ON ABSOLUTE MATRIX SUMMABILITY METHODS

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We have proved a theorem on $|T, p_n|_k$ summability methods. This theorem includes a known theorem.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. By $(w^\delta_n)$, we denote the $n$th Cesàro means of order $\delta (\delta > -1)$ of the sequence $(s_n)$. The series $\sum a_n$ is said to be summable $|C, \delta|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} \left| w^\delta_n - w^\delta_{n-1} \right|^k < \infty. \quad (1.1)$$

In the special case for $\delta = 1$, $|C, \delta|_k$ summability reduces to $|C, 1|_k$ summability.

Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \ i \geq 1). \quad (1.2)$$

The sequence-to-sequence transformation

$$\vartheta_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (1.3)$$

defines the sequence $(\vartheta_n)$ of the $(\tilde{N}, p_n)$ means of the sequence $(s_n)$, generated by the sequence of coefficients $(p_n)$ (see [4]). The series $\sum a_n$ is said to be summable $|\tilde{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} \left| \vartheta_n - \vartheta_{n-1} \right|^k < \infty. \quad (1.4)$$

If we take $p_n = 1$ for all values of $n$, then $|\tilde{N}, p_n|_k$ summability is the same as $|C, 1|_k$ summability.
Given a normal matrix \( T = (t_{nk}) \), we associate two lower semimatrices \( \tilde{T} = (\tilde{t}_{nk}) \) and \( \hat{T} = (\hat{t}_{nk}) \) as follows:

\[
\tilde{t}_{nk} = \sum_{i=k}^{n} t_{ni}, \quad n, k = 0, 1, \ldots, \\
\hat{t}_{00} = \tilde{t}_{00} = t_{00}, \quad \hat{t}_{nk} = \tilde{t}_{nk} - \tilde{t}_{n-1,k}, \quad n = 1, 2, \ldots.
\]

It may be noted that \( T \) and \( \hat{T} \) are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

\[
T_n(s) = \sum_{v=0}^{n} t_{nv}s_v = \sum_{v=0}^{n} \tilde{t}_{nv}a_v,
\]

\[
\Delta T_n(s) = \sum_{v=0}^{n} \hat{t}_{nv}a_v.
\]

The series \( \sum a_n \) is said to be summable \( |T, p_n|_k \), \( k \geq 1 \), if (see [5])

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{P_n} \right)^{k-1} |\Delta T_n(s)|^k < \infty.
\]

In the special case, for \( t_{nv} = p_v/P_n \), \( |T, p_n|_k \) summability is the same as \( |N, p_n|_k \) summability.

2. The main result

The object of this paper is to prove the following theorem.

**Theorem 2.1.** Let \( k \geq 1 \). Let \( (s_n) \) be a bounded sequence and suppose that \( (\lambda_n) \) is a sequence such that

\[
\sum_{n=0}^{m} \left( \frac{p_n}{P_n} \right)^{k-1} |\lambda_n|^k |t_{nn}|^k = O(1) \quad \text{as} \quad m \to \infty,
\]

\[
\sum_{n=0}^{m} |\Delta \lambda_n| = O(1) \quad \text{as} \quad m \to \infty.
\]

If

\[
\frac{1}{|t_{nn}|} \sum_{v=0}^{n-1} |\Delta_v (\hat{t}_{nv})| = O(1) \quad \text{as} \quad n \to \infty,
\]

\[
\sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{k-1} |\Delta_v \hat{t}_{nv}| |t_{nn}|^{k-1} = O \left( \left( \frac{P_v}{p_v} \right)^{k-1} |t_{vv}|^k \right) \quad \text{as} \quad m \to \infty,
\]
\[
\frac{1}{|t_{nn}|} \sum_{v=0}^{n-1} |\Delta \lambda_v| |\hat{t}_{n,v+1}| = O(1) \quad \text{as } n \to \infty, \tag{2.4}
\]

\[
\sum_{n=v+1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} |\hat{t}_{n,v+1}| |\hat{t}_{nn}|^{k-1} = O(1) \quad \text{as } m \to \infty, \tag{2.5}
\]

then the series \( \sum a_n \lambda_n \) is summable \(|T, p_n|_k\).

**Proof.** Let \( (y_n) \) be the \( T \)-transform of the series \( \sum a_n \lambda_n \). Then we have, by (1.6),

\[
Y_n = y_n - y_{n-1} = \sum_{v=0}^{n} \hat{t}_{nv} a_v \lambda_v. \tag{2.6}
\]

Since \( \hat{t}_{nn} = t_{nn} \), by Abel’s transformation, we get that

\[
Y_n = \sum_{v=0}^{n-1} \Delta v (\hat{t}_{nv} \lambda_v) s_v + \hat{t}_{nn} \lambda_n s_n
\]

\[
= \sum_{v=0}^{n-1} \Delta \lambda_v \hat{t}_{n,v+1} s_v + \sum_{v=0}^{n-1} \lambda_v \Delta v (\hat{t}_{nv}) s_v + s_n t_{nn} \lambda_n
\]

\[
= Y_n(1) + Y_n(2) + Y_n(3). \tag{2.7}
\]

Using Minkowski’s inequality, it is sufficient to show that

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{P_n} \right)^{k-1} |Y_n(r)|^k < \infty \quad \text{for } r = 1, 2, 3. \tag{2.8}
\]

Since \( (s_n) \) is bounded, when \( k > 1 \), applying Hölder’s inequality with indices \( k \) and \( k' \), where \( 1/k + 1/k' = 1 \), we have that

\[
\sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} |Y_n(1)|^k \leq \sum_{n=1}^{m+1} \left[ \sum_{v=0}^{n-1} \left( \frac{P_n}{P_n} \right)^{k-1} |\Delta \lambda_v| |\hat{t}_{n,v+1}| |s_v| \right]^k
\]

\[
= O(1) \sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \left[ \sum_{v=0}^{n-1} \Delta \lambda_v |\hat{t}_{n,v+1}| t_{nn}^{-1} \right]^{k-1}
\]

\[
= O(1) \sum_{n=1}^{m+1} \left[ \sum_{v=0}^{n-1} \Delta \lambda_v |\hat{t}_{n,v+1}| t_{nn}^{-1} \right]^{k-1}
\]

\[
= O(1) \sum_{v=0}^{m} \Delta \lambda_v \sum_{n=v+1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} |\hat{t}_{n,v+1}| t_{nn}^{-1}^{k-1}
\]

\[
= O(1) \sum_{v=0}^{m} \Delta \lambda_v = O(1) \quad \text{as } m \to \infty, \tag{2.9}
\]

by virtue of the hypothesis of Theorem 2.1.
Again using Hölder's inequality, we have

\[
\sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} |Y_n(2)|^k \leq \sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \left\{ \sum_{v=0}^{n-1} |\lambda_v| |\Delta v_{tvv}||s_v| \right\}^k
\]

\[
= O(1) \sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \sum_{v=0}^{n-1} |\lambda_v|^k |\Delta v_{tvv}||t_{nn}|^{k-1}
\]

\[
\times \left\{ \frac{1}{|t_{nn}|} \sum_{v=0}^{n-1} |\Delta v_{tvv}| \right\}^{k-1}
\]

\[
= O(1) \sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \sum_{v=0}^{n-1} |\lambda_v|^k |\Delta v_{tvv}||t_{nn}|^{k-1}
\]

\[
= O(1) \sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \sum_{v=0}^{n-1} \left( \frac{P_n}{P_n} \right)^{k-1} |\Delta v_{tvv}||t_{nn}|^{k-1}
\]

\[
= O(1) \sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} |\lambda_v|^k |t_{tvv}|^{k-1} = O(1) \quad \text{as } m \to \infty,
\]

by virtue of the hypothesis of Theorem 2.1.

Finally, we have that

\[
\sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{k-1} |Y_n(3)|^k = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{k-1} |t_{nn}|^k |\lambda_n|^k = O(1) \quad \text{as } m \to \infty,
\]

(2.10)

by virtue of the hypothesis of Theorem 2.1.

Therefore, we get that

\[
\sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{k-1} |Y_n(r)|^k = O(1) \quad \text{as } m \to \infty, \quad \text{for } r = 1, 2, 3.
\]

(2.11)

This completes the proof of Theorem 2.1.

\[\square\]

### 3. An application

Now we will prove the following corollary.

**Corollary 3.1** (see [2]). Let \( k \geq 1 \). If the sequence \( (s_n) \) is bounded and \( (\lambda_n) \) is a sequence such that

\[
\sum_{n=1}^{m} \frac{P_n}{P_n} |\lambda_n|^k = O(1) \quad \text{as } m \to \infty,
\]

(3.1)

\[
\sum_{n=1}^{m} |\Delta \lambda_n| = O(1) \quad \text{as } m \to \infty,
\]

then the series \( \sum a_n \lambda_n \) is summable \( \|N, p_n\|_k \).
Proof. In Theorem 2.1, let $t_{nv} = \frac{p_v}{P_n}$. Then to prove the corollary, it is sufficient to show that the conditions of Theorem 2.1 are satisfied.

If $t_{nn} = \frac{p_n}{P_n}$, (2.1) are automatically satisfied.

Since

$$\Delta_v \hat{t}_{nv} = \hat{t}_{nv} - \hat{t}_{n,v+1}$$

$$= \hat{t}_{nv} - \hat{t}_{n-1,v} - \hat{t}_{n,v+1} + \hat{t}_{n-1,v+1}$$

$$= \sum_{i=v}^{n} t_{ni} - \sum_{i=v}^{n-1} t_{n-1,i} - \sum_{i=v+1}^{n} t_{ni} + \sum_{i=v+1}^{n-1} t_{n-1,i}$$

$$= \frac{1}{P_n} \sum_{i=v}^{n} P_i - \frac{1}{P_{n-1}} \sum_{i=v}^{n-1} P_i - \frac{1}{P_n} \sum_{i=v+1}^{n} P_i + \frac{1}{P_{n-1}} \sum_{i=v+1}^{n-1} P_i$$

$$= -\frac{p_n p_v}{P_n P_{n-1}},$$

we get

$$\frac{1}{|t_{nn}|} \sum_{v=0}^{n-1} |\Delta_v \hat{t}_{nv}| = \frac{P_n}{P_n} \sum_{v=0}^{n-1} \frac{p_n p_v}{P_n P_{n-1}} = O(1) \quad \text{as} \quad n \to \infty. \quad (3.3)$$

Thus condition (2.2) is satisfied.

Using $\Delta_v \hat{t}_{nv}$ and $t_{nn}$,

$$\sum_{n=v+1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} |\Delta_v \hat{t}_{nv}| |t_{nn}|^{k-1} = \sum_{n=v+1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} \frac{p_n p_v}{P_n P_{n-1}} \left( \frac{P_n}{P_n} \right)^{k-1}$$

$$= p_v \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = \frac{p_v}{P_v} \quad (3.4)$$

$$= \left( \frac{P_v}{p_v} \right)^{k-1} |t_{vv}|^{k} \quad \text{as} \quad m \to \infty,$$

condition (2.3) is satisfied.

Since

$$\hat{t}_{nv} = \hat{t}_{nv} - \hat{t}_{n-1,v} = \sum_{i=v}^{n} t_{ni} - \sum_{i=v}^{n-1} t_{n-1,i}$$

$$= \frac{1}{P_n} \sum_{i=v}^{n} P_i - \frac{1}{P_{n-1}} \sum_{i=v}^{n-1} P_i$$

$$= p_v \left( -\frac{1}{P_n} + \frac{1}{P_{n-1}} \right) = p_v \frac{P_n}{P_n P_{n-1}}$$,
On absolute matrix summability methods

\[
\frac{1}{|\mathcal{I}_{mn}|} \sum_{v=0}^{n-1} |\Delta \lambda_v| |\mathcal{I}_{nv,v+1}| = \frac{P_n}{P_n} \sum_{v=0}^{n-1} |\Delta \lambda_v| P_v \frac{P_n}{P_n P_{n-1}}
\]

\[
= \frac{1}{P_{n-1}} \sum_{v=0}^{n-1} |\Delta \lambda_v| P_v = O(1) \sum_{v=0}^{n-1} |\Delta \lambda_v| = O(1) \quad \text{as } n \to \infty,
\]

(3.5)

and condition (2.4) is satisfied.

Finally,

\[
\sum_{n=v+1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} |\mathcal{I}_{nv,v+1}| |\mathcal{I}_{mn}|^{k-1} = \sum_{n=v+1}^{m+1} \left( \frac{P_n}{P_n} \right)^{k-1} P_v \frac{P_n}{P_n P_{n-1}} \left( \frac{P_n}{P_n} \right)^{k-1}
\]

\[
= P_v \sum_{n=v+1}^{m+1} \frac{P_n}{P_n P_{n-1}} = O(1) \quad \text{as } m \to \infty,
\]

(3.6)

so condition (2.5) is satisfied.

This completes the proof of the corollary. \(\square\)

References


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