] \in \mathcal{M}_\#^0(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$.

Arguments similar to those used in the proof of Theorem 4.1 can be used to prove that $\mathcal{M}_\#^0(\mathcal{S}^r(\mathcal{B}),(\mathcal{AS}))$ is also a Banach space.

Since the predual of $B(\ell^2)$ is the trace-class operators, which is the class of matrices that are the trace norm limits of their upper left-hand corner truncations, we define, analogously, the space $\mathcal{M}_\#(\mathcal{S}^r(\mathcal{B}),(\mathcal{A}S))$ as the space of all matrices $B \in \mathcal{M}_\#^0(\mathcal{S}^r(\mathcal{B}),(\mathcal{A}B))$ such

that $\|B - B_{\underline{n}}\| \to 0$ as $n \to \infty$. It is not hard to see that $\mathcal{M}_{\#}(\mathcal{S}^r(\mathcal{B}),(\mathcal{A}S))$ is a closed subspace of $\mathcal{M}^0_{\#}(\mathcal{S}^r(\mathcal{B}),(\mathcal{A}S))$ under the norm defined above. For brevity of notation, denote by $\mathcal{M}$ the space $\mathcal{M}_{\#}(\mathcal{S}^r(\mathcal{B}),(\mathcal{A}S))$ with the induced norm. Then $\mathcal{M}$ is a Banach space. We show that $\mathcal{S}^r(\mathcal{B})$ is the dual of $\mathcal{M}$.

Theorem 4.4. *Under the standing assumption that $\mathcal{B}$ has a predual $\mathcal{B}_{\#}$, and the notations defined above, the dual of $\mathcal{M}$ is isometrically isomorphic to $\mathcal{S}^r(\mathcal{B})$ (as Banach spaces).*

Proof. By the definition of $\mathcal{M}$, it is clear that each $A = [a_{jk}] \in \mathcal{S}^r(\mathcal{B})$ can be used to define a bounded linear functional ϕ_A on $\mathcal{M}$ as

$$\phi_A(M) = \sum_{j,k=1}^{\infty} a_{jk}(m_{jk}) \quad \forall M = [m_{jk}] \in \mathcal{M}. \tag{4.27}$$

Notice that by the definition (4.26) of the norm on $\mathcal{M}$, we also have $|\phi_A(M)| \le \|A\|\|M\|$. Thus $\|\phi_A\| \le \|A\|$. Therefore, $\mathcal{S}^r(\mathcal{B})$ can be regarded as a subspace of $\mathcal{M}^{\#}$. Denote by $\mathcal{S}_1$ the closed unit ball of $\mathcal{S}^r(\mathcal{B})$. Then $\mathcal{S}_1$ separates points in $\mathcal{M}$. We show that $\mathcal{S}_1$ is complete in the weak* topology, σ, it inherits from $\mathcal{M}^{\#}$. To this end, let $\{A^\alpha = [a^\alpha_{jk}]\}_{\alpha\in\Lambda}$ be a σ-Cauchy net in $\mathcal{S}_1$. We show that for each $(\mu,\nu) \in \mathbb{N}\times\mathbb{N}$, the net of the (μ,ν)-entries $\{a^\alpha_{\mu\nu}\}$ of $\{A^\alpha\}$ is a weak* Cauchy net in $\mathcal{B}$. Let $m \in \mathcal{B}_{\#}$. Choose M to be the matrix whose (μ,ν)-entry is m and zero for all others. Then we see that $M \in \mathcal{M}$ and $\{\phi_{A^\alpha}(M) = a^\alpha_{\mu\nu}(m)\}$ is a Cauchy net in $\mathbb{C}$. Thus $\{a^\alpha_{\mu\nu}\}_\alpha$ is a weak* Cauchy net in $\mathcal{B}$. Since $\|A^\alpha\| \le 1$ for all α and $\|a^\alpha_{\mu\nu}\| \le \|A^\alpha\|$, $\{a^\alpha_{\mu\nu}\}_{\alpha\in\Lambda}$ is a weak* Cauchy net in the closed unit ball of $\mathcal{B}$. Since the closed unit ball is weak* compact, by the Alaoglu theorem, there is $a_{\mu\nu} \in \mathcal{B}$ such that $a^\alpha_{\mu\nu} \to a_{\mu\nu}$ in the weak* topology. Let $A = [a_{jk}]$. We show that $A \in \mathcal{S}_1$ and that $A^\alpha \to A$ in the weak* topology of $\mathcal{M}^{\#}$ induced on $\mathcal{S}_1$. Let $\epsilon > 0$; and let $M = [m_{jk}] \in \mathcal{M}$. There is an α_0 such that $|\phi_{A^\alpha}(M) - \phi_{A^\beta}(M)| < \epsilon/2$ for all $\alpha,\beta \succeq \alpha_0$. By definition of $\mathcal{M}$, there is an N such that $\|M - M_{\underline{n}}\| < \epsilon/4$ for all $n \ge N$. Thus, for $n \ge N$ and $\alpha,\beta \succeq \alpha_0$, we have

$$\begin{aligned}
\left|\left((\phi_{(A^\alpha)})_{\underline{n}} - (\phi_{(A^\beta)})_{\underline{n}}\right)(M)\right| &= \left|\left(\phi_{A^\alpha} - \phi_{A^\beta}\right)\left(M_{\underline{n}}\right)\right| \\
&\le \left|\left(\phi_{A^\alpha} - \phi_{A^\beta}\right)(M_{\underline{n}} - M)\right| + \left|\left(\phi_{A^\alpha} - \phi_{A^\beta}\right)(M)\right| \\
&< \|\phi_{A^\alpha} - \phi_{A^\beta}\|\|M_{\underline{n}} - M\| + \frac{\epsilon}{2} \\
&\le \|A^\alpha - A^\beta\|\frac{\epsilon}{4} + \frac{\epsilon}{2} \le \epsilon.
\end{aligned} \tag{4.28}$$

For a fixed $n \ge N$, $(A^\alpha)_{\underline{n}} \to A_{\underline{n}}$ in the weak* topology σ. Taking limit in β, we have

$$\left|\left((\phi_{(A^\alpha)})_{\underline{n}} - (\phi_A)_{\underline{n}}\right)(M)\right| \le \epsilon \quad \forall n \ge n_0,\ \alpha \succeq \alpha_0. \tag{4.29}$$

Since this is true for all $n \ge N$, we may take the limit as $n \to \infty$ to obtain

$$\left|(\phi_{A^\alpha} - \phi_A)(M)\right| \le \epsilon \quad \forall \alpha \succeq \alpha_0. \tag{4.30}$$

This shows that $A^\alpha \to A$ in the weak* topology, and hence $\mathcal{S}_1$ is σ-complete.

We claim that the norm on $\mathcal{M}^{\#}$ is the same as that on $\mathcal{S}^r(\mathcal{B})$, and $\mathcal{M}^{\#} = \mathcal{S}^r(\mathcal{B})$. For each $A \in \mathcal{S}_1$, $||\phi_A|| \le ||A|| \le 1$. Thus $\mathcal{S}_1$ is contained in the closed unit ball of $\mathcal{M}^{\#}$. Suppose this inclusion is proper. Then there is $A_0 \in \mathcal{M}^{\#}$ with $||A_0|| \le 1$ such that $A_0 \notin \mathcal{S}_1$. Since $\mathcal{S}_1$ is weak* closed and convex, by [3, Theorem V 2.10, page 417], there is a weak* continuous linear functional on $\mathcal{M}^{\#}$, that is, an element $M \in \mathcal{M}$, such that

$$\mathcal{R}e(\phi_M(A)) \le c - \epsilon < c \le \mathcal{R}e(\phi_M(A_0)) \quad \forall A \in \mathcal{S}_1, \tag{4.31}$$

for some constants c and $\epsilon > 0$.

For each $A \in \mathcal{S}_1$, let $\widetilde{A} = \zeta A$, where $\zeta \in \mathbb{C}$ is chosen such that $|\zeta| = 1$ and $\phi_M(\widetilde{A}) = |\phi_M(A)|$. Since $\widetilde{A} \in \mathcal{S}_1$ for each $A \in \mathcal{S}_1$, we have, by the definition (4.26) of the norm on $\mathcal{M}$,

$$||M|| = \sup\{|\phi_M(A)| = \mathcal{R}e(\phi_M(\widetilde{A})) : A \in \mathcal{S}_1\} \le c - \epsilon < c \le \mathcal{R}e(\phi_M(A_0)) \le ||M||, \tag{4.32}$$

a contradiction. Therefore, $\mathcal{S}_1$ is the unit ball of $\mathcal{M}^{\#}$.

Let $A \in \mathcal{S}^r(\mathcal{B})$. As a linear functional ϕ_A on $\mathcal{M}$,

$$||\phi_A|| = \sup\{|\phi_A(M)| : M \in \mathcal{M}, ||M|| \le 1\} \le ||A||. \tag{4.33}$$

If $||\phi_A|| = 1$, then A is in the unit ball of $\mathcal{M}^{\#}$, which is just $\mathcal{S}_1$. Thus $||A|| \le 1 \le ||\phi_A|| \le ||A||$, and hence $||A|| = ||\phi_A||$. □

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