MATRIX TRANSFORMATIONS AND WALSH'S EQUICONVERGENCE THEOREM

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In 1977, Jacob defines G_α , for any $0 \leq \alpha < \infty$, as the set of all complex sequences x such that $\limsup |x_k|^{1/k} \leq \alpha$. In this paper, we apply $G_u - G_v$ matrix transformation on the sequences of operators given in the famous Walsh's equiconvergence theorem, where we have that the difference of two sequences of operators converges to zero in a disk. We show that the $G_u - G_v$ matrix transformation of the difference converges to zero in an arbitrarily large disk. Also, we give examples of such matrices.

1. Introduction

If $x = (x_k)$ is a complex number sequence and $A = [a_{nk}]$ is an infinite matrix, then Ax is the sequence whose n th term is given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k. \quad (1.1)$$

The matrix A is called $X - Y$ matrix if Ax is in the set Y whenever x is in X . For $0 \leq \alpha < \infty$, let $G_\alpha = \{x : \limsup |x_k|^{1/k} \leq \alpha\}$. For various values of α , this sequence space has been studied extensively by many authors (see [3, 8, 9]). In particular, Jacob [5, page 186] proves the following result.

THEOREM 1.1. *An infinite matrix A is a $G_u - G_v$ matrix if and only if for each number w such that $0 < w < 1/v$, there exist numbers B and s such that $0 < s < 1/u$ and*

$$|a_{nk}| w^n \leq B s^k \quad (1.2)$$

for all n and k .

2. Preliminaries

Let f be an analytic function in the disk $\mathbf{D}_R = \{z \in \mathbf{C} : |z| < R\}$ for some $R > 1$. If $f(z)$ has the Taylor series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then for each positive integer n , let

$$S_n(z; f) = \sum_{k=0}^n a_k z^k \quad (2.1)$$

be the n th partial sum of $f(z)$. Also, let $L_n(z; f)$ denote the unique Lagrange interpolation polynomial of degree at most n which interpolates $f(z)$ in the $(n+1)$ st roots of unity, that is,

$$L_n(\omega^k; f) = f(\omega^k) \quad \text{for } k = 0, 1, \dots, n, \quad (2.2)$$

where $\omega = e^{2\pi i/(n+1)}$. Then the well-known Walsh's equiconvergence theorem [10] states that

$$\lim_{n \rightarrow \infty} [L_n(z; f) - S_n(z; f)] = 0 \quad \text{for } z \in D_{R^2}, \quad (2.3)$$

the convergence being uniform and geometric on any closed subdisk of D_{R^2} .

This theorem has been extended in various ways by several authors. In [7], Price used certain arithmetical means and in [6], Lou used commutators of interpolation operators to enlarge the disk $\mathbf{D}_{R^2}$ of equiconvergence. In [1], Brück applied certain summability methods to the difference $L_n - S_n$ in order to enlarge the disk $\mathbf{D}_{R^2}$. Also, in [2], the authors extended the disk of convergence by substituting the n th partial sum $S_n(z; f)$ by polynomials

$$Q_{l,n}(z; f) = \sum_{k=0}^n \sum_{j=0}^{l-1} a_{k+j(n+1)} z^k, \quad (2.4)$$

where l is a fixed positive integer.

Our aim is to apply a certain class of matrices to L_n and S_n and enlarge the disk $\mathbf{D}_{R^2}$ of Walsh's equiconvergence to $\mathbf{D}_\rho$ for any $\rho > R^2$.

Throughout this paper, we let Γ be any circle $|t| = r$ with $1 < r < R$. For any function f analytic in $\mathbf{D}_R$, we have by Cauchy integral formula

$$\begin{aligned} L_n(z; f) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{t^{n+1} - z^{n+1}}{t^{n+1} - 1} \frac{f(t)}{t - z} dt \\ &= \frac{1}{2\pi i} \int_{\Gamma} \left[1 - \left(\frac{z}{t} \right)^{n+1} \right] \frac{t^{n+1}}{t^{n+1} - 1} \frac{f(t)}{t - z} dt. \end{aligned} \quad (2.5)$$

Since $|t| = r > 1$, we get that

$$L_n(z; f) = \frac{1}{2\pi i} \int_{\Gamma} \left[1 - \left(\frac{z}{t} \right)^{n+1} \right] \sum_{j=0}^{\infty} \left(\frac{1}{t^{n+1}} \right)^j \frac{f(t)}{t - z} dt. \quad (2.6)$$

Interchanging the summation and the integral, we see that

$$\begin{aligned} L_n(z; f) &= \frac{1}{2\pi i} \int_{\Gamma} \left[1 - \left(\frac{z}{t} \right)^{n+1} \right] \frac{f(t)}{t - z} dt \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma} \left[1 - \left(\frac{z}{t} \right)^{n+1} \right] \sum_{j=1}^{\infty} \frac{1}{t^{j(n+1)}} \frac{f(t)}{t - z} dt. \end{aligned} \quad (2.7)$$

Similarly, we can express $S_n(z; f)$ as follows:

$$S_n(z; f) = \frac{1}{2\pi i} \int_{\Gamma} \left[1 - \left(\frac{z}{t} \right)^{n+1} \right] \frac{f(t)}{t-z} dt. \quad (2.8)$$

Therefore,

$$L_n(z; f) = S_n(z; f) + \frac{1}{2\pi i} \int_{\Gamma} \left[1 - \left(\frac{z}{t} \right)^{n+1} \right] \sum_{j=1}^{\infty} \frac{1}{t^{j(n+1)}} \frac{f(t)}{t-z} dt. \quad (2.9)$$

For simplicity, we will denote $L_n(z; f)$ by $L_n(z)$ and $S_n(z; f)$ by $S_n(z)$.

3. Main result

For $1 < r < R$, choose $\rho > R^2$, $u > \rho/r$, and $0 < v < 1$. Let A be a $G_u - G_v$ matrix. Therefore, by Theorem 1.1, for any w such that $1 < w < 1/v$, there exist numbers B and s such that $0 < s < 1/u$ and

$$|a_{nk}| w^n \leq B s^k \quad \forall n, k. \quad (3.1)$$

Consequently, the matrix A is a summability matrix which transforms null sequences into null sequences. This is because

$$\begin{aligned} \sum_{k=0}^{\infty} |a_{nk}| &\leq \frac{B}{(1-s)w^n} \leq \frac{B}{(1-s)}, \\ \sum_{k=0}^{\infty} a_{nk} &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad a_{nk} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (3.2)$$

We define $\lambda_n(z) = \sum_{k=0}^{\infty} a_{nk} L_k(z)$ and $\sigma_n(z) = \sum_{k=0}^{\infty} a_{nk} S_k(z)$. Then, for $|z| < \rho$, we obtain that

$$\begin{aligned} \sigma_n(z) &= \sum_{k=0}^{\infty} a_{nk} \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} \left[1 - \left(\frac{z}{t} \right)^{k+1} \right] dt \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} \left[\sum_{k=0}^{\infty} a_{nk} - \left(\frac{z}{t} \right) \sum_{k=0}^{\infty} a_{nk} \left(\frac{z}{t} \right)^k \right] dt. \end{aligned} \quad (3.3)$$

The interchange of the integral and the summation is justified by showing that the series $\sum_k a_{nk}$ and $\sum_k a_{nk} (z/t)^k$ converge absolutely as follows. Using (3.1), we get that the series

$$\sum_{k=0}^{\infty} |a_{nk}| \leq \frac{B}{w^n} \sum_{k=0}^{\infty} s^k, \quad (3.4)$$

which converges for each n since $s < 1/u < 1$ and that the series

$$\begin{aligned} \sum_{k=0}^{\infty} |a_{nk}| \left| \frac{z}{t} \right|^k &\leq \frac{B}{w^n} \sum_{k=0}^{\infty} \left(\frac{|z|s}{|t|} \right)^k, \quad t \in \Gamma, \\ &= \frac{B}{w^n} \sum_{k=0}^{\infty} \left(\frac{|z|s}{r} \right)^k, \end{aligned} \quad (3.5)$$

which also converges for each n , since $|z|s/r < |z|/ru < |z|/\rho < 1$. Also,

$$\begin{aligned} \lambda_n(z) &= \sum_{k=0}^{\infty} a_{nk} \left[S_k(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} \left(1 - \left(\frac{z}{t} \right)^{k+1} \right) \sum_{j=1}^{\infty} \frac{1}{t^{j(k+1)}} dt \right] \\ &= \sigma_n(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} \sum_{j=1}^{\infty} \left[\sum_{k=0}^{\infty} a_{nk} \frac{1}{t^{j(k+1)}} - \sum_{k=0}^{\infty} a_{nk} \left(\frac{z}{t} \right)^{k+1} \frac{1}{t^{j(k+1)}} \right] dt. \end{aligned} \quad (3.6)$$

The interchange of the integral and the summation is justified as follows. Using (3.1), we see that for each n and each j ,

$$\begin{aligned} \sum_{k=0}^{\infty} |a_{nk}| \frac{1}{|t|^{j(k+1)}} &\leq \frac{B}{w^n r^j} \sum_{k=0}^{\infty} \left(\frac{s}{r^j} \right)^k \\ &\leq \frac{B}{w^n r^j} \frac{r^j}{(r^j - s)} = \frac{B}{w^n (r^j - s)} \end{aligned} \quad (3.7)$$

because $s/r^j < 1/ur^j < 1/\rho r^{j-1} < 1$, and similarly

$$\begin{aligned} \sum_{k=0}^{\infty} |a_{nk}| \left| \frac{z}{t} \right|^{k+1} \frac{1}{|t|^{j(k+1)}} &\leq \frac{B|z|}{w^n r^{j+1}} \sum_{k=0}^{\infty} \left(\frac{|z|s}{r^{j+1}} \right)^k \\ &\leq \frac{B|z|}{w^n r^{j+1}} \frac{r^{j+1}}{(r^{j+1} - |z|s)} \\ &= \frac{B|z|}{w^n (r^{j+1} - |z|s)} \end{aligned} \quad (3.8)$$

because $|z|s/r^{j+1} < |z|s/r < 1$.

THEOREM 3.1. Let $\rho > R^2$. Choose $u > \rho/r$, where $1 < r < R$ and $0 < v < 1$ and let A be a $G_u - G_v$ matrix. Then

$$\lim_{n \rightarrow \infty} [\lambda_n(z) - \sigma_n(z)] = 0 \quad \forall z \in D_{\rho}. \quad (3.9)$$

Proof. Using the expressions obtained for $\lambda_n(z)$ and $\sigma_n(z)$, we get that

$$\lambda_n(z) - \sigma_n(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} \sum_{j=1}^{\infty} \left[\sum_{k=0}^{\infty} a_{nk} \frac{1}{t^{j(k+1)}} - \sum_{k=0}^{\infty} a_{nk} \left(\frac{z}{t} \right)^{k+1} \frac{1}{t^{j(k+1)}} \right] dt. \quad (3.10)$$

Therefore using (3.7) and (3.8), for each n , we have that

$$|\lambda_n(z) - \sigma_n(z)| \leq \frac{B}{2\pi w^n} \int_{\Gamma} \frac{|f(t)|}{|t - z|} \left[\sum_{j=1}^{\infty} \frac{1}{r^j - s} + \sum_{j=1}^{\infty} \frac{|z|}{(r^{j+1} - |z|s)} \right] dt. \quad (3.11)$$

It can be easily proved that the two series on the right-hand side of the above inequality converge by using the ratio test. Therefore, $w > 1$ implies that

$$\lim_{n \rightarrow \infty} [\lambda_n(z) - \sigma_n(z)] = 0 \quad (3.12)$$

for each $|z| < \rho$. □

4. Examples

First, we give below an obvious example for such a matrix A . Choose $u > \rho/r$ and v such that $0 < v < 1$. Define the matrix A by

$$a_{nk} = \frac{v^n}{t^k}, \quad t > u. \quad (4.1)$$

For each w so that $0 < w < 1/v$, we have

$$|a_{nk}| w^n = \frac{(vw)^n}{t^k} < \frac{1}{t^k}, \quad (4.2)$$

where $1/t < 1/u$. Hence by Theorem 1.1, A is a $G_u - G_v$ matrix.

Our next example is the Sonnenschein matrix $A(g) = [a_{nk}]$ which is defined by [4, page 257]

$$[g(z)]^n = \sum_{k=0}^{\infty} a_{nk} z^k \quad \text{for } n \geq 1, \quad (4.3)$$

where g is analytic at $z = 0$ and $a_{00} = 1$, and $a_{0k} = 0$ for $k \geq 1$. Clearly, for each $n \geq 1$,

$$a_{nk} = \frac{1}{k!} \frac{d^k}{dz^k} [g(z)]^n \Big|_{z=0}. \quad (4.4)$$

As we easily see that the first $(n-1)$ derivatives of $[g(z)]^n$ contains $g(z)$ as its factor. So, if $g(0) = 0$, then the first $(n-1)$ terms of the series $\sum_{k=0}^{\infty} a_{nk} z^k$ vanish and the matrix $A(g) = [a_{nk}]$ reduces to an upper triangular matrix.

Now, for $u > \rho/r$ and $0 < v < 1$, choose

$$l > \max \left\{ u \left(1 + \frac{1}{v} \right), \frac{3}{2v} \right\}. \quad (4.5)$$

Let $g(z) = 1/(z - 2l) + 1/2l$ so that $g(0) = 0$. Therefore, the Sonnenschein matrix $A(g) = [a_{nk}]$ is an upper triangular matrix. Since $g(z)$ is analytic at $z = 0$ and on D_{2l} , $[g(z)]^n$ is analytic on D_{2l} . Let $C = \{z : |z| = l\}$. Then on C ,

$$|g(z)| \leq \frac{1}{|z - 2l|} + \frac{1}{2l} \leq \frac{3}{2l}. \quad (4.6)$$

Therefore by Cauchy integral formula,

$$\begin{aligned} |a_{nk}| &= \left| \frac{1}{2\pi i} \int_C \frac{[g(z)]^n}{t^{k+1}} dt \right| \\ &\leq \left(\frac{3}{2l} \right)^n \frac{1}{l^k} \quad \text{for } k \geq n > 0. \end{aligned} \quad (4.7)$$

Then for any w such that $0 < w < 1/v$, we have

$$\begin{aligned} |a_{nk}| w^n &\leq \left(\frac{3}{2l} \right)^n \frac{w^n}{l^k} \\ &\leq \left(\frac{3}{2l} \right)^n \left(\frac{1}{vl} \right)^k \quad \text{for } k \geq n \quad (0 < v < 1) \\ &< v^n \left(\frac{1}{vl} \right)^k \quad \text{since } l > \frac{3}{2v}, \\ &< (1+v)^n \left(\frac{1}{vl} \right)^k \\ &= \left(\frac{1+v}{vl} \right)^k \quad \text{for } k \geq n, \end{aligned} \quad (4.8)$$

where $(1+v)/vl = (1/l)(1+1/v) < 1/u$. Therefore by Theorem 1.1, $A(g)$ is a $G_u - G_v$ matrix.

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