THE INCIDENCE CHROMATIC NUMBER OF SOME GRAPH

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The concept of the incidence chromatic number of a graph was introduced by Brualdi and Massey (1993). They conjectured that every graph G can be incidence colored with $\Delta(G)+2$ colors. In this paper, we calculate the incidence chromatic numbers of the complete k-partite graphs and give the incidence chromatic number of three infinite families of graphs.

1. Introduction

Throughout the paper, all graphs dealt with are finite, simple, undirected, and loopless. Let G be a graph, and let $V(G)$, $E(G)$, $\Delta(G)$, respectively, denote vertex set, edge set, and maximum degree of G. In 1993, Brualdi and Massey [3] introduced the concept of incidence coloring. The order of G is the cardinality $|v(G)|$. The size of G is the cardinality $|E(G)|$. Let

$$I(G) = \{(v,e) \mid v \in V,\ e \in E,\ v \text{ is incident with } e\} \tag{1.1}$$

be the set of incidences of G. We say that two incidences (v,e) and (w,f) are adjacent provided one of the following holds:

(i) $v = w$;
(ii) $e = f$;
(iii) the edge $vw = e$ or $vw = f$.

Figure 1.1 shows three cases of two incidences being adjacent.

An incidence coloring σ of G is a mapping from $I(G)$ to a set C such that no two adjacent incidences of G have the same image. If $\sigma : I(G) \to C$ is an incidence coloring of G and $|C| = k$, k is a positive integer, then we say that G is k-incidence colorable.

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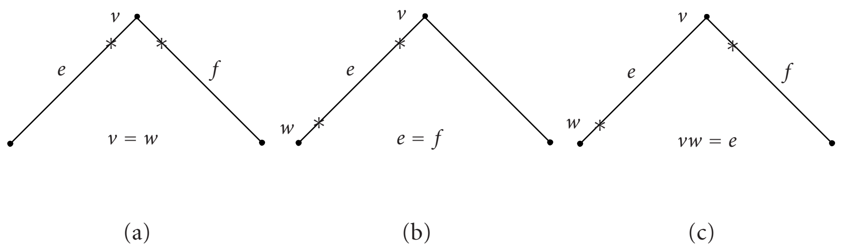

Figure 1.1. Cases of two incidences being adjacent.

The minimum cardinality of C for which there exists an incidence coloring $\sigma : I(G) \to C$ is called the incidence chromatic number of G, and is denoted by $\text{inc}(G)$. A partition $\{I_1, I_2, \dots, I_k\}$ of $I(G)$ is called an independence partition of $I(G)$ if each I_i is independent in $I(G)$ (i.e., no two incidences of I_i are adjacent in $I(G)$). Clearly, for $k' \geq \text{inc}(G)$, G is k'-incidence colorable.

We may consider G as a digraph by splitting each edge uv into two opposite arcs (u,v) and (v,u). Let $e = uv$. We identify (u,e) with the arc (u,v). So $I(G)$ may be identified with the set of all arcs $A(G)$. Two distinct arcs (incidences) (u,v) and (x,y) are adjacent if one of the following holds (see Figure 1.2):

(1') $u = x$;

(2') $u = y$ and $v = x$;

(3') $v = x$.

This concept was first developed by Brualdi and Massey [3] in 1993. They posed the incidence coloring conjecture (ICC), which states that for every graph G, $\text{inc}(G) \leq \Delta + 2$. In 1997, Guiduli [5] showed that incidence coloring is a special case of directed star arboricity, introduced by Algor and Alon [1]. They pointed out that the ICC was solved in the negative following an example in [1]. Following the analysis in [1], they showed that $\text{inc}(G) \geq \Delta + \Omega(\log \Delta)$, where $\Omega = 1/8 - \circ(1)$. Making use of a tight upper bound for directed star arboricity, they obtained the upper bound $\text{inc}(G) \leq \Delta + O(\log \Delta)$.

Brualdi and Massey determined the incidence chromatic numbers of trees, complete graphs, and complete bipartite graphs [3]; Chen, Liu, and Wang determined the incidence chromatic numbers of paths, cycles, fans, wheels, adding-edge wheels, and complete 3-partite graphs [4].

In this paper, we will consider the incidence chromatic number for complete k-partite graphs. We will give the incidence chromatic number of complete k-partite graphs, and also give the incidence chromatic number of three infinite families of graphs. Let k be positive integer, put $[k] = 1, 2, \dots, k$. We state first the following definitions.

Definition 1.1. For a graph $G(V,E)$ with vertex set V and edge set E, the incidence graph $I(G)$ of G is defined as the graph with vertex set $V(I(G))$ and edge set $E(I(G))$.

Definitions not given here may be found in [2].

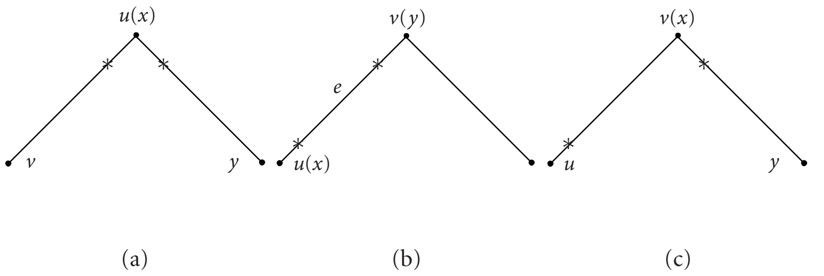

Figure 1.2. Cases of two arcs (incidences) are adjacent.

2. Some useful lemmas and properties of incidence chromatic number

LEMMA 2.1. *Let T be a tree of order $n \geq 2$ with maximum degree Δ. Then* $\operatorname{inc}(T) = \Delta + 1$.

LEMMA 2.2. *A graph G is k-incidence colorable if and only if its incidence graph $I(G)$ is k-vertex colorable, that is,* $\operatorname{inc}(G) = \chi(I(G))$.

Let $M = \{(ue, ve) \mid e = uv \in E(G),\ (ue, ve) \in E(I(G))\}$, then M forms a perfect matching of incidence graph $I(G)$. The following lemmas are obvious.

LEMMA 2.3. *The incidence graph $I(G)$ of a graph G is a graph with a perfect matching.*

LEMMA 2.4. *For a graph G, $v \in V(G)$, let $B_v = \{(u, uv) \mid uv \in E(G),\ u \in V(G)\}$, $A_v = \{(v, vu) | uv \in E(G),\ u \in V(G)\}$, then $\{B_v\}$ is an independence-partition of incidence graph $I(G)$, and the induced subgraph $G[A_v]$ of $I(G)$ is a clique graph.*

By the definition of incidence graph, it is easy to give the proof.

LEMMA 2.5. *Let Δ be the maximum degree of graph G, $I(G)$ the incidence graph, then complete graph $K_{\Delta+1}$ is a subgraph of $I(G)$.*

Proof. Let $d(u) = \Delta$, $p = \Delta$, and $e_k = uv_1$, $e_2 = uv_2$, $\ldots$, $e_p = uv_1$ be the edges of G. p incidences in $I_u = \{(u, e_1), (u, e_2), \ldots, (u, e_p)\}$ are adjacent to each other. For an incidence in $I_v = \{(v_i, v_i u) \mid uv_i \in G,\ 1 \leq i \leq p\}$, $(v_i, v_i u)$ is adjacent to all incidences in I_u. Since $p + 1$ incidences $(u, e_2), \ldots, (u, e_p), (v_i, v_i u)$ are vertices of $I(G)$, by the definition of incidence graph, we can complete the proof. □

LEMMA 2.6. *For a simple graph G with order n,* $\operatorname{inc}(G) = n = \Delta(G) + 1$, *when* $\Delta(G) = n - 1$.

Proof. Let $|V(G)| = \Delta + 1 = \nu(G)$, by Lemma 2.4, $\{B_v\}$ is an independence-partition of incidence graph $I(G)$, then $\chi(I(G)) \leq \nu(G)$. By Lemma 2.5, $K_{\Delta+1}$ is the subgraph of $I(G)$, thus $\chi(I(G)) \geq \nu(G)$, then $\operatorname{inc}(G) = \chi(I(G)) = \Delta + 1$, as required. □

The following corollaries can be easily verified.

COROLLARY 2.7. *Let G be a graph with order n ($n \geq 2$), then* $\operatorname{inc}(G) \leq \Delta + 2$, *when* $\Delta(G) = n - 2$.

In fact, for graphs G with order n, we can give each incidence in $I(G)$ proper incidence coloring as follows. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set and $C = \{1, 2, \dots, n\}$ the color set. For $i, j = 1, 2, \dots, n$, we let $\sigma(v_i, v_iv_j) = j$. It is easy to see that the coloring above is an incidence coloring of G only with n colors. That is, $\mathrm{inc}(G) \le \Delta + 2$, when $\Delta \ge n - 2$.

Corollary 2.8. *Let W_n be the wheel graph with order $n+1$. Then* $\mathrm{inc}(W_n) = n + 1$.

Lemma 2.9. *Let H be a subgraph of G, then* $\mathrm{inc}(H) \le G$.

Lemma 2.10. *Let G be union of disjoint graphs $G_1, G_2, \dots,$ and G_t. If G_i has an m-incidence coloring for all $i = 1, 2, \dots, t$, then G has an m-incidence coloring. That is* $\mathrm{inc}(G) = \max\{\mathrm{inc}(G_i) \mid i = 1, 2, \dots, t\}$.

Proof. To prove this lemma, we only need to prove that $G_1 \cup G_2$ has an m-incidence coloring. Let $\{I_1, I_2, \dots, I_m\}$ be an independence partition of $I(G_1)$, and $\{I'_1, I'_2, \dots, I'_m\}$ an independence partition of $I(G_2)$. Then $\{I_1 \cup I'_1, I_2 \cup I'_2, \dots, I_m \cup I'_m\}$ forms an independence-partition of $I(G_1) \cup I(G_2)$. Hence G has an m-incidence coloring. The proof of the lemma is complete. □

Theorem 2.11. *Let G be a graph with maximum degree $\Delta(G) = n-2$ and minimum degree $\delta(G) \le [n/2] - 1$, then* $\mathrm{inc}(G) = n - 1 = \Delta(G) + 1$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$, $d(v_1) = \delta(G)$, and $u \notin V(G)$. Consider the auxiliary graph G' with vertex set $V(G') = V(G) \cup \{u\}$ and edge set $E(G') = E(G) \cup \{uv_i \mid i = 2, 3, \dots, n\}$. It follows that $\Delta(G') = n - 1$. Let $G'' = G' - \{v_1\}$, then $\Delta(G'') = n - 1$, by Lemma 2.5, $\mathrm{inc}(G'') = n$. For color set $C = \{1, 2, \dots, n\}$, suppose that σ' is the n-incidence coloring of G'' with color set C. Without loss of generality, let $\sigma'(v_i, v_iv_j) = j$ $(v_iv_j \in E(G))$ and $\sigma'(v_i, v_iu) = 1$ $(i = 2, 3, \dots, n)$, $\sigma'(u, uv_i) = i$ $(i = 2, 3, \dots, n)$. In incidence set $I(G)$, incidences $(v_i, v_iv_j)(i, j = 2, 3, \dots, n,$ and $i \ne j)$ are all adjacent to (v_i, v_iu) and (v_j, v_ju), thus the color n cannot be used to color any incidence in $I(G'' - \{u\})$. Denote by $N(v_1) = \{v_{i_1}, v_{i_2}, \dots, v_{i_\delta}\}$ the vertices adjacent to v_1. The incidence coloring σ' of graph G'' may be extended to an incidence coloring σ of graph G. For $x, y \in V(G)$ and $x, y \notin \{v_1\} \cup N(v_1)$, let $\sigma(x, xy) = \sigma'(x, xy)$. Because $\Delta(G) = n - 2$, for vertex v_{i_k} $(k = 1, 2, \dots, \delta)$, there exists a vertex $v_{t_k} \in V(G)$ such that $v_{i_k}v_{t_k} \notin E(G)$. Let $\sigma(v_{i_k}, v_{i_k}v_1) = t_k$. At last, we give incidences $(v_1, v_1, v_{i_k}) \in I(G)$ $(k = 1, 2, \dots, \delta)$ the color used to color incidence $(u, uv_i) \in I(G'')$ $(i = 2, 3, \dots, n)$. Since $d(v_1) = \delta \le [n/2] - 1$, then $2d(v_1) \le 2[n/2] - 2 \le n - 2$, that is, $d(v_1) \le n - 2 - d(v_1)$, thus we can select $d(v_1)$ colors to incidence color, thus σ is a proper n-incidence coloring of G. The proof is completed. □

For the general case, using the way similar to Theorem 2.11, we can give a stronger result.

Theorem 2.12. *For graph G with order n and maximum degree $\Delta(G) = n - k$,* $\mathrm{inc}(G) = n - k + 1 = \Delta(G) + 1$, *when minimum degree $\delta(G) \le [(n - k + 2)/2] - 1$.*

For a graph G, if there exists two vertices $u, v \in V(G) \backslash v_1$ such that $d(u) = n - 3$, $d(v) \le n - 4$, and $uv \notin E(G)$, we say that G is with the property P.

Theorem 2.13. *For graph G with order n and maximum degree* $\Delta(G) = n-3$, $\mathrm{inc}(G) \le \Delta(G)+2 (n \ge 4)$, *when minimum degree* $\delta(G) \le [n/2]-1$.

Proof. By $V_n = \{v_1, v_2, \ldots, v_n\}$ we denote a labeling of the vertices of G and let $d(v_1) = \delta(G)$. For $n = 4,5$, the desired result follows from Lemma 2.1.

For the case $n \ge 6$, the proof can be divided into two cases.

Case 1. G is with the property P. Consider the auxiliary graph $G' = G + uv$. Since $\Delta(G') = n-2$ and $\delta(G') \le [n/2]-1$, by Theorem 2.11, $\mathrm{inc}(G') = n-1 = \Delta(G')+1$. Thus $\mathrm{inc}(G) \le \mathrm{inc}(G') = \Delta(G)+2$.

Case 2. G not with the property P. For two vertices $u, v \in V(G)\backslash v_1$, let $V_1(G) = \{v \in V(G) \mid d_G(v) = n-3\}$ and $V_2(G) = V(G)\backslash\{V_1(G) \cup \{v_1\}\}$.

Subcase 1. $V_2(G) = \varnothing$. Let $w \notin V(G)$ and $G' = G + w + \{wv \mid v \in V_1(G)\}$, then $\Delta(G') = n-1$ and $\delta(G') \le [n/2]-1$. By Theorem 2.11, using similar methods as in the proof of Theorem 2.11, we can prove the desired result $\mathrm{inc}(G) \le n-1$.

Subcase 2. $V_2(G) \ne \varnothing$. Let x be the arbitrary vertex in $V_1(G)$, then $N(x) = V_2(G)$. For arbitrary vertex $v \in V_2(G)$, since $d(v) \le n-4$, then $|V_2(G)| \ge 3$, and there exists two vertices u_1', v_1' in $V_2(G)$ such that $u_1'v_1' \notin E(G)$. Let $G_1 = G + u_1'v_1'$. If G_1 is with the property P, then $\mathrm{inc}(G) \le \mathrm{inc}(G_1) \le n-1$. Otherwise let $V_1(G_1) = \{v \in V(G_1) \mid d_{G_1}(v) = n-3\}$ and $V_2(G_1) = V(G_1)\backslash\{V_1(G_1) \cup \{v_1\}\}$. If $V_2(G_1) = \varnothing$, then $\mathrm{inc}(G_1) \le \Delta(G_1)+2$. If $V_2(G_1) \ne \varnothing$, then $|V_2(G_1)| \ge 3$; there exists two vertices u_1', v_1' in $V_2(G_1)$ such that $u_2'v_1' \notin E(G_1)$. Let $G_2 = G_1 + u_2'v_2'$. If G_2 is not with the property P, then $|V_2(G_2)| \ge 3$ when $V_2(G_2) \ne \varnothing$. We can also construct graph G_3 that is not with the property P. In that way, we can obtain a serial of graphs $G, G_1, G_2, \ldots, G_k, \ldots$ such that all the graphs are not with the property P and $|V_2(G_k)| \ge 3$. Let $D(G) = \sum_{v \in G} d(v)$, then $D(G) \le D(G_1) \le D(G_2) \le \cdots \le D(G_k) \le \cdots$. Because G is the finite graph, there exists a graph G_{k_0} such that $|V_2(G_{k_0})| = 3$. Suppose that $V_2(G_{k_0}) = \{u_1, u_2, u_3\}$ and $v' \in V_1(G_{k_0})$, then $V_2(G_{k_0}) = N(v')$. Thus $d_{G_{k_0}}(u_1) = d_{G_{k_0}}(u_2) = d_{G_{k_0}}(u_3) = n-4$, and u_1, u_2, u_3 are without edge and adjacent to each other. Let $\widehat{G} = G_{k_0} + u_1u_2$, then $u_1, u_3 \in \widehat{G}\backslash v_1$, $d(u_1) = n-3$, $d(u_3) \le n-4$, and $u_1u_3 \notin E(\widehat{G})$, then $\widehat{G}$ is with the property P, thus $\mathrm{inc}(G) \le \mathrm{inc}(G_1) \le \mathrm{inc}(G_2) \le \cdots \le \mathrm{inc}(G_{k_0}) \le \mathrm{inc}(\widehat{G}) \le n-1$. The proof is complete. □

Theorem 2.14. *Let* $u, v \in V(G)$ *such that* $uv \notin E(G)$ *and* $N_G(u) = N_G(v)$, *then* $\mathrm{inc}(G) \ge \Delta + 2$.

Proof. The proof is by contradiction. Suppose that the graph G has an $(\Delta+1)$-incidence coloring with color set $C = \{1, 2, \ldots, \Delta+1\}$. Let $N_G(u) = \{x_1, x_2, \ldots, x_\Delta\}$ and $N_G(v) = \{y_1, y_2, \ldots, y_\Delta\}$. Then each of the incidences (x_i, x_iu) $(1 \le i \le \Delta)$ is colored the same, as are the incidences (y_i, y_iv). Without loss of generality, suppose k the color that (y_i, y_iv) has. Because $N_G(u) = N_G(v)$ and (u, x_1x_1) is adjacent to (y_1, y_1v), then (u, ux_1) has a color other than k. Because (u, ux_2) is adjacent to $(y_2, y_2v), \ldots, (u, ux_\Delta)$ which is adjacent to $(y_\Delta, y_\Delta v)$, then $(u, ux_2), \ldots, (u, ux_\Delta)$ also has a color other than k, respectively. Further, the Δ incidences (u, ux_i) $(1 \le i \le \Delta)$ have different colors, so the color k is different from that of incidences (u, ux_i). On the other hand, (y_1, y_1v) and (x_1, x_1u) are neighborly incidences, so the color k is different from that of (x_1, x_1u). Thus $k \notin C$, this gives a contradiction! Hence $\mathrm{inc}(G) \ge \Delta + 2$. □

3. The incidence chromatic number of complete k-partite graph

THEOREM 3.1. *Let* $G = K_{n_1,n_2,\dots,n_k}$ *be a complete k-partite graph* $(k \geq 2)$. *Then*

$$\operatorname{inc}(G) = \begin{cases} \Delta+1, & \Delta(G) = n-1, \\ \Delta(G)+2, & \textit{otherwise}. \end{cases} \tag{3.1}$$

Proof. Let $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ and $|V_i| = n_i$ $(i = 1,2,\dots,k)$. V_i is the i-part vertex set and $V_i = \{v_1^i, v_2^i, \dots, v_{n_i}^i\}$ $(i = 1,2,\dots,k)$. Without loss of generality, we let $n_1 \geq n_2 \geq \cdots \geq n_k$. Thus $\Delta(G) = \sum_{m=1}^{k-1} n_m$. The proof can be divided into the following two cases.

Case 3. There exists $i \in \{1,2,\dots,k\}$ such that $n_i = 1$. We let the vertex set of G be $V(G) = \{v_i, v_2, \dots, v_m\}$, where $m = \sum_{i=1}^{k} n_i$. By Lemma 2.6, it easy to draw the conclusion.

Case 4. $n_i \geq 2$ $(1 \leq i \leq k)$. To complete the proof, we give an incidence coloring just with $\Delta + 2$ colors firstly.

For $j,t = 1,2,\dots,k$, $i = 1,2,\dots,n_j$, and $s = 1,2,\dots,n_t$, we let

$$\sigma\left(v_i^j, v_i^j v_s^t\right) = \begin{cases} \sum\limits_{m=0}^{t-1}(n_m+s), & i \neq s, t < j \text{ or } i = s, t > j, \\ \sum\limits_{m=0}^{t-2}(n_m+s), & i \neq s, t > j \text{ or } i = s, t < j, \\ \Delta+1, & i = s, t = 1, \\ \Delta+2, & i = s, t = k. \end{cases} \tag{3.2}$$

To complete the proof, it suffices to prove that G cannot be colored with $\Delta + 1$ colors. It is obvious that each of the vertices in V_1 is the maximum-degree vertex. For $n_1 \geq 2$, let $u, v \in V_1$, then $uv \notin E(G)$ and $N(u) \neq N(v)$. Hence $\operatorname{inc}(G) \geq \Delta + 2$ follows from Theorem 2.14. Therefore $\operatorname{inc}(G) = \Delta + 2$, and the proof is completed. □

By Theorem 3.1, it is easy to obtain the theorem in [3, 4]. In fact, the incidence coloring σ given to determine the incidence chromatic number for complete 3-partite graphs is a special case of the coloring above. Hence, we obtain some corollaries as follows.

COROLLARY 3.2. *Let* K_n *be complete graph. Then* $\operatorname{inc}(K_n) = n$.

The incidence coloring of $K_{3,4}$ and K_5 is given in Figure 3.1.

4. Incidence chromatic number of three families of graphs

The planar graph Q_n, which is called triangular prism, is defined by $Q_n = G(V(G), E(G))$, where the vertex set $V(G) = u_1, u_2, \dots, u_n \cup v_1, v_2, \dots, v_n$, and the edges set $E(Q_n)$ consists

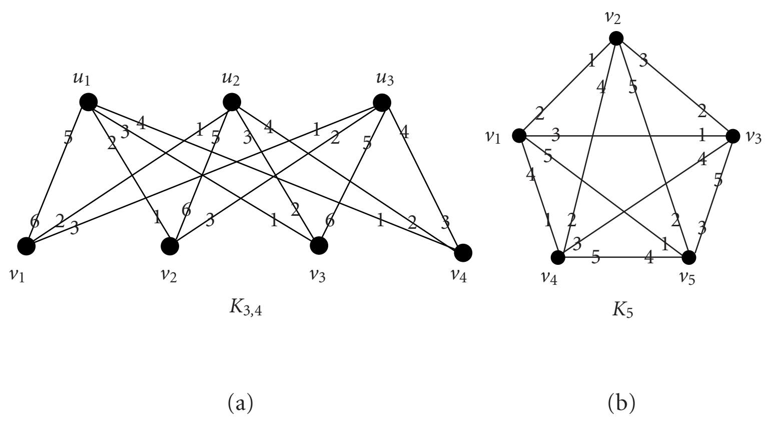

Figure 3.1. An incidence coloring of $K_{3,4}$ and K_5, respectively.

of two n-cycles $u_1,u_2,\dots,u_n$ and $v_1,v_2,\dots,v_n$, and $2n$ edges $(u_i,v_i),(u_i,v_{i+1})$ for all $i\in[n](v_1=v_{n+1})$.

Theorem 4.1. *For any integer $n\geq 3$,*

$$\operatorname{inc}(Q_n)=\begin{cases}\Delta+1=5, & n=0\bmod(5),\\ \Delta+2=6, & \textit{otherwise.}\end{cases}\tag{4.1}$$

Proof. Because $\Delta(G)=4$, we know that $\operatorname{inc}(Q_n)\geq\Delta+1=5$. When $n=5k$, $k\geq 1$, we give a 5-incidence coloring σ of Q_{5k}. For $i=1,2,\dots,5k$, let $(u_i,u_iu_i^*)$ be the incidence set $\{u_i,u_iw\mid w=v_{i+1},u_{i\pm1},v_i\}$. Let

$$\begin{gathered}\sigma(u_i,u_iu_i^*)=\{1+2(i-1)(\operatorname{mod}5),\ 2+2(i-1)(\operatorname{mod}5),\\ 3+2(i-1)(\operatorname{mod}5),\ 4+2(i-1)(\operatorname{mod}5)\},\\ \sigma(v_{i\pm1},v_{i\pm1}v_i)=\sigma(u_i,u_iv_i),\qquad \sigma(w,wu_i)=\sigma(u_{imp1},u_{i\mp1}u_i),\\ \sigma(v_i,v_iu_i)=\sigma(u_{i+1}u_{i+1}v_{i+1}).\end{gathered}\tag{4.2}$$

It is easy to see that the coloring above is a proper 5-incidence coloring of Q_n. Thus, we can only consider the case $n\neq 5k$. We will first prove that Q_n is 6-incidence colorable by explicitly giving a 6-incidence coloring σ of Q_n for any integer $n\geq 3$. At last, we will give the proof that Q_n cannot be incidence coloring just with colors $1,2,3,4,5$. The proof can be divided into the following three cases.

Case 5. $n=3k$ $(k\geq 1)$. Let $i=3s+t$ $(t\leq 2)$, $i=1,2,\dots,n$, then Q_n has an incidence coloring using 6 colors from the color set $C=\{1,2,\dots,n+r+1\}$, as follows: for $i=1,2,\dots,n$, let

$$\begin{aligned}
\sigma(v_i, v_iv_{i+1}) &= \sigma(u_i, u_iu_{i+1}) = \begin{cases} t, & t \neq 0, \\ 3, & t = 0, \end{cases} \\
\sigma(v_i, v_iv_{i-1}) &= \sigma(u_i, u_iu_{i-1}) = t+1, \\
\sigma(u_i, u_iv_{i+1}) &= \sigma(v_i, v_iu_{i-1}) = \sigma(u_i, u_iu_{i+1}) + 3, \\
\sigma(u_i, u_iv_i) &= \sigma(v_{i+1}, v_{i+1}u_{i+1}) = \sigma(u_{i+1}, u_{i+1}u_i) + 3.
\end{aligned} \tag{4.3}$$

Case 6. $n = 3k+1 (k \geq 1)$. Let $i = 3s+t (t \leq 2)$. For $i = 1, 2, \ldots, n$, let

$$\begin{gathered}
\sigma(v_i, v_iv_{i+1}) = \begin{cases} 3, & t = 0, \\ 4, & i = 1, \\ 5, & i = 2, \\ t, & \text{otherwise}, \end{cases} \qquad \sigma(u_i, u_iu_{i+1}) = \begin{cases} 3, & t = 0, \\ 6, & i = 1, \\ t, & \text{otherwise}, \end{cases} \\
\sigma(v_i, v_iv_{i-1}) = \begin{cases} 6, & i = 1, \\ 2, & i = 2, \\ t+1, & \text{otherwise}, \end{cases} \qquad \sigma(u_i, u_iu_{i-1}) = \begin{cases} 5, & i = 1, \\ 4, & i = n, \\ t+1, & \text{otherwise}, \end{cases} \\
\sigma(u_i, u_iv_i) = \sigma(v_{i+1}, v_{i+1}u_{i+1}) = \begin{cases} \sigma(u_{i+1}, u_{i+1}u_i) + 3, & i \neq 1, n, \\ 5, & i = n, \end{cases} \\
\sigma(u_1u_1v_1) = \sigma(v_2v_2v_1) = 2, \qquad \sigma(v_1v_1u_3) = 3.
\end{gathered} \tag{4.4}$$

Case 7. $n = 3k+2$ $(k \geq 1)$. Let $i = 3s+t$ $(t \leq 2)$, for $i = 1, 2, \ldots, n$, and $w_{n+1} = w_1$, $w = u, v$; $w_0 = w_n$, $w = u, v$. We let

$$\begin{gathered}
\sigma(u_i, u_iu_{i+1}) = \begin{cases} 3, & t = 0, \\ 5, & i = n, \\ 6, & i = 1, \\ t, & \text{otherwise}, \end{cases} \\
\sigma(u_i, u_iv_i) = \sigma(v_{i+1}, v_{i+1}u_{i+1}) = \begin{cases} \sigma(u_{i+1}, u_{i+1}u_i) + 3, & i \neq 1, n, \\ 3, & i = 1, \\ 5, & i = n, \end{cases}
\end{gathered}$$

$$\sigma(v_i, v_iv_{i+1}) = \begin{cases} 2, & i = n, \\ 3, & t = 0, \\ 4, & i = 1, \\ t, & \text{otherwise}, \end{cases}$$

$$\sigma(u_i, u_iv_i) = \sigma(u_i, u_iu_{i+1}) + 3 = \begin{cases} 2, & i = 1, \\ 5, & i = n-1, \\ \sigma(u_i, u_iu_{i+1}) + 3, & i \neq 1, n, \end{cases}$$

$$\sigma(u_i, u_iu_{i-1}) = \sigma(v_i, v_iv_{i-1}) = \begin{cases} 1, & i = 1, \\ t+1, & \text{otherwise}, \end{cases}$$

$$\sigma(v_i, v_iu_{i-1}) = \begin{cases} \sigma(u_i, u_iu_{i+1}) + 3, & i \neq 1, n \\ 2, & i = n, \\ 6, & i = 1, \end{cases}$$

$$\sigma(u_i, u_iv_{i+1}) = \begin{cases} \sigma(u_i, u_iu_{i+1}) + 3, & i \neq 1, n, \\ 2, & i = n, \\ 4, & i = 1, \end{cases}$$

$$\sigma(v_i, v_iu_{i-1}) = \begin{cases} \sigma(u_i, u_iu_{i+1}) + 3, & i \neq 1, n, \\ 2, & i = n, \\ 6, & i = 1. \end{cases} \tag{4.5}$$

It is easy to show that Q_n is 6-incidence colorable. To complete the proof, it remains to be shown that there do not exist an incidence coloring using only 5 colors. Assume, on the contrary, that Q_n is 5-incident colorable. For each vertex $v_i \in Q_n$, $d(v_i) = \Delta(Q_n)$. Thus, four incidences (u_i, u_iv_i), $(u_{i-1}, u_{i-1}v_i)$, $(v_{i\pm1}, v_{i\pm1}v_i)$ have the same color, without loss of generality, 1. For $i = 1, 2, \ldots, n$, the case is the same. Because there are 5 colors that can be used in incidence coloring, and the degree of each vertex v_i in cycle $v_1v_2 \cdots v_nv_1$ is 4, thus the two incidences (v_i, v_iv_{i+1}) and $(v_{i+4}, v_{i+4}v_{i+5})$ (or $(v_{i-4}, v_{i-4}v_{i-5})$) have the same color. If $n \neq 5k$, form the proof above, it is easy to obtain a contradiction. Thus, we have completed the prove. □

Theorem 4.2. *Let G be a Hamilton graph with order $n \geq 3$ and degree $\Delta \leq 3$. Then* $\operatorname{inc}(G) \leq \Delta + 2$.

Proof. When $\Delta \leq 2$, by Lemma 2.2, $\operatorname{inc}(G) \leq \Delta + 2$. When $\Delta = 3$, by Lemma 2.3, we can only consider the case $d(v) = 3$ $(\forall v \in V(G))$. Let $\{v_1, v_2, \ldots, v_n, v_1\}$ be the Hamilton cycle and $S = E(G) \setminus \{v_iv_{i+1} \mid 1 \leq i \leq n-1\}$. The proof can be divided into the following three cases.

Case 8. $n = 0 \operatorname{mod}(3)$. For $i = 1, 2, \ldots, n$, we let $\sigma(v_i, v_iv_{i+1}) = 2i - 1 (\operatorname{mod} 3)$ and $\sigma(v_{i+1}, v_{i+1}v_i) = 2i (\operatorname{mod} 3)$, where $v_{n+1} = v_1$. Because the edges $e \in S$ form a matching, thus we can incidence color the incidence uncolored with two new colors $3, 4$. Then, we have given G an incidence coloring with colors $0, 1, \ldots, 4$.

Case 9. $n \neq 0 \bmod (3)$. Let $v_j \in A_{v_1}$ $(j \neq 1, n)$ and $v_k \in A_{v_n}$ $(n \neq 1, n-1)$. For $i = 1, 2, \ldots, n$ and $v_{n+1} = v_1$, we let

$$\sigma(v_i, v_i v_{i+1}) = \begin{cases} 2i - 1 (\bmod 3), & i \neq 1, j, \\ 4, & i = j = k+1, \\ 3, & \text{otherwise}, \end{cases}$$
$$\sigma(v_{i+1}, v_{i+1} v_i) = \begin{cases} 2i (\bmod 3), & i \neq 1,\ j-1, \\ 3, & i = j-1 = k, \\ 4, & \text{otherwise}, \end{cases} \tag{4.6}$$
$$\sigma(v_j, v_j v_1) = \begin{cases} 1, & n = 1 \bmod (3) \text{ and } j = 1 \bmod (3), \\ 0, & n = 2 \bmod (3) \text{ and } j \neq 0 \bmod (3), \\ 2, & \text{otherwise}, \end{cases}$$
$$\sigma(v_1, v_1 v_j) = n - 1 \bmod (3).$$

Since the edges $e \in S \setminus \{v_1 v_k\}$ form a matching, thus we can incidence color the incidence uncolored with two new colors $3, 4$. Thus, we have given G an incidence coloring with colors $0, 1, \ldots, 4$. $\square$

The plane check graph $C_{m,n}$ is defined by $V(C_{m,n}) = \{v_{i,j} \mid i \in [m];\ j \in [n]\}$; $E(C_{m,n}) = \{v_{i,j} v_{i,j+1} \mid i \in [m];\ j \in [n-1]\} \cup \{v_{i,j} v_{i+1,j} \mid i \in [m-1];\ j \in [n]\}$, which is the Cartesian product of path P_m and P_n,

Theorem 4.3. *For plane graph $C_{m,n}$, we have* $\operatorname{inc}(C_{m,n}) = 5$.

Proof. $\Delta(C_{m,n}) = 4$, then $\operatorname{inc}(C_{m,n}) \geq 5$. We now give a 5-incidence coloring σ of $C_{m,n}$ as follows: $(i \in [m];\ j \in [n])$

$$\begin{aligned} \sigma(v_{i,j}, v_{i,j} v_{i,j+1}) &= j + 3(i-1)(\bmod (5))\ (j \neq n), \\ \sigma(v_{i,j+1}, v_{i,j+1} v_{i,j}) &= j + 4(i-1)(\bmod (5))\ (j \neq n), \\ \sigma(v_{i,j}, v_{i,j} v_{i+1,j}) &= j + 2(i-1)(\bmod (5))\ (i \neq m), \\ \sigma(v_{i+1,j}, v_{i+1,j} v_{i,j}) &= j + 4(i-1)(\bmod (5))\ (i \neq m). \end{aligned} \tag{4.7}$$

It is easy to see that the coloring above is an incidence coloring of $C_{m,n}$. Thus $\operatorname{inc}(C_{m,n}) = 5$. $\square$

Remark 4.4. It is difficult to obtain the incidence chromatic number for some graphs. We have presented a hybrid genetic algorithm for the incidence coloring on graphs in [6]. The experimental results indicate that a hybrid genetic algorithm can obtain solutions of excellent quality of problem instances with different size.

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