THE INCIDENCE CHROMATIC NUMBER OF SOME GRAPH

LIU XIKUI AND LI YAN

Received 1 April 2003 and in revised form 5 December 2003

The concept of the incidence chromatic number of a graph was introduced by Brualdi and Massey (1993). They conjectured that every graph $G$ can be incidence colored with $\Delta(G) + 2$ colors. In this paper, we calculate the incidence chromatic numbers of the complete $k$-partite graphs and give the incidence chromatic number of three infinite families of graphs.

1. Introduction

Throughout the paper, all graphs dealt with are finite, simple, undirected, and loopless. Let $G$ be a graph, and let $V(G)$, $E(G)$, $\Delta(G)$, respectively, denote vertex set, edge set, and maximum degree of $G$. In 1993, Brualdi and Massey [3] introduced the concept of incidence coloring. The order of $G$ is the cardinality $|v(G)|$. The size of $G$ is the cardinality $|E(G)|$. Let

$$I(G) = \{(v,e) \mid v \in V, e \in E, v \text{ is incident with } e\} \quad (1.1)$$

be the set of incidences of $G$. We say that two incidences $(v,e)$ and $(w,f)$ are adjacent provided one of the following holds:

(i) $v = w$;
(ii) $e = f$;
(iii) the edge $vw = e$ or $vw = f$.

Figure 1.1 shows three cases of two incidences being adjacent.

An incidence coloring $\sigma$ of $G$ is a mapping from $I(G)$ to a set $C$ such that no two adjacent incidences of $G$ have the same image. If $\sigma : I(G) \rightarrow C$ is an incidence coloring of $G$ and $|C| = k$, $k$ is a positive integer, then we say that $G$ is $k$-incidence colorable.
The minimum cardinality of $C$ for which there exists an incidence coloring $\sigma : I(G) \to C$ is called the incidence chromatic number of $G$, and is denoted by $\text{inc}(G)$. A partition $\{I_1, I_2, \ldots, I_k\}$ of $I(G)$ is called an independence partition of $I(G)$ if each $I_i$ is independent in $I(G)$ (i.e., no two incidences of $I_i$ are adjacent in $I(G)$). Clearly, for $k' \geq \text{inc}(G)$, $G$ is $k'$-incidence colorable.

We may consider $G$ as a digraph by splitting each edge $uv$ into two opposite arcs $(u, v)$ and $(v, u)$. Let $e = uv$. We identify $(u, e)$ with the arc $(u, v)$. So $I(G)$ may be identified with the set of all arcs $A(G)$. Two distinct arcs (incidences) $(u, v)$ and $(x, y)$ are adjacent if one of the following holds (see Figure 1.2):

1. $u = x$; 
2. $u = y$ and $v = x$; 
3. $v = x$.

This concept was first developed by Brualdi and Massey [3] in 1993. They posed the incidence coloring conjecture (ICC), which states that for every graph $G$, $\text{inc}(G) \leq \Delta + 2$. In 1997, Guiduli [5] showed that incidence coloring is a special case of directed star arboricity, introduced by Algor and Alon [1]. They pointed out that the ICC was solved in the negative following an example in [1]. Following the analysis in [1], they showed that $\text{inc}(G) \geq \Delta + \Omega(\log \Delta)$, where $\Omega = 1/8 - o(1)$. Making use of a tight upper bound for directed star arboricity, they obtained the upper bound $\text{inc}(G) \leq \Delta + O(\log \Delta)$.

Brualdi and Massey determined the incidence chromatic numbers of trees, complete graphs, and complete bipartite graphs [3]; Chen, Liu, and Wang determined the incidence chromatic numbers of paths, cycles, fans, wheels, adding-edge wheels, and complete 3-partite graphs [4].

In this paper, we will consider the incidence chromatic number for complete $k$-partite graphs. We will give the incidence chromatic number of complete $k$-partite graphs, and also give the incidence chromatic number of three infinite families of graphs. Let $k$ be a positive integer, put $[k] = 1, 2, \ldots, k$. We state first the following definitions.

Definition 1.1. For a graph $G(V, E)$ with vertex set $V$ and edge set $E$, the incidence graph $I(G)$ of $G$ is defined as the graph with vertex set $V(I(G))$ and edge set $E(I(G))$.

Definitions not given here may be found in [2].
2. Some useful lemmas and properties of incidence chromatic number

**Lemma 2.1.** Let $T$ be a tree of order $n \geq 2$ with maximum degree $\Delta$. Then $\text{inc}(T) = \Delta + 1$.

**Lemma 2.2.** A graph $G$ is $k$-incidence colorable if and only if its incidence graph $I(G)$ is $k$-vertex colorable, that is, $\chi(I(G)) = k$.

Let $M = \{(ue, ve) \mid e = uv \in E(G), (ue, ve) \in E(I(G))\}$, then $M$ forms a perfect matching of incidence graph $I(G)$. The following lemmas are obvious.

**Lemma 2.3.** The incidence graph $I(G)$ of a graph $G$ is a graph with a perfect matching.

**Lemma 2.4.** For a graph $G$, $v \in V(G)$, let $B_v = \{(u, uv) \mid uv \in E(G), u \in V(G)\}$, $A_v = \{(v, vu) \mid uv \in E(G), u \in V(G)\}$, then $\{B_v\}$ is an independence-partition of incidence graph $I(G)$, and the induced subgraph $G[A_v]$ of $I(G)$ is a clique graph.

By the definition of incidence graph, it is easy to give the proof.

**Lemma 2.5.** Let $\Delta$ be the maximum degree of graph $G$, $I(G)$ the incidence graph, then complete graph $K_{\Delta+1}$ is a subgraph of $I(G)$.

**Proof.** Let $d(u) = \Delta$, $p = \Delta$, and $e_1 = uv_1, e_2 = uv_2, \ldots, e_p = uv_1$ be the edges of $G$. $p$ incidences in $I_\Delta = \{(u, e_1), (u, e_2), \ldots, (u, e_p)\}$ are adjacent to each other. For an incidence in $I_v = \{(v, u) \mid uv_i \in E(G), 1 \leq i \leq p\}$, $(v, u)$ is adjacent to all incidences in $I_\Delta$. Since $p + 1$ incidences $(u, e_2), \ldots, (u, e_p)$ are vertices of $I(G)$, by the definition of incidence graph, we can complete the proof. \qed

**Lemma 2.6.** For a simple graph $G$ with order $n$, $\text{inc}(G) = n = \Delta(G) + 1$, when $\Delta(G) = n - 1$.

**Proof.** Let $|V(G)| = \Delta + 1 = v(G)$, by Lemma 2.4, $B_v$ is an independence-partition of incidence graph $I(G)$, then $\chi(I(G)) \leq v(G)$. By Lemma 2.5, $K_{\Delta+1}$ is the subgraph of $I(G)$, thus $\chi(I(G)) \geq v(G)$, then $\text{inc}(G) = \chi(I(G)) = \Delta + 1$, as required. \qed

The following corollaries can be easily verified.

**Corollary 2.7.** Let $G$ be a graph with order $n (n \geq 2)$, then $\text{inc}(G) \leq \Delta + 2$, when $\Delta(G) = n - 2$. 

![Figure 1.2. Cases of two arcs (incidences) are adjacent.](image)
In fact, for graphs $G$ with order $n$, we can give each incidence in $I(G)$ proper incidence coloring as follows. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set and $C = \{1, 2, \ldots, n\}$ the color set. For $i, j = 1, 2, \ldots, n$, we let $\sigma(v_i, v_j) = j$. It is easy to see that the coloring above is an incidence coloring of $G$ only with $n$ colors. That is, $\text{inc}(G) \leq \Delta + 2$, when $\Delta \geq n - 2$.

**Corollary 2.8.** Let $W_n$ be the wheel graph with order $n + 1$. Then $\text{inc}(W_n) = n + 1$.

**Lemma 2.9.** Let $H$ be a subgraph of $G$, then $\text{inc}(H) \leq G$.

**Lemma 2.10.** Let $G$ be union of disjoint graphs $G_1, G_2, \ldots, G_t$. If $G_i$ has an $m$-incidence coloring for all $i = 1, 2, \ldots, t$, then $G$ has an $m$-incidence coloring. That is $\text{inc}(G) = \max\{\text{inc}(G_i) \mid i = 1, 2, \ldots, t\}$.

**Proof.** To prove this lemma, we only need to prove that $G_1 \cup G_2$ has an $m$-incidence coloring. Let $\{I_1, I_2, \ldots, I_m\}$ be an independence partition of $I(G_1)$, and $\{I'_1, I'_2, \ldots, I'_m\}$ an independence partition of $I(G_2)$. Then $\{I_1 \cup I'_1, I_2 \cup I'_2, \ldots, I_m \cup I'_m\}$ forms an independence-partition of $I(G_1) \cup I(G_2)$. Hence $G$ has an $m$-incidence coloring. The proof of the lemma is complete.

**Theorem 2.11.** Let $G$ be a graph with maximum degree $\Delta(G) = n - 2$ and minimum degree $\delta(G) \leq \lceil n/2 \rceil - 1$, then $\text{inc}(G) = n - 1 = \Delta(G) + 1$.

**Proof.** Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, $d(v_1) = \delta(G)$, and $u \notin V(G)$. Consider the auxiliary graph $G'$ with vertex set $V(G') = V(G) \cup \{u\}$ and edge set $E(G') = E(G) \cup \{uv_i \mid i = 2, 3, \ldots, n\}$. It follows that $\Delta(G') = n - 1$. Let $G'' = G' - \{v_1\}$, then $\Delta(G'') = n - 1$, by Lemma 2.5, $\text{inc}(G'') = n$. For color set $C = \{1, 2, \ldots, n\}$, suppose that $\sigma'$ is the $n$-incidence coloring of $G''$ with color set $C$. Without loss of generality, let $\sigma'(v_i, v_iv_j) = j (v_i, v_j \in E(G))$ and $\sigma'(v_i, v_iu) = 1 (i = 2, 3, \ldots, n)$, $\sigma'(u, uv_i) = i (i = 2, 3, \ldots, n)$. In incidence set $I(G)$, incidences $(v_i, v_iv_j) (i, j = 2, 3, \ldots, n, i \neq j)$ are all adjacent to $(v_i, v_iu)$ and $(v_j, v_iu)$, thus the color $n$ cannot be used to color any incidence in $I(G'' - \{u\})$. Denote by $N(v_1) = \{v_1, v_2, \ldots, v_i\}$ the vertices adjacent to $v_1$. The incidence coloring $\sigma'$ of graph $G'$ may be extended to an incidence coloring $\sigma$ of graph $G$. For $x, y \notin \{v_1\} \cup N(v_1)$, let $\sigma(x, xy) = \sigma'(x, xy)$. Because $\Delta(G) = n - 2$, for vertex $v_k (k = 1, 2, \ldots, \delta)$, there exists a vertex $v_k \in V(G)$ such that $v_kv_k \notin E(G)$. Let $\sigma(v_k, v_kv_1) = t_k$. At last, we give incidences $(v_1, v_1v_k) \in I(G) (k = 1, 2, \ldots, \delta)$ the color used to color incidence $(u, uv_1) \in I(G'') (i = 2, 3, \ldots, n)$. Since $d(v_1) = \delta \leq \lceil n/2 \rceil - 1$, then $2d(v_1) \leq 2\lceil n/2 \rceil - 2 \leq n - 2$, that is, $d(v_1) \leq n - 2 - d(v_1)$, thus we can select $d(v_1)$ colors to incidence color, thus $\sigma$ is a proper $n$-incidence coloring of $G$. The proof is completed.

For the general case, using the way similar to Theorem 2.11, we can give a stronger result.

**Theorem 2.12.** For graph $G$ with order $n$ and maximum degree $\Delta(G) = n - k$, $\text{inc}(G) = n - k + 1 = \Delta(G) + 1$, when minimum degree $\delta(G) \leq \lceil (n - k + 2)/2 \rceil - 1$.

For a graph $G$, if there exists two vertices $u, v \in V(G) \setminus v_1$ such that $d(u) = n - 3, d(v) \leq n - 4$, and $uv \notin E(G)$, we say that $G$ is with the property $P$. 
Theorem 2.13. For graph G with order n and maximum degree $\Delta(G) = n - 3$, inc($G$) $\leq \Delta(G) + 2(n \geq 4)$, when minimum degree $\delta(G) \leq \lfloor n/2 \rfloor - 1$.

Proof. By $V_n = \{v_1, v_2, \ldots, v_n\}$ we denote a labeling of the vertices of $G$ and let $d(v_1) = \delta(G)$. For $n = 4, 5$, the desired result follows from Lemma 2.1. For the case $n \geq 6$, the proof can be divided into two cases.

Case 1. $G$ is with the property $P$. Consider the auxiliary graph $G' = G + uv$. Since $\Delta(G') = n - 2$ and $\delta(G') \leq \lfloor n/2 \rfloor - 1$, by Theorem 2.11, inc($G'$) = $n - 1 = \Delta(G') + 1$. Thus inc($G$) $\leq$ inc($G'$) = $\Delta(G) + 2$.

Case 2. $G$ not with the property $P$. For two vertices $u, v \in V(G) \backslash v_1$, let $V_1(G) = \{v \in V(G) \mid d_G(v) = n - 3\}$ and $V_2(G) = (V(G) \backslash \{V_1(G) \cup \{v_1\})$.

Subcase 1. $V_2(G) = \emptyset$. Let $w \notin V(G)$ and $G' = G + w + \{vw \mid v \in V_1(G)\}$, then $\Delta(G') = n - 1$ and $\delta(G') \leq \lfloor n/2 \rfloor - 1$. By Theorem 2.11, using similar methods as in the proof of Theorem 2.11, we can prove the desired result inc($G$) $\leq n - 1$.

Subcase 2. $V_2(G) \neq \emptyset$. Let $x$ be the arbitrary vertex in $V_1(G)$, then $N(x) = V_2(G)$. For arbitrary vertex $v \in V_2(G)$, since $d(v) \leq n - 4$, then $|V_2(G)| \geq 3$, and there exists two vertices $u_1, v_1$ in $V_2(G)$ such that $u_1, v_1 \notin E(G)$. Let $G_1 = G + u_1v_1$. If $G_1$ is with the property $P$, then inc($G$) $\leq$ inc($G_1$) $\leq n - 1$. Otherwise let $V_1(G_1) = \{v \in V(G_1) \mid d_{G_1}(v) = n - 3\}$ and $V_2(G_1) = V(G_1) \backslash \{V_1(G_1) \cup \{v_1\}\}$. If $V_2(G_1) = \emptyset$, then inc($G_1$) $\leq \Delta(G_1) + 2$. If $V_2(G_1) \neq \emptyset$, then $|V_2(G_1)| \geq 3$; there exists two vertices $u_1', v_1'$ in $V_2(G_1)$ such that $u_1', v_1' \notin E(G_1)$.

Let $G_2 = G_1 + u_1'v_1'$. If $G_2$ is not with the property $P$, then $|V_2(G_2)| \geq 3$ when $V_2(G_2) \neq \emptyset$. We can also construct graph $G_3$ that is not with the property $P$. In that way, we can obtain a serial of graphs $G_1, G_2, \ldots, G_k, \ldots$ such that all the graphs are not with the property $P$ and $|V_2(G_k)| \geq 3$. Let $D(G) = \sum_{v \in G} d(v)$, then $D(G) \leq D(G_1) \leq D(G_2) \leq \cdots \leq D(G_k) \leq \cdots$. Because $G$ is the finite graph, there exists a graph $G_k$ such that $|V_2(G_k)| = 3$. Suppose that $V_2(G_k) = \{u_1, u_2, u_3\}$ and $v' \in V_1(G_k)$, then $V_2(G_k) = N(v')$. Thus $d_{G_k}(u_1) = d_{G_k}(u_2) = d_{G_k}(u_3) = n - 4$, and $u_1, u_2, u_3$ are without edge and adjacent to each other. Let $\hat{G} = G_k + u_1u_2$, then $u_1, u_2 \in \hat{G \backslash v_1}$, $d(u_1) = n - 3$, $d(u_3) \leq n - 4$, and $u_1u_3 \notin E(\hat{G})$, then $\hat{G}$ is with the property $P$, thus inc($G$) $\leq$ inc($G_1$) $\leq$ inc($G_2$) $\leq \cdots \leq$ inc($G_k$) $\leq$ inc($\hat{G}$) $\leq n - 1$. The proof is complete.

Theorem 2.14. Let $u, v \in V(G)$ such that $uv \notin E(G)$ and $N_G(u) = N_G(v)$, then inc($G$) $\geq \Delta + 2$.

Proof. The proof is by contradiction. Suppose that the graph $G$ has an $(\Delta + 1)$-incidence coloring with color set $C = \{1, 2, \ldots, \Delta + 1\}$. Let $N_G(u) = \{x_1, x_2, \ldots, x_\Delta\}$ and $N_G(v) = \{y_1, y_2, \ldots, y_\Delta\}$. Then each of the incidences $(x_i, x_iu) (1 \leq i \leq \Delta)$ is colored the same, as are the incidences $(y_i, y_iv)$.

Without loss of generality, suppose $k$ the color that $(y_i, y_iv)$ has. Because $N_G(u) = N_G(v)$ and $(u, x_1x_1)$ is adjacent to $(y_1, y_1v)$, then $(u, ux_1)$ has a color other than $k$. Because $(u, ux_2)$ is adjacent to $(y_2, y_2v), \ldots, (u, ux_\Delta)$ which is adjacent to $(y_\Delta, y_\Deltav)$, then $(u, ux_2), \ldots, (u, ux_\Delta)$ also has a color other than $k$, respectively. Further, the $\Delta$ incidences $(u, ux_i) (1 \leq i \leq \Delta)$ have different colors, so the color $k$ is different from that of incidences $(u, ux_i)$. On the other hand, $(y_1, y_1v)$ and $(x_1, x_1u)$ are neighborly incidences, so the color $k$ is different from that of $(x_1, x_1u)$. Thus $k \notin C$, this gives a contradiction! Hence inc($G$) $\geq \Delta + 2$. □
3. The incidence chromatic number of complete \( k \)-partite graph

**Theorem 3.1.** Let \( G = K_{n_1,n_2,...,n_k} \) be a complete \( k \)-partite graph \((k \geq 2)\). Then

\[
\text{inc}(G) = \begin{cases} 
\Delta + 1, & \Delta(G) = n - 1, \\
\Delta(G) + 2, & \text{otherwise.}
\end{cases}
\] (3.1)

**Proof.** Let \( V(G) = V_1 \cup V_2 \cup \cdots \cup V_k \) and \(|V_i| = n_i\) \((i = 1,2,...,k)\). \( V_i \) is the \( i \)-part vertex set and \( V_i = \{v_i^1, v_i^2, ..., v_i^{n_i} \} \) \((i = 1,2,...,k)\). Without loss of generality, we let \( n_1 \geq n_2 \geq \cdots \geq n_k \). Thus \( \Delta(G) = \sum_{m=1}^{k-1} n_m \). The proof can be divided into the following two cases.

**Case 3.** There exists \( i \in \{1,2,...,k\} \) such that \( n_i = 1 \). We let the vertex set of \( G \) be \( V(G) = \{v_1, v_2, ..., v_m \} \), where \( m = \sum_{i=1}^{k} n_i \). By Lemma 2.6, it easy to draw the conclusion.

**Case 4.** \( n_i \geq 2 \) \((1 \leq i \leq k)\). To complete the proof, we give an incidence coloring just with \( \Delta + 2 \) colors firstly.

For \( j,t = 1,2,...,k \), \( i = 1,2,...,n_j \), and \( s = 1,2,...,n_t \), we let

\[
\sigma(v_i^j, v_i^t, v_s^t) = \begin{cases} 
\sum_{m=0}^{t-1} (n_m + s), & i \neq s, t < j \text{ or } i = s, t > j, \\
\sum_{m=0}^{t-2} (n_m + s), & i \neq s, t > j \text{ or } i = s, t < j, \\
\Delta + 1, & i = s, t = 1, \\
\Delta + 2, & i = s, t = k.
\end{cases}
\] (3.2)

To complete the proof, it suffices to prove that \( G \) cannot be colored with \( \Delta + 1 \) colors. It is obvious that each of the vertices in \( V_1 \) is the maximum-degree vertex. For \( n_1 \geq 2 \), let \( u,v \in V_1 \), then \( uv \notin E(G) \) and \( N(u) \neq N(v) \). Hence \( \text{inc}(G) \geq \Delta + 2 \) follows from Theorem 2.14. Therefore \( \text{inc}(G) = \Delta + 2 \), and the proof is completed.

By Theorem 3.1, it is easy to obtain the theorem in [3, 4]. In fact, the incidence coloring \( \sigma \) given to determine the incidence chromatic number for complete 3-partite graphs is a special case of the coloring above. Hence, we obtain some corollaries as follows.

**Corollary 3.2.** Let \( K_n \) be complete graph. Then \( \text{inc}(K_n) = n \).

The incidence coloring of \( K_{3,4} \) and \( K_5 \) is given in Figure 3.1.

4. Incidence chromatic number of three families of graphs

The planar graph \( Q_n \), which is called triangular prism, is defined by \( Q_n = G(V(G), E(G)) \), where the vertex set \( V(G) = u_1, u_2, ..., u_n \cup v_1, v_2, ..., v_n \), and the edges set \( E(Q_n) \) consists
of two $n$-cycles $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$, and $2n$ edges $(u_i, v_i), (u_i, v_{i+1})$ for all $i \in [n] (v_1 = v_{n+1})$.

**Theorem 4.1.** For any integer $n \geq 3$,

$$ \text{inc}(Q_n) = \begin{cases} \Delta + 1 = 5, & n = 0 \mod(5), \\ \Delta + 2 = 6, & \text{otherwise}. \end{cases} \quad (4.1) $$

**Proof.** Because $\Delta(G) = 4$, we know that $\text{inc}(Q_n) \geq \Delta + 1 = 5$. When $n = 5k$, $k \geq 1$, we give a 5-incidence coloring $\sigma$ of $Q_{5k}$. For $i = 1, 2, \ldots, 5k$, let $(u_i, u_i u_i^*)$ be the incidence set $\{u_i, u_i w | w = v_{i+1}, u_i v_i \}$. Let

$$ \sigma(u_i, u_i u_i^*) = \{1 + 2(i - 1)(\mod 5), 2 + 2(i - 1)(\mod 5), \\ 3 + 2(i - 1)(\mod 5), 4 + 2(i - 1)(\mod 5)\}, \quad (4.2) $$

$$ \sigma(v_{i+1}, v_i v_i) = \sigma(u_i, u_i v_i), \quad \sigma(w, w u_i) = \sigma(u_{im} u_i, u_i v_i), \quad \sigma(v_i, v_i u_i) = \sigma(u_{i+1}, u_{i+1} v_{i+1}). $$

It is easy to see that the coloring above is a proper 5-incidence coloring of $Q_n$. Thus, we can only consider the case $n \neq 5k$. We will first prove that $Q_n$ is 6-incidence colorable by explicitly giving a 6-incidence coloring $\sigma$ of $Q_n$ for any integer $n \geq 3$. At last, we will give the proof that $Q_n$ cannot be incidence coloring just with colors $1, 2, 3, 4, 5$. The proof can be divided into the following three cases.

**Case 5.** $n = 3k$ ($k \geq 1$). Let $i = 3s + t$ ($t \leq 2$), $i = 1, 2, \ldots, n$, then $Q_n$ has an incidence coloring using 6 colors from the color set $C = \{1, 2, \ldots, n + r + 1\}$, as follows: for $i = 1, 2, \ldots, n$,
810  The incidence chromatic number of some graph

let

\[\sigma(v_i, v_{i+1}) = \sigma(u_i, u_{i+1}) = \begin{cases} t, & t \neq 0, \\ 3, & t = 0, \end{cases}\]

\[\sigma(v_i, v_{i-1}) = \sigma(u_i, u_{i-1}) = t + 1,\]

\[\sigma(u_i, u_{i+1}) = \sigma(v_i, v_{i+1}) = \sigma(u_i, u_{i+1}) + 3,\]

\[\sigma(u_i, u_i) = \sigma(v_{i+1}, v_{i+1}u_{i+1}) = \sigma(u_{i+1}, u_{i+1}u_i) + 3.\]  \hfill (4.3)

**Case 6.** \(n = 3k + 1 (k \geq 1).\) Let \(i = 3s + t (t \leq 2).\) For \(i = 1, 2, \ldots, n,\) let

\[\sigma(v_i, v_{i+1}) = \begin{cases} 3, & t = 0, \\ 4, & i = 1, \\ 5, & i = 2, \\ t, & \text{otherwise}, \end{cases}\]

\[\sigma(u_i, u_{i+1}) = \begin{cases} 3, & t = 0, \\ 6, & i = 1, \\ t, & \text{otherwise}, \end{cases}\]

\[\sigma(v_i, v_{i-1}) = \begin{cases} 6, & i = 1, \\ 2, & i = 2, \\ t + 1, & \text{otherwise}, \end{cases}\]

\[\sigma(u_i, u_{i-1}) = \begin{cases} 5, & i = 1, \\ 4, & i = n, \\ t + 1, & \text{otherwise}, \end{cases}\]

\[\sigma(u_i, u_i) = \sigma(v_{i+1}, v_{i+1}u_{i+1}) = \begin{cases} \sigma(u_{i+1}, u_{i+1}u_i) + 3, & i \neq 1, n, \\ 5, & i = n, \end{cases}\]

\[\sigma(u_1u_1v_1) = \sigma(v_2v_2v_1) = 2, \quad \sigma(v_1v_1u_3) = 3.\]  \hfill (4.4)

**Case 7.** \(n = 3k + 2 (k \geq 1).\) Let \(i = 3s + t (t \leq 2),\) for \(i = 1, 2, \ldots, n,\) and \(w_{n+1} = w_1, w = u, v;\)

\(w_0 = w_{n}, w = u, v.\) We let

\[\sigma(u_i, u_{i+1}) = \begin{cases} 3, & t = 0, \\ 5, & i = n, \\ 6, & i = 1, \\ t, & \text{otherwise}, \end{cases}\]

\[\sigma(u_i, u_i) = \sigma(v_{i+1}, v_{i+1}u_{i+1}) = \begin{cases} \sigma(u_{i+1}, u_{i+1}u_i) + 3, & i \neq 1, n, \\ 3, & i = 1, \\ 5, & i = n, \end{cases}\]
It is easy to show that $Q_n$ is 6-incidence colorable. To complete the proof, it remains to be shown that there do not exist an incidence coloring using only 5 colors. Assume, on the contrary, that $Q_n$ is 5-incident colorable. For each vertex $v_i \in Q_n$, $d(v_i) = \Delta(Q_n)$. Thus, four incidences $(u_i, u_i v_i), (u_{i-1}, u_{i-1} v_i), (v_i v_{i+1}, v_i v_{i+1})$ have the same color, without loss of generality, 1. For $i = 1, 2, \ldots, n$, the case is the same. Because there are 5 colors that can be used in incidence coloring, and the degree of each vertex $v_i$ in cycle $v_1 v_2 \cdots v_nv_1$ is 4, thus the two incidences $(v_i v_i v_{i+1})$ and $(v_{i+4}, v_{i+4} v_{i+5})$ (or $(v_{i-4}, v_{i-4} v_{i-5})$) have the same color. If $n \neq 5k$, form the proof above, it is easy to obtain a contradiction. Thus, we have completed the prove. 

**Theorem 4.2.** Let $G$ be a Hamilton graph with order $n \geq 3$ and degree $\Delta \leq 3$. Then $\text{inc}(G) \leq \Delta + 2$.

**Proof.** When $\Delta \leq 2$, by Lemma 2.2, $\text{inc}(G) \leq \Delta + 2$. When $\Delta = 3$, by Lemma 2.3, we can only consider the case $d(v) = 3 \ (\forall v \in V(G))$. Let $\{v_1, v_2, \ldots, v_n, v_1\}$ be the Hamilton cycle and $S = E(G) \setminus \{v_i v_{i+1} | 1 \leq i \leq n-1\}$. The proof can be divided into the following three cases.

**Case 8.** $n = 0 \bmod(3)$. For $i = 1, 2, \ldots, n$, we let $\sigma(v_i v_i v_{i+1}) = 2i - 1 \bmod 3$ and $\sigma(v_{i+1}, v_{i+1} v_i) = 2i \bmod 3$, where $v_{n+1} = v_1$. Because the edges $e \in S$ form a matching, thus we can incidence color the incidence uncolored with two new colors 3, 4. Then, we have given $G$ an incidence coloring with colors 0, 1, 2, 3, 4.
The incidence chromatic number of some graph

Case 9. \( n \neq 0 \mod(3) \). Let \( v_j \in A_{v_1} (j \neq 1, n) \) and \( v_k \in A_{v_n} (n \neq 1, n - 1) \). For \( i = 1, 2, \ldots, n \) and \( v_{n+1} = v_1 \), we let

\[
\sigma(v_i, v_iv_{i+1}) = \begin{cases} 
2i - 1 \mod(3), & i \neq 1, j, \\
4, & i = j = k + 1, \\
3, & \text{otherwise}, 
\end{cases}
\]

\[
\sigma(v_{i+1}, v_{i+1}v_i) = \begin{cases} 
2i \mod(3), & i \neq 1, j - 1, \\
3, & i = j - 1 = k, \\
4, & \text{otherwise},
\end{cases}
\] (4.6)

\[
\sigma(v_j, v_jv_1) = \begin{cases} 
1, & n = 1 \mod(3) \text{ and } j = 1 \mod(3), \\
0, & n = 2 \mod(3) \text{ and } j \neq 0 \mod(3), \\
2, & \text{otherwise},
\end{cases}
\]

\[
\sigma(v_1, v_1v_j) = n - 1 \mod(3).
\]

Since the edges \( e \in S \setminus \{v_1v_k\} \) form a matching, thus we can incidence color the incidence uncolored with two new colors 3,4. Thus, we have given \( G \) an incidence coloring with colors 0,1,\ldots,4. \( \square \)

The plane check graph \( G_{m,n} \) is defined by \( V(G_{m,n}) = \{v_{i,j} | i \in [m]; j \in [n]\}; E(G_{m,n}) = \{v_{i,j}v_{i,j+1} | i \in [m]; j \in [n - 1]\} \cup \{v_{i,j}v_{i+1,j} | i \in [m - 1]; j \in [n]\} \), which is the Cartesian product of path \( P_m \) and \( P_n \).

**Theorem 4.3.** For plane graph \( G_{m,n} \), we have \( \text{inc}(G_{m,n}) = 5 \).

**Proof.** \( \Delta(G_{m,n}) = 4 \), then \( \text{inc}(G_{m,n}) \geq 5 \). We now give a 5-incidence coloring \( \sigma \) of \( G_{m,n} \) as follows: \( (i \in [m]; j \in [n]) \)

\[
\sigma(v_{i,j}, v_{i,j+1}) = j + 3(i - 1)(\mod(5)) \ (j \neq n),
\]

\[
\sigma(v_{i,j+1}, v_{i,j+1}v_{i,j}) = j + 4(i - 1)(\mod(5)) \ (j \neq n),
\]

\[
\sigma(v_{i,j}, v_{i,j}v_{i+1,j}) = j + 2(i - 1)(\mod(5)) \ (i \neq m),
\]

\[
\sigma(v_{i+1,j}, v_{i+1,j}v_{i,j}) = j + 4(i - 1)(\mod(5)) \ (i \neq m).
\] (4.7)

It is easy to see that the coloring above is an incidence coloring of \( G_{m,n} \). Thus \( \text{inc}(G_{m,n}) = 5 \). \( \square \)

**Remark 4.4.** It is difficult to obtain the incidence chromatic number for some graphs. We have presented a hybrid genetic algorithm for the incidence coloring on graphs in [6]. The experimental results indicate that a hybrid genetic algorithm can obtain solutions of excellent quality of problem instances with different size.

**Acknowledgment**

This work is supported by China National Science Foundation.
References


Liu Xikui: College of Information & Engineering, Shandong University of Science and Technology, Qingdao 266510, Shandong, China

Current address: School of Professional Technology, Xuzhou Normal University, Xuzhou 221011, Jiangsu, China

E-mail address: xkliubs@eyou.com

Li Yan: College of Information & Engineering, Shandong University of Science and Technology, Qingdao 266510, Shandong, China

E-mail address: liyan7511@eyou.com
Submit your manuscripts at
http://www.hindawi.com