A NONCOMMUTATIVE GENERALIZATION OF AUSLANDER'S LAST THEOREM

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We show that every finitely generated left R-module in the Auslander class over an n-perfect ring R having a dualizing module and admitting a Matlis dualizing module has a Gorenstein projective cover.

In 1966 [1], Auslander introduced a class of finitely generated modules having a certain complete resolution by projective modules. Then using these modules, he defined the G-dimension (G ostensibly for Gorenstein) of finitely generated modules. It seems appropriate then to call the modules of G-dimension 0 the Gorenstein projective modules. In [4], Gorenstein projective modules (whether finitely generated or not) were defined. In the same paper, the dual notion of a Gorenstein projective module was defined and so a relative theory of Gorenstein modules was initiated (cf. [2, 5] and references therein). In [12], Grothendieck introduced the notion of a dualizing complex. A dualizing module for R is one whose deleted injective resolution is a dualizing complex. Then a local Noetherian ring R is Gorenstein if and only if R is itself a dualizing module for R. In this case, Auslander announced the result that over such a ring, every finitely generated module has a finitely generated Gorenstein projective cover (or equivalently, a minimal maximal Cohen-Macaulay approximation). In [9], this result was generalized to the situation where R is a local Cohen-Macaulay ring having a dualizing module. More recently, in [13], Jørgensen has shown the existence of Gorenstein projective precovers for every module over a commutative Noetherian ring with a dualizing complex. Using Christensen [3], we here introduce the notion of a dualizing bimodule associated with a pair of Noetherian rings (but not necessarily commutative ones). In [6], it was shown that in this situation, every module in the Auslander class defined by the pair of rings admits a Gorenstein projective precover. Now we give examples where the dualizing bimodule has a double structure over the same noncommutative Noetherian ring and that in this case, if the ring also admits a Matlis dualizing module, (cf. [8] or [10]), we particularize the result to the existence of a stronger approximation, that is, every finitely generated module in the Auslander class has a finitely generated Gorenstein projective cover.

Given a class of R-modules $\mathscr{F}$, an $\mathscr{F}$-precover of a left R-module M is a morphism $F \xrightarrow{\varphi} M$ with $F \in \mathscr{F}$ and such that if $F' \xrightarrow{f} M$ is a morphism with $F' \in \mathscr{F}$, then there is

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a morphism $F' \xrightarrow{g} F$ such that $\varphi g = f$. If whenever $F = F'$ and $f = \varphi$, then g is always an automorphism, and we say that $F \xrightarrow{\varphi} M$ is an $\mathcal{F}$-cover. $\mathcal{F}$-preenvelopes and $\mathcal{F}$-envelopes are defined dually.

A left R-module M is said to be Gorenstein projective if there is an exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \tag{1}$$

of projective left R-modules which remains exact whenever $\mathrm{Hom}_R(-,P)$ is applied to it for every projective module P and such that $M = \mathrm{Ker}(P^0 \to P^1)$. Gorenstein injectives are defined dually (cf. [5]).

Definition 1 [6, Definition 2.1]. Let R be a right and left Noetherian ringand let ${}_RV_R$ be an $R-R$-bimodule such that $\mathrm{End}({}_RV) = R$ and $\mathrm{End}(V_R) = R$. Then V is said to be a dualizing module if it satisfies the following three conditions:

(i) $\mathrm{id}({}_RV) \le r$ and $\mathrm{id}(V_R) \le r$ for some integer r,

(ii) $\mathrm{Ext}^i_R({}_RV, {}_RV) = \mathrm{Ext}^i_R(V_R, V_R) = 0$ for all $i \ge 1$,

(iii) ${}_RV$ and V_R are finitely generated.

The preceding definition is given in [6] for a bimodule ${}_SV_R$, where S and R are left and right Noetherian rings, respectively, but through this paper, we will consider the case $S = R$.

Examples. If R is a Cohen-Macaulay local ring of Krull dimension d admitting a dualizing module Ω (see [7]), then Ω is a dualizing module in this sense.

If R is an n-Gorenstein ring (cf. [5, Definition 9.1.9]), then ${}_RR_R$ is a dualizing module.

Let $R = \oplus_{g\in G} R_g$ be a strongly graded ring over a finite group G, right and left Noetherian and let ${}_{R_e}V_{R_e}$ be a dualizing module (for R_e, $e \in G$ is the neutral element in G). Then $W = R \otimes_{R_e} V \otimes_{R_e} R$ is a dualizing module (for R).

Let R be right and left Noetherian and let ${}_RV_R$ be a dualizing module. Then ${}_{R[[x]]}V[[x]]_{R[[x]]}$ is a dualizing module.

In [11], the authors defined Auslander and Bass classes of modules over a Cohen-Macaulay ring admitting a dualizing module. We now use the bimodule V to introduce the corresponding classes in a noncommutative setting.

Definition 2. Let R be right and left Noetherian and let ${}_RV_R$ be a dualizing module. Define the left Auslander class $\mathcal{A}^l(R)$ (relative to V) as those left R-modules M such that $\mathrm{Tor}^R_i(V,M) = 0$ and $\mathrm{Ext}^i_R(V, V \otimes_R M) = 0$ for all $i \ge 1$ and such that the natural morphism $M \to \mathrm{Hom}_R(V, V \otimes_R M)$ is an isomorphism. The right Auslander class $\mathcal{A}^r(R)$ is the class of right R-modules M such that $\mathrm{Tor}^R_i(M,V) = 0$ and $\mathrm{Ext}^i_R(V, M \otimes_R V) = 0$ for all $i \ge 1$ and such that the natural morphism $M \to \mathrm{Hom}_R(V, M \otimes_R V)$ is an isomorphism.

The left Bass class $\mathcal{B}^l(R)$ (relative to V) is defined as those left R-modules N such that $\mathrm{Ext}^i_R(V,N) = 0$ and $\mathrm{Tor}^R_i(V, \mathrm{Hom}_R(V,N)) = 0$ for all $i \ge 1$ and such that the natural morphism $V \otimes_R \mathrm{Hom}_R(V,N) \to N$ is an isomorphism. The right Bass class $\mathcal{B}^r(R)$ is defined as those right R-modules N such that $\mathrm{Ext}^i_R(V,N) = 0$ and $\mathrm{Tor}^R_i(\mathrm{Hom}_R(V,N), V) = 0$ for all $i \ge 1$ and such that the natural morphism $\mathrm{Hom}_R(V,N) \otimes_R V \to N$ is an isomorphism.

We recall the following definition from [8].

Definition 3. A ring R has a Matlis dualizing module if there is an (R,R)-bimodule E such that ${}_RE$ and E_R are both injective cogenerators and such that the canonical maps $R \to \mathrm{Hom}_R({}_RE_R,\ {}_RE_R)$ and $R \to \mathrm{Hom}_R(E_R, E_R)$ are both bijections. E will be called a Matlis dualizing module for R.

Several examples of Matlis dualizing modules are given in [8]. We now give some additional examples.

Examples. If R is left and right Noetherian having a Matlis dualizing module E, then $E[x^{-1}]$ is a Matlis dualizing module for $R[[x]]$.

If R is a strongly graded ring over a finite group, right and left Noetherian, and ${}_{R_e}E_{R_e}$ is a dualizing module (for R_e), then $W = R \otimes_{R_e} E \otimes_{R_e} R$ is a dualizing module (for R).

In what follows, R will always be a right and left Noetherian ring and if E is a Matlis dualizing module for R, we will denote $M^\vee = \mathrm{Hom}_R(M,E)$ for $M \in R$-Mod or $M \in$ Mod-R.

Proposition 4. *Let R be a ring and let V and E be a dualizing module and a Matlis dualizing module for R, respectively. If $M \in R$-Mod is finitely generated, then $M \in \mathcal{A}^l(R)$ if and only if $M^\vee \in \mathcal{B}^r(R)$.*

Proof. Suppose that $M \in \mathcal{A}^l(R)$. Since $\mathrm{Tor}_i^R(V,M) = 0$, then

$$\mathrm{Ext}_R^i\left(V, M^\vee\right) \cong \left(\mathrm{Tor}_i^R(V,M)\right)^\vee = 0 \quad \forall i \geq 1. \tag{2}$$

On the other hand, $(\mathrm{Tor}_i^R(\mathrm{Hom}_R(V,M^\vee),V))^\vee \cong \mathrm{Ext}_R^i(V,(\mathrm{Hom}_R(V,M^\vee))^\vee)$. But $(\mathrm{Hom}_R(V,M^\vee))^\vee = \mathrm{Hom}_R(\mathrm{Hom}_R(V,\mathrm{Hom}_R(M,E)),E) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(V \otimes_R M,E),E) \cong (V \otimes_R M)^{vv}$ and since $V \otimes_R M$ is finitely generated, $(V \otimes_R M)^{\vee\vee} \cong V \otimes_R M$ (cf. [8]), and so we get that

$$\mathrm{Ext}_R^i\left(V, \left(\mathrm{Hom}_R\left(V, M^\vee\right)\right)^\vee\right) \cong \mathrm{Ext}_R^i\left(V, V \otimes_R M\right) = 0 \quad \forall i \geq 1. \tag{3}$$

Therefore, $\mathrm{Tor}_i(\mathrm{Hom}_R(V,M^\vee),V) = 0$ for all $i \geq 1$.

Finally, by hypothesis $M \cong \mathrm{Hom}_R(V, V \otimes_R M)$ and so $\mathrm{Hom}_R(V, V \otimes_R M)^\vee \cong M^\vee$ is an isomorphism. We also know that $\mathrm{Hom}_R(V,M^\vee) \otimes_R V \cong (V \otimes_R M)^\vee \otimes_R V$. Therefore, we only have to show that $(V \otimes_R M)^\vee \otimes_R V \to \mathrm{Hom}_R(V, V \otimes_R M)^\vee$ is an isomorphism to get that $\mathrm{Hom}_R(V,M^\vee) \otimes_R V \cong M^\vee$.

The functors $(V \otimes_R M)^\vee \otimes_R -$ and $\mathrm{Hom}_R(-, V \otimes_R M)^\vee$ are both right exact and the natural morphism

$$\left(V \otimes_R M\right)^\vee \otimes_R R^n \longrightarrow \mathrm{Hom}_R\left(R^n, V \otimes_R M\right)^\vee \tag{4}$$

is an isomorphism, and so the morphism is also an isomorphism for finitely generated modules, in particular for V.

Conversely, let now $N = M^\vee$ and suppose that $N \in \mathcal{B}^r(R)$. Since M is finitely generated, we get that $N^\vee \cong M$. Now $\mathrm{Tor}_i^R(V,M)^\vee \cong \mathrm{Ext}_R^i(V,M^\vee) = \mathrm{Ext}_R^i(V,N) = 0$ for all $i \geq 1$ and so $\mathrm{Tor}_i^R(V,M) = 0$ for all $i \geq 1$.

Moreover, $\mathrm{Tor}_i^R(\mathrm{Hom}_R(V,N),V) = 0$ for all $i \geq 1$ and so

$$0 = \mathrm{Tor}_i^R\left(\mathrm{Hom}_R(V,N),V\right)^\vee \cong \mathrm{Ext}_R^i\left(V, \mathrm{Hom}_R(V,N)^\vee\right). \tag{5}$$

But $\mathrm{Hom}_R(V,N)^\vee = \mathrm{Hom}_R(V,M^\vee)^\vee \cong (V\otimes_R M)^{\vee\vee} \cong V\otimes_R M$, and therefore $\mathrm{Ext}^i_R(V, V\otimes_R M) = 0$ for all $i \geq 1$. It only remains to show that $M \to \mathrm{Hom}_R(V, V\otimes_R M)$ is an isomorphism.

Since $N^\vee \in \mathcal{B}^r(R)$, then $\mathrm{Hom}_R(V,N)\otimes_R V \to N$ is an isomorphism, and therefore

$$N^\vee \cong \left(\mathrm{Hom}_R(V,N)\otimes_R V\right)^\vee \cong \mathrm{Hom}_R\left(V, \mathrm{Hom}_R(V,N)^\vee\right). \tag{6}$$

Then consider the natural transformation

$$-\otimes_R \mathrm{Hom}_R(N,E) \longrightarrow \mathrm{Hom}_R\left(\mathrm{Hom}_R(-,N),E\right). \tag{7}$$

This gives an isomorphism for R^n and since both functors are right exact, it follows that $V\otimes_R \mathrm{Hom}_R(N,E) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(V,N),E)$ and so

$$M = N^\vee \longrightarrow \mathrm{Hom}_R\left(V, \mathrm{Hom}_R(V,N)^\vee\right) \cong \mathrm{Hom}_R\left(V, V\otimes_R M\right) \tag{8}$$

is an isomorphism. □

We now recall from [6] that a ring R is said to be *left (right) n-perfect* if every left (right) flat R-module has projective dimension less than or equal to n.

Left perfect rings, commutative Noetherian rings of finite Krull dimension, the universal enveloping algebra $\mathcal{U}(g)$ of a Lie algebra of dimension n, and n-Gorenstein rings are all examples of left n-perfect rings. Also, if R is left n-perfect, then $R[x]$, $R[[x]]$, the crossed product $R*\mathcal{U}(g)$, and the Weyl algebra $A_k(R)$ are left k-perfect for some k (cf. [6]).

Proposition 5. *Let R be a right and left n-perfect ring, let V and E be a dualizing module of finite left and right injective dimension r and a Matlis dualizing module for R, respectively, and let $G \in R$-Mod be finitely generated. Then G is Gorenstein projective if and only if $G^\vee$ is Gorenstein injective.*

Proof. If G is Gorenstein projective, by [5, Proposition 10.2.6], there exists an exact sequence

$$0 \longrightarrow G \longrightarrow P_{r+n} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{9}$$

where every P_i is a finitely generated projective. Now since G and P_i, $i = 0,\ldots,r+n$, are in $\mathcal{A}^l(R)$ by [6, Proposition 2.3], then $M \in \mathcal{A}^l(R)$ by [6, Proposition 2.7]. Now, $P_i^\vee$ is injective for every $i = 0,\ldots,r+n$ and by the preceding proposition $M^\vee \in \mathcal{B}^r(R)$, so by [6, Theorem 2.11], $G^\vee$ is Gorenstein injective.

Conversely, let $G^\vee$ be Gorenstein injective. Since G is finitely generated, there exists a flat preenvelope $G \to F$ which factors via a finitely generated free module R^k, so we can assume that F is finitely generated free. But then, since $R^\vee = E$, we get that $E^n \to G^\vee$ is an injective precover, and so the injective cover of $G^\vee$ is Artinian. Then there is an exact sequence in Mod-R,

$$0 \longrightarrow N \longrightarrow E_{r-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow G^\vee \longrightarrow 0, \tag{10}$$

where every E_i is Artinian and injective. Then $N \in \mathscr{B}^r(R)$ since $G^\vee$ and so are E_i, $i = 0,\dots,r-1$. In this way, we see that

$$0 \longrightarrow G^{\vee\vee} = G \longrightarrow E_0^\vee \longrightarrow \cdots \longrightarrow E_{r-1}^\vee \longrightarrow N^\vee \longrightarrow 0 \tag{11}$$

is exact with $N^\vee \in \mathscr{A}^l(R)$ and $E_i^\vee$ is projective for every $i = 0,\dots,r-1$ and therefore by [6, Theorem 2.14], G is Gorenstein projective. □

The following result appears in [6] but we include a proof here for completeness.

Theorem 6. *Let R be a left n-perfect ring and ${}_RV_R$ a dualizing module for R such that* $\mathrm{id}({}_RV),\mathrm{id}(V_R) \le r$. *If $M \in \mathscr{A}^l(R)$, then it has a Gorenstein projective precover $G \xrightarrow{\varphi} M \to 0$ such that* $\mathrm{pd}(\mathrm{Ker}(\varphi)) \le r-1$.

Proof. Let $0 \to C \to F_{r-1} \to \cdots \to F_0 \to M \to 0$ be a (partial) projective resolution of M. Then, by [6, Lemma 2.12], C is Gorenstein projective. Now let

$$\cdots P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \tag{12}$$

be an exact sequence of projective modules such that $C = \mathrm{Ker}(P^0 \to P^1)$ and it remains exact whenever $\mathrm{Hom}_R(-,P)$ is applied for every projective P. We consider

$$0 \longrightarrow P^0 \longrightarrow \cdots \longrightarrow P^{r-1} \longrightarrow D \longrightarrow 0 \tag{13}$$

exact. Then we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & P^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & P^{r-1} \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] & F_{r-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd} \tag{14}$$

The associated complex to this diagram (i.e., the mapping complex)

$$0 \longrightarrow C \longrightarrow C \oplus P^0 \longrightarrow \cdots \longrightarrow F_0 \oplus D \longrightarrow M \longrightarrow 0 \tag{15}$$

is exact and has as a subcomplex the exact sequence $0 \to C \to C \to 0$. Then quotient complex

$$0 \longrightarrow P^0 \longrightarrow \cdots \longrightarrow F_0 \oplus D \longrightarrow M \longrightarrow 0 \tag{16}$$

is exact and all of its terms are projective except perhaps $F_0 \oplus D$. Now if $0 \to L \to F_0 \oplus D \to M \to 0$ is exact with $\mathrm{pd}(L) < \infty$, then $\mathrm{pd}(L) \le r-1$. Since $F_0 \oplus D$ is Gorenstein projective and $\mathrm{Ext}_R^1(X,L) = 0$ for every Gorenstein projective X, it follows that $F_0 \oplus D \to M$ is the desired precover. □

Given a class $\mathscr{C}$ of R-modules, we let ${}^\perp C$ be the class of R-modules F such that $\mathrm{Ext}_R^1(F, C) = 0$ for every $C \in \mathscr{C}$. We let $C^\perp$ be the class of R-modules F such that $\mathrm{Ext}_R^1(C,F) = 0$ for every $C \in \mathscr{C}$. A pair of classes of R-modules $(\mathscr{F},\mathscr{C})$ is called a *cotorsion theory* if $\mathscr{F}^\perp = \mathscr{C}$ and ${}^\perp\mathscr{C} = \mathscr{F}$. A cotorsion theory is said to be *complete* if for every R-module M, there is an exact sequence $0 \to M \to C \to F \to 0$ such that $C \in \mathscr{C}$ and $F \in \mathscr{F}$, or equivalently if there is an exact sequence $0 \to C \to F \to M \to 0$ such that $C \in \mathscr{C}$ and $F \in \mathscr{F}$, which is equivalent to say that every R-module has a special $\mathscr{F}$-precover and a special $\mathscr{C}$-preenvelope (cf. [5]). A cotorsion theory is said to be *perfect* if every R-module has an $\mathscr{F}$-cover and a $\mathscr{C}$-envelope.

Now since R is left Noetherian, then $\mathrm{Hom}(-,-)$ is left balanced by $\mathrm{Inj} \times \mathrm{Inj}$ on R-Mod $\times$ R-Mod, and therefore we can compute left derived functors of $\mathrm{Hom}_R(-,-)$ using left injective resolutions in the second variable constructed with injective covers or right injective resolutions in the first one (cf. [5, Example 8.3.5]). We will denote them by $\mathrm{Ext}_i(-,-)$ $i \geq 0$ and $\overline{\mathrm{Ext}}^0(M,N)$, and $\overline{\mathrm{Ext}}_0(M,N)$ will denote the cokernel and the kernel of the natural morphism

$$\mathrm{Ext}_0^R(M,N) \longrightarrow \mathrm{Hom}_R(M,N). \tag{17}$$

Theorem 7. *Let R be a right n-perfect ring and let ${}_RV_R$ be a dualizing module for R such that* $\mathrm{id}({}_RV), \mathrm{id}(V_R) \leq r$. *If $\mathscr{L}$ and* GorInj *denote the classes of right R-modules of finite injective dimension and Gorenstein injective, then* $(\mathscr{L}, \mathrm{GorInj})$ *is a perfect cotorsion theory of* $\mathscr{B}^r(R)$.

Proof. Suppose that $L \in \mathscr{B}^r(R) \cap {}^{\perp}\mathrm{GorInj}$. Then $\mathrm{Ext}_R^1(L,G) = 0$ for every G Gorenstein injective. Now if G is Gorenstein injective, then there exists an exact sequence $0 \to G' \to E_0 \to G \to 0$ with E_0 injective and G' Gorenstein injective. By [5, Theorem 8.2.7], $\overline{\mathrm{Ext}}_R^0(L, G) \cong \mathrm{Ext}_R^1(L,G') = 0$ for every Gorenstein injective G. Analogously, $\mathrm{Ext}_1^R(L,G) \cong \overline{\mathrm{Ext}}_0^R(L, G') = 0$ and by induction, $\mathrm{Ext}_i^R(L,G) = 0$ for all $i \geq 1$ and for every Gorenstein injective G.

Now let $0 \to L \to E^0 \to \cdots \to E^{r+n} \to C \to 0$ be a (partial) injective resolution of L. By [6, Lemma 2.9], C is Gorenstein injective and so $\mathrm{Ext}_{n+r}^R(L,G) = 0$ by the above. Therefore

$$\mathrm{Hom}_R\left(E^{r+n+1}, C\right) \longrightarrow \mathrm{Hom}_R\left(E^{r+n}, C\right) \longrightarrow \mathrm{Hom}_R\left(E^{r+n-1}, C\right) \tag{18}$$

is exact and so C is a direct summand of E^{r+n} which shows that $\mathrm{id}(L) < \infty$. If $L \in \mathscr{L}$, then it is immediate that $\mathrm{Ext}_R^1(L,G) = 0$ for every Gorenstein injective G.

Suppose now that $G \in \mathscr{B}^r(R) \cap L^{\perp}$. Then by [6, Theorem 2.11], G is Gorenstein injective. If G is Gorenstein injective, then it is immediate that $G \in \mathscr{L}^{\perp}$.

Therefore $(\mathscr{L}, \mathrm{GorInj})$ is a cotorsion theory. By [6, Theorem 2.16], it is complete. Finally, since R is right Noetherian, $\mathscr{L}$ is closed under direct limits and so by [5, Theorem 7.2.6], $(\mathscr{L}, \mathrm{GorInj})$ is perfect. □

Theorem 8. *Let R be a left and right n-perfect ring admitting a Matlis dualizing module and let ${}_RV_R$ be a dualizing module for R such that* $\mathrm{id}({}_RV), \mathrm{id}(V_R) \leq r$. *If $M \in \mathscr{A}^l(R)$ is finitely generated, then M has a Gorenstein projective cover $G \xrightarrow{\varphi} M$ such that G is finitely generated and* $\mathrm{pd}(\mathrm{Ker}(\varphi)) \leq r - 1$.

Proof. By Theorem 6, there is an exact sequence $0 \to L \to G \to M \to 0$ with G Gorenstein projective and $\mathrm{pd}(L) \leq r - 1$, which can be supposed finitely generated. Then if $0 \to M^{\vee} \to G^{\vee} \to L^{\vee} \to 0$ is exact with $G^{\vee}$ Gorenstein injective by Proposition 5 and $\mathrm{id}(L^{\vee}) < \infty$, therefore

$$\mathrm{Hom}_R\left(G^{\vee}, N\right) \longrightarrow \mathrm{Hom}_R\left(M^{\vee}, N\right) \longrightarrow \mathrm{Ext}_R^1\left(L^{\vee}, N\right) = 0 \tag{19}$$

is exact for every Gorenstein injective N, which gives that $M^\vee \to G^\vee$ is a Gorenstein injective preenvelope. By the preceding theorem, $M^\vee$ has a Gorenstein injective envelope $\varphi : M^\vee \to C$, which is a summand of $G^\vee$ and so C is artinian. But $\mathrm{Coker}(\varphi)$ is also a direct summand of $L^\vee$ and so $\mathrm{id}(\mathrm{Coker}(\varphi)) \le r-1$. Then $\mathrm{pd}((\mathrm{Coker}(\varphi))^\vee) < \infty$ and therefore we have an exact sequence

$$0 \longrightarrow \big(\mathrm{Coker}(\varphi)\big)^\vee \longrightarrow C^\vee \xrightarrow{\varphi^\vee} M \longrightarrow 0, \tag{20}$$

where $C^\vee$ is Gorenstein projective by Proposition 5 and $\mathrm{pd}((\mathrm{Coker}(\varphi))^\vee) \le r-1$. Since $\mathrm{pd}((\mathrm{Coker}(\varphi))^\vee) < \infty$, it follows that $C^\vee \xrightarrow{\varphi^\vee} M$ is a Gorenstein projective precover. Finally, $C^\vee \xrightarrow{\varphi^\vee} M$ is the desired cover since $C^\vee$ and M are reflexive. □

Corollary 9. *Let R and M be as in the previous theorem and let $G \to M$ be a Gorenstein projective cover. Then* $\mathrm{pd}(M) < \infty$ *if and only if $G \to M$ is a projective cover.*

Proof. Suppose that $\mathrm{pd}(M) < \infty$. Then $\mathrm{pd}(C) < \infty$ and let

$$0 \longrightarrow P_k \longrightarrow \cdots \longrightarrow P_0 \longrightarrow G \longrightarrow 0 \tag{21}$$

be a projective resolution of G. Since $\mathrm{Ext}^1_R(C,P) = 0$ for every projective P, it follows that the sequence splits and so G is projective. The converse is immediate. □

Corollary 10. *Let R and V be as in the previous theorem and let $M \in \mathcal{A}^l(R)$ be finitely generated. Then the minimal Gorenstein projective resolution of M is of the form*

$$0 \longrightarrow P_k \longrightarrow \cdots \longrightarrow P_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0, \tag{22}$$

where P_i is projective for every $i = 1,\dots,k$ and $k \le r$.

Proof. By [6, Corollary 2.13], $M \in \mathcal{A}^r(R)$ if and only if there is an exact sequence

$$0 \longrightarrow G_k \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0, \tag{23}$$

where every G_i, $i = 0,\dots,k$, is Gorenstein projective and $k \le r$. Now the result follows from the preceding corollary. □

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