ON THE L^p -CONVERGENCE FOR MULTIDIMENSIONAL ARRAYS OF RANDOM VARIABLES

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For a d -dimensional array of random variables $\{X_n, n \in \mathbb{Z}_+^d\}$ such that $\{|X_n|^p, n \in \mathbb{Z}_+^d\}$ is uniformly integrable for some $0 < p < 2$, the L^p -convergence is established for the sums $(1/|n|^{1/p})(\sum_{j < n}(X_j - a_j))$, where $a_j = 0$ if $0 < p < 1$, and $a_j = EX_j$ if $1 \leq p < 2$.

1. Introduction

Let $\mathbb{Z}_+^d$, where d is an integer, denote the positive integer d -dimensional lattice points. The notation $m < n$, where $m = (m_1, m_2, \dots, m_d)$ and $n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}_+^d$, means that $m_i \leq n_i$, $1 \leq i \leq d$, $|n|$ is used for $\prod_{i=1}^d n_i$.

Gut [2] proved that if $\{X, X_n, n \in \mathbb{Z}_+^d\}$ is a d -dimensional array of i.i.d. random variables with $E|X|^p < \infty$ ($0 < p < 2$) and $EX = 0$ if $1 \leq p < 2$, then

$$\frac{\sum_{j < n} X_j}{|n|^{1/p}} \rightarrow 0 \text{ in } L^p \quad \text{as } \min_{1 \leq i \leq d} n_i \rightarrow \infty, \quad (1.1)$$

where $(n_1, n_2, \dots, n_d) = n \in \mathbb{Z}_+^d$.

In 1999, Hong and Hwang [3] proved that if $\{X_{mn}, m \geq 1, n \geq 1\}$ is a double array of pairwise independent random variables such that

$$P\{|X_{mn}| > t\} \leq P\{|X| > t\}, \quad t \geq 0, m \geq 1, n \geq 1, \quad (1.2)$$

where X is a random variable, then the condition $E(|X|^p \log^+ |X|) < \infty$ ($1 < p < 2$) implies that

$$\frac{\sum_{k=1}^m \sum_{l=1}^n (X_{kl} - EX_{kl})}{(mn)^{1/p}} \rightarrow 0 \text{ in } L^1 \quad \text{as } \max\{m, n\} \rightarrow \infty. \quad (1.3)$$

In this note, we provide conditions for $(1/|n|^{1/p})(\sum_{j < n}(X_j - a_j)) \rightarrow 0$ in L^p as $|n| \rightarrow \infty$, where $n \in \mathbb{Z}_+^d$, $j \in \mathbb{Z}_+^d$, $a_j = 0$ if $0 < p < 1$, and $a_j = EX_j$ if $1 \leq p < 2$.

2. Result

THEOREM 2.1. *Let $\{X_n, n \in \mathbb{Z}_+^d\}$ be a d -dimensional array of random variables such that $\{|X_n|^p, n \in \mathbb{Z}_+^d\}$ is uniformly integrable for some $0 < p < 2$. Assume that $\{X_n, n \in \mathbb{Z}_+^d\}$ is pairwise independent if $p = 1$ and $\{X_n, n \in \mathbb{Z}_+^d\}$ is independent if $1 < p < 2$. Then,*

$$\frac{\sum_{j < n} (X_j - a_j)}{|n|^{1/p}} \longrightarrow 0 \quad \text{in } L^p \text{ as } |n| \longrightarrow \infty, \quad (2.1)$$

where $a_j = 0$ if $0 < p < 1$, and $a_j = EX_j$ if $1 \leq p < 2$.

Proof. For arbitrary $\epsilon > 0$, there exists $M > 0$ such that

$$E(|X_n|^p I(|X_n| > M)) < \epsilon \quad \forall n \in \mathbb{Z}_+^d. \quad (2.2)$$

Set

$$\begin{aligned} X'_n &= X_n I(|X_n| \leq M), \quad n \in \mathbb{Z}_+^d, \\ X''_n &= X_n I(|X_n| > M), \quad n \in \mathbb{Z}_+^d. \end{aligned} \quad (2.3)$$

For all $n \in \mathbb{Z}_+^d$,

$$E|X''_n - EX''_n|^p \leq 4E|X''_n|^p < 4\epsilon. \quad (2.4)$$

If $0 < p < 1$, then

$$\begin{aligned} E \left| \sum_{j < n} X_j \right|^p &\leq E \left| \sum_{j < n} X'_j \right|^p + E \left| \sum_{j < n} X''_j \right|^p \leq E \left| \sum_{j < n} X'_j \right|^p + \sum_{j < n} E|X''_j|^p \\ &\leq (|n|M)^p + |n|\epsilon \quad (\text{by (2.2)}). \end{aligned} \quad (2.5)$$

The conclusion (2.1) follows from (2.5).

If $p = 1$ and $\{X_n, n \in \mathbb{Z}_+^d\}$ is pairwise independent, then

$$\begin{aligned} E \left| \sum_{j < n} (X_j - EX_j) \right| &\leq E \left| \sum_{j < n} (X'_j - EX'_j) \right| + \sum_{j < n} E|X''_j - EX''_j| \\ &\leq \left[E \left| \sum_{j < n} (X'_j - EX'_j) \right|^2 \right]^{1/2} + \sum_{j < n} E|X''_j - EX''_j| \\ &\quad (\text{by the Jensen inequality (see [1, page 103])}) \\ &\leq \left[\sum_{j < n} E(X'_j - EX'_j)^2 \right]^{1/2} + 4|n|\epsilon \quad (\text{by (2.4)}) \\ &\leq (|n|M^2)^{1/2} + 4|n|\epsilon \\ &\quad \left(\text{since } E(X'_j - EX'_j)^2 = E(X'_j)^2 - (EX'_j)^2 \leq M^2, j \in \mathbb{Z}_+^d \right) \\ &= o(|n|) \quad \text{as } |n| \longrightarrow \infty. \end{aligned} \quad (2.6)$$

If $1 < p < 2$ and $\{X_n, n \in \mathbb{Z}_+^d\}$ is independent, then

$$\begin{aligned}
 E \left| \sum_{j < n} (X_j - EX_j) \right|^p &\leq 2^{p-1} \left[E \left| \sum_{j < n} (X'_j - EX'_j) \right|^p + E \left| \sum_{j < n} (X''_j - EX''_j) \right|^p \right] \\
 &\leq 2^{p-1} \left[\left(E \left| \sum_{j < n} (X'_j - EX'_j) \right|^2 \right)^{p/2} + 2 \sum_{j < n} E |X''_j - EX''_j|^p \right] \\
 &\quad \text{(by the Jensen inequality [1] and the von Bahr-Esseen} \\
 &\quad \text{inequality [4])} \\
 &\leq 2^{p-1} \left(\sum_{j < n} E (X'_j - EX'_j)^2 \right)^{p/2} + 2^{p+2} |n| \epsilon \quad \text{(by (2.4))} \\
 &\leq 2^{p-1} (|n| M^2)^{p/2} + 2^{p+2} |n| \epsilon \\
 &\quad \left(\text{since } E (X'_j - EX'_j)^2 = E (X'_j)^2 - (EX'_j)^2 \leq M^2, j \in \mathbb{Z}_+^d \right) \\
 &= o(|n|) \quad \text{as } |n| \rightarrow \infty,
 \end{aligned} \tag{2.7}$$

again establishing (2.1). $\square$

Note that if $\{X, X_n, n \in \mathbb{Z}_+^d\}$ are random variables such that $E|X|^p < \infty$ ($p > 0$) and $\sup_{n \in \mathbb{Z}_+^d} P\{|X_n| > t\} \leq P\{|X| > t\}$ for all $t \geq 0$, then $\{|X_n|^p, n \in \mathbb{Z}_+^d\}$ is uniformly integrable. The following corollary follows immediately from Theorem 2.1.

COROLLARY 2.2. *Let $\{X, X_n, n \in \mathbb{Z}_+^d\}$ be random variables such that $E|X|^p < \infty$ for some $0 < p < 2$, and $\sup_{n \in \mathbb{Z}_+^d} P\{|X_n| > t\} \leq P\{|X| > t\}$ for all $t \geq 0$. Assume that $\{X_n, n \in \mathbb{Z}_+^d\}$ is pairwise independent if $p = 1$ and $\{X_n, n \in \mathbb{Z}_+^d\}$ is independent if $1 < p < 2$. Then,*

$$\frac{\sum_{j < n} (X_j - a_j)}{|n|^{1/p}} \rightarrow 0 \text{ in } L^p \quad \text{as } |n| \rightarrow \infty, \tag{2.8}$$

where $a_j = 0$ if $0 < p < 1$, and $a_j = EX_j$ if $1 \leq p < 2$.

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