ON π - s -IMAGES OF METRIC SPACES

ZHAOWEN LI

Received 6 June 2004 and in revised form 24 November 2004

We establish the characterizations of metric spaces under compact-covering (resp., pseudo-sequence-covering, sequence-covering) π - s -maps by means of cfp-covers (resp., sfp-covers, cs-covers) and σ -strong networks.

1. Introduction and definitions

In 1966, Michael [11] introduced the concept of compact-covering maps. Since many important kinds of maps are compact-covering, such as closed maps on paracompact spaces, much work has been done to seek the characterizations of metric spaces under various compact-covering maps, for example, compact-covering (open) s -maps, pseudo-sequence-covering (quotient) s -maps, sequence-covering (quotient) s -maps, and compact-covering (quotient) s -maps, see [3, 9, 12, 15, 16]. π -map is another important map which was introduced by Ponomarev [13] in 1960 and correspondingly, many spaces, including developable spaces, weak Cauchy spaces, g -developable spaces, and semimetrizable spaces, were characterized as the images of metric spaces under certain quotient π -maps, see [1, 4, 6, 7].

The purpose of this paper is to establish the characterizations of metric spaces under compact-covering (resp., pseudo-sequence-covering, sequence-covering) π - s -maps by means of cfp-covers (resp., sfp-covers, cs-covers) and σ -strong networks.

In this paper, all spaces are Hausdorff, and all maps are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers. ω denotes $\mathbb{N} \cup \{0\}$. $\tau(X)$ denotes a topology on X . For a collection $\mathcal{P}$ of subsets of a space X and a map $f : X \rightarrow Y$, denote $\{f(P) : P \in \mathcal{P}\}$ by $f(\mathcal{P})$. For the usual product space $\prod_{i \in \mathbb{N}} X_i$, π_i denotes the projective $\prod_{i \in \mathbb{N}} X_i$ onto X_i . For a sequence $\{x_n\}$ in X , denote $\langle x_n \rangle = \{x_n : n \in \mathbb{N}\}$.

Definition 1.1. Let $f : X \rightarrow Y$ be a map.

- (1) f is called a compact-covering map [11] if each compact subset of Y is the image of some compact subset of X .
- (2) f is called a sequence-covering map [14] if whenever $\{y_n\}$ is a convergent sequence in Y , then there exists a convergent sequence $\{x_n\}$ in X such that each $x_n \in f^{-1}(y_n)$.

- (3) f is called a pseudo-sequence-covering map [3] if each convergent sequence (including its limit point) of Y is the image of some compact subset of X .
- (4) f is called an s -map, if $f^{-1}(y)$ is separable in X for any $y \in Y$.
- (5) f is called a π -map [13], if (X, d) is a metric space, and for each $y \in Y$ and its open neighborhood V in Y , $d(f^{-1}(y), M \setminus f^{-1}(V)) > 0$.
- (6) f is called a π - s -map, if f is both π -map and s -map.

It is easy to check that compact maps on metric spaces are π - s -maps.

Definition 1.2. Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$.

- (1) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is called a σ -strong network [5] for X if for each $x \in X$, $\langle \text{st}(x, \mathcal{P}_n) \rangle$ is a local network of x in X . If every $\mathcal{P}_n$ satisfies property P , then $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is called a σ -strong network consisting of P -covers.
- (2) $\{\mathcal{P}_n\}$ is called a weak development for X if for each $x \in X$, $\langle \text{st}(x, \mathcal{P}_n) \rangle$ is a weak neighborhood base of x in X .

Definition 1.3 [2]. Let X be a space.

- (1) Let $\{x_n\}$ be a convergent sequence in X , and $P \subset X$. $\{x_n\}$ is eventually in P if whenever $\{x_n\}$ converges to x , then $\{x\} \cup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$.
- (2) Let $x \in P \subset X$. P is called a sequential neighborhood of x in X if whenever a sequence $\{x_n\}$ in X converges to x , then $\{x_n\}$ is eventually in P .
- (3) Let $P \subset X$. P is called a sequentially open subset in X if P is a sequential neighborhood of x in X for any $x \in P$.
- (4) X is called a sequential space if each sequentially open subset in X is open.

Definition 1.4 [10]. Let $\mathcal{P}$ be a collection of subsets of a space X .

- (1) $\mathcal{P}$ is called a cfp-cover (i.e., compact-finite-partition cover) of compact subset K in X if there are a finite collection $\{K_\alpha : \alpha \in J\}$ of closed subsets of K and $\{P_\alpha : \alpha \in J\} \subset \mathcal{P}$ such that $K = \bigcup\{K_\alpha : \alpha \in J\}$ and each $K_\alpha \subset P_\alpha$.
- (2) $\mathcal{P}$ is called a cfp-cover for X if for any compact subset K of X , there exists a finite subcollection $\mathcal{P}^* \subset \mathcal{P}$ such that $\mathcal{P}^*$ is a cfp-cover of K in X .
- (3) $\mathcal{P}$ is called an sfp-cover (i.e., sequence-finite-partition cover) for X if for any convergent sequence (including its limit point) K in X , there exists a finite subcollection $\mathcal{P}^* \subset \mathcal{P}$ such that $\mathcal{P}^*$ is a cfp-cover of K in X .
- (4) $\mathcal{P}$ is called a cs-cover for X , if every convergent sequence in X is eventually in some element of $\mathcal{P}$.

2. Results

THEOREM 2.1. A space X is the compact-covering π - s -image of a metric spaces if and only if X has a σ -strong network consisting of point-countable cfp-covers.

Proof. To prove the only if part, suppose $f : (M, d) \rightarrow X$ is a compact-covering π - s -map, where (M, d) is a metric space. For each $n \in \mathbb{N}$, put $\mathcal{F}_n = \{f(B(z, 1/n)) : z \in M\}$, where $B(z, 1/n) = \{y \in M : d(z, y) < 1/n\}$. Obviously, $\mathcal{F}_{n+1}$ refines $\mathcal{F}_n$. We claim that $\bigcup\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a σ -strong network for X . In fact, for each $x \in X$, and its open neighborhood U , since f is a π -map, then there exists $n \in \mathbb{N}$ such that $d(f^{-1}(x), M \setminus f^{-1}(U)) > 1/n$.

We can pick $m \in \mathbb{N}$ such that $m \geq 2n$. If $z \in M$ with $x \in f(B(z, 1/m))$, then

$$f^{-1}(x) \cap B(z, 1/m) \neq \emptyset. \quad (2.1)$$

If $B(z, 1/m) \not\subset f^{-1}(U)$, then

$$d(f^{-1}(x), M \setminus f^{-1}(U)) \leq \frac{2}{m} \leq \frac{1}{n}, \quad (2.2)$$

which is a contradiction. Thus $B(z, 1/m) \subset f^{-1}(U)$, so $f(B(z, 1/m)) \subset U$. Hence $\text{st}(x, \mathcal{F}_m) \subset U$. Therefore $\bigcup \{\mathcal{F}_n : n \in \mathbb{N}\}$ is a σ -strong network for X .

For each $n \in \mathbb{N}$, let $\mathcal{B}_n$ be a locally finite open refinement of $\{B(z, 1/n) : z \in M\}$. Since locally finite collections are closed under finite intersections, we can assume that $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$ for each $n \in \mathbb{N}$. Put $\mathcal{P}_n = f(\mathcal{B}_n)$. Obviously, $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$. Since f is an s -map, each $\mathcal{P}_n$ is point-countable in X . Because $\mathcal{P}_n$ refines $\mathcal{F}_n$ for each $n \in \mathbb{N}$, then $\bigcup \{\mathcal{P}_n : n \in \mathbb{N}\}$ is also a σ -strong network for X .

We now show that each $\mathcal{P}_n$ is a cfp-cover for X . Suppose K is compact in X , since f is compact-covering, then $f(L) = K$ for some compact subset L of M . Since $\mathcal{B}_n$ is an open cover of L in M , $\mathcal{B}_n$ have a finite subcover $\mathcal{B}_n^L$. Thus $\mathcal{B}_n^L$ can be precisely refined by some finite cover of L consisting of closed subsets of L , denoted by $\{L_\alpha : \alpha \in J_n\}$. Put $\mathcal{P}_n^K = f(\mathcal{B}_n^L)$, since $\mathcal{P}_n^K$ is precisely refined by closed cover $\{f(L_\alpha) : \alpha \in J_n\}$ of K , then $\mathcal{P}_n^K$ is a cfp-cover of K in X . Hence each $\mathcal{P}_n$ is a cfp-cover for X .

To prove the if part, suppose $\bigcup \{\mathcal{P}_i : i \in \mathbb{N}\}$ is a σ -strong network for X consisting of point-countable cfp-covers. For each $i \in \mathbb{N}$, $\mathcal{P}_i$ is a point-countable cfp-cover for X . Let $\mathcal{P}_i = \{P_\alpha : \alpha \in \Lambda_i\}$, endow Λ_i with the discrete topology, then Λ_i is a metric space. Put

$$M = \left\{ \alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i : \langle P_{\alpha_i} \rangle \text{ forms a local network at some point } x_\alpha \text{ in } X \right\}, \quad (2.3)$$

and endow M with the subspace topology induced from the usual product topology of the collection $\{\Lambda_i : i \in \mathbb{N}\}$ of metric spaces, then M is a metric space. Since X is Hausdorff, x_α is unique in X . For each $\alpha \in M$, we define $f : M \rightarrow X$ by $f(\alpha) = x_\alpha$. For each $x \in X$ and $i \in \mathbb{N}$, there exists $\alpha_i \in \Lambda_i$ such that $x \in P_{\alpha_i}$. Since $\bigcup \{\mathcal{P}_i : i \in \mathbb{N}\}$ is a σ -strong network for X , then $\{P_{\alpha_i} : i \in \mathbb{N}\}$ is a local network of x in X . Put $\alpha = (\alpha_i)$, then $\alpha \in M$ and $f(\alpha) = x$. Thus f is surjective. Suppose $\alpha = (\alpha_i) \in M$ and $f(\alpha) = x \in U \in \tau(X)$, then there exists $n \in \mathbb{N}$ such that $P_{\alpha_n} \subset U$. Put

$$V = \{\beta \in M : \text{the } n\text{th coordinate of } \beta \text{ is } \alpha_n\}, \quad (2.4)$$

then V is an open neighborhood of α in M , and $f(V) \subset P_{\alpha_n} \subset U$. Hence f is continuous. For each $\alpha, \beta \in M$, we define

$$d(\alpha, \beta) = \begin{cases} 0, & \alpha = \beta, \\ \max \{1/k : \pi_k(\alpha) \neq \pi_k(\beta)\}, & \alpha \neq \beta, \end{cases} \quad (2.5)$$

then d is a distance on M . Because the topology of M is the subspace topology induced from the usual product topology of the collection $\{\Lambda_i : i \in \mathbb{N}\}$ of discrete spaces, thus d

is a metric on M . For each $x \in U \in \tau(X)$, there exists $n \in \mathbb{N}$ such that $\text{st}(x, \mathcal{P}_n) \subset U$. For $\alpha \in f^{-1}(x)$, $\beta \in M$, if $d(\alpha, \beta) < 1/n$, then $\pi_i(\alpha) = \pi_i(\beta)$ whenever $i \leq n$. So $x \in P_{\pi_n(\alpha)} = P_{\pi_n(\beta)}$. Thus,

$$f(\beta) \in \bigcap_{i \in \mathbb{N}} P_{\pi_i(\beta)} \subset P_{\pi_n(\beta)} \subset U. \quad (2.6)$$

Hence

$$d(f^{-1}(x), M \setminus f^{-1}(U)) \geq \frac{1}{n}. \quad (2.7)$$

Therefore f is a π -map.

For each $x \in X$, it follows from the point-countable property of $\mathcal{P}_i$ that $\{\alpha \in \Lambda_i : x \in P_\alpha\}$ is countable. Put

$$L = \left(\prod_{i \in \mathbb{N}} \{\alpha \in \Lambda_i : x \in P_\alpha\} \right) \bigcap M, \quad (2.8)$$

then L is a hereditarily separable subspace of M , and $f^{-1}(x) \subset L$. Thus $f^{-1}(x)$ is separable in M , that is, f is an s -map.

We will prove that f is compact-covering. Suppose K is compact in X . Since each $\mathcal{P}_n$ is a cfp-cover for X , there exists finite subcollection $\mathcal{P}_n^K$ such that it is a cfp-cover of K in X . Thus there are a finite collection $\{K_\alpha : \alpha \in J_n\}$ of closed subsets of K and $\{P_\alpha : \alpha \in J_n\} \subset \mathcal{P}_n^K$ such that $K = \bigcup \{K_\alpha : \alpha \in J_n\}$ and each $K_\alpha \subset P_\alpha$. Obviously, each K_α is compact in X . Put

$$L = \left\{ (\alpha_i) : \alpha_i \in J_i, \bigcap_{i \in \mathbb{N}} K_{\alpha_i} \neq \emptyset \right\}, \quad (2.9)$$

then

(i) L is compact in M .

In fact, for all $(\alpha_i) \notin L$, $\bigcap_{i \in \mathbb{N}} K_{\alpha_i} = \emptyset$. From $\bigcap_{i \in \mathbb{N}} K_{\alpha_i} = \emptyset$, there exists $n_0 \in \mathbb{N}$ such that $\bigcap_{i=1}^{n_0} K_{\alpha_i} = \emptyset$. Put

$$W = \{(\beta_i) : \beta_i \in J_i, \beta_i = \alpha_i, 1 \leq i \leq n_0\}, \quad (2.10)$$

then W is an open neighborhood of (α_i) in $\prod_{i \in \mathbb{N}} J_i$, and $W \cap L = \emptyset$. Thus L is closed in $\prod_{i \in \mathbb{N}} J_i$. Since $\prod_{i \in \mathbb{N}} J_i$ is compact in $\prod_{i \in \mathbb{N}} \Lambda_i$, L is compact in M .

(ii) $L \subset M$, $f(L) = K$.

In fact, for all $(\alpha_i) \in L$, $\bigcap_{i \in \mathbb{N}} K_{\alpha_i} \neq \emptyset$. Pick $x \in \bigcap_{i \in \mathbb{N}} K_{\alpha_i}$, then $\langle P_{\alpha_i} \rangle$ is a local network of x in X , so $(\alpha_i) \in M$. This implies $L \subset M$.

For all $x \in K$, for each $i \in \mathbb{N}$, pick $\alpha_i \in J_i$ such that $x \in K_{\alpha_i}$. Thus $f((\alpha_i)) = x$, so $K \subset f(L)$. Obviously, $f(L) \subset K$. Hence $f(L) = K$.

In a word, f is compact-covering. $\square$

COROLLARY 2.2. *A space X is the compact-covering, quotient, and π - s -image of a metric space if and only if X has a weak-development consisting of point-countable cfp-covers.*

Proof. To prove the only if part, suppose X is the compact-covering, quotient, and π - s -image of a metric space M . From Theorem 2.1, X has a σ -strong network consisting of point-countable cfp-covers $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$. For each $x \in X$, $\text{st}(x, \mathcal{P}_n)$ is a sequential neighborhood of x in X . Obviously, X is a sequential space. Thus $\text{st}(x, \mathcal{P}_n)$ is a weak neighborhood base of x in X . Hence $\{\mathcal{P}_n\}$ is a weak-development for X .

To prove the if part, suppose X has a weak development consisting of point-countable cfp-covers. From Theorem 2.1, X is the image of a metric space under a compact-covering π - s -map f . Obviously, X is sequential. By [8, Proposition 2.1.16], f is quotient. $\square$

Similar to the proofs of Theorem 2.1 and Corollary 2.2, we have the following theorem.

THEOREM 2.3. *A space X is the pseudo-sequence-covering π - s -image of a metric space if and only if X has a σ -strong network consisting of point-countable sfp-covers.*

COROLLARY 2.4. *A space X is the pseudo-sequence-covering, quotient, and π - s -image of a metric space if and only if X has a weak-development consisting of point-countable sfp-covers.*

THEOREM 2.5. *A space X is the sequence-covering π - s -image of a metric space if and only if X has a σ -strong network consisting of point-countable cs-covers.*

Proof. To prove the only if part, suppose $f : (M, d) \rightarrow X$ is a sequence-covering π - s -map, where (M, d) is a metric space. Similar to the proof of Theorem 2.1, we can show that $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ -strong network consisting of point-countable covers. It suffices to show that each $\mathcal{P}_n$ is a cs-cover for X . Suppose $\{x_n\}$ converges to $x \in X$ in X . Since f is sequence-covering, then there exists a convergent sequence $\{z_i\}$ such that each $z_i \in f^{-1}(x_i)$. Suppose $\{z_i\} \rightarrow z$, then $z \in f^{-1}(x)$ and $z \in B$ for some $B \in \mathcal{B}_n$. Thus $\{z_i\}$ is eventually in B , so $\{x_i\}$ is eventually in $f(B) \in \mathcal{P}_n$. Hence each $\mathcal{P}_n$ is a cs-cover for X .

To prove the if part, suppose $\bigcup\{\mathcal{P}_i : i \in \mathbb{N}\}$ is a σ -strong network consisting of point-countable cs-covers for X . For each $i \in \mathbb{N}$, $\mathcal{P}_i$ is a point-countable cs-cover for X . Let $\mathcal{P}_i = \{P_\alpha : \alpha \in \Lambda_i\}$. Similar to the proof of Theorem 2.1, we can show that f is a π - s -map. It suffices to show that f is sequence-covering. Suppose $\{x_n\}$ converges to x in X . For each $i \in \mathbb{N}$, since $\mathcal{P}_i$ is a cs-cover for X , then there exists $P_{\alpha_i} \in \mathcal{P}_i$ such that $\{x_n\}$ is eventually in P_{α_i} . For each $n \in \mathbb{N}$, if $x_n \in P_{\alpha_i}$, let $\alpha_{in} = \alpha_i$; if $x_n \notin P_{\alpha_i}$, pick $\alpha_{in} \in \Lambda_i$ such that $x_n \in P_{\alpha_{in}}$. Thus there exists $n_i \in \mathbb{N}$ such that $\alpha_{in} = \alpha_i$ for all $n > n_i$. So $\{\alpha_{in}\}$ converges to α_i . For each $n \in \mathbb{N}$, put

$$\beta_n = (\alpha_{in}) \in \prod_{i \in \mathbb{N}} \Lambda_i, \quad (2.11)$$

then $(\beta_n) \in f^{-1}(x)$ and $\{\beta_n\}$ converges to x . Thus f is sequence-covering. $\square$

Similar to the proof of Corollary 2.2, we have the following corollary.

COROLLARY 2.6. *A space X is the sequence-covering, quotient, and π - s -image of a metric space if and only if X has a weak-development consisting of point-countable cs-covers.*

We give examples to illustrate the theorems of this paper.

Example 2.7. Let Z be the topological sum of the unit interval $[0,1]$, and the collection $\{S(x) : x \in [0,1]\}$ of 2^ω convergent sequence $S(x)$. Let X be the space obtained from Z by identifying the limit point of $S(x)$ with $x \in [0,1]$, for each $x \in [0,1]$. Then, from [8, Example 2.9.27], or see [3, Example 9.8], we have the following facts.

- (1) X is the compact-covering, quotient compact image of a locally compact metric space.
- (2) X has no point-countable cs-network.

The above facts together with [9, Theorem 1] yield the following conclusion: compact-covering (quotient) π - s -images of metric spaces are not sequence-covering (quotient) π - s -images of metric spaces.

Example 2.8. Let X be a sequential fan S_ω (see [8, Example 1.8.7]), then X is a Fréchet and $\aleph_0$ -space. So X is the sequence-covering s -image of a metric space. Because X is not g -first countable, thus X is not the pseudo-sequence-covering π -image of a metric space. Hence the following holds: sequence-covering (resp., pseudo-sequence-covering) s -images of metric spaces are not sequence-covering (resp., pseudo-sequence-covering) π - s -images of metric spaces.

Example 2.9. Let X be a Gillman-Jerison space $\psi(\mathbb{N})$ (see [8, Example 1.8.4]). Since X is developable, then X is the sequence-covering, quotient π -image of a metric space by [10, Corollary 3.1.12]. But X has no point-countable cs^* -networks. Then, it follows from [8, Theorem 2.7.5] that X is not the pseudo-sequence-covering s -image of a metric space. Thus,

- (1) sequence-covering (quotient) π -images of metric spaces are not sequence-covering (quotient) π - s -images of metric spaces,
- (2) pseudo-sequence-covering (quotient) π -images of metric spaces are not pseudo-sequence-covering (quotient) π - s -images of metric spaces.

Acknowledgments

The author would like to thank the referee for valuable suggestions. This work is supported by the NSF of Hunan Province in China (No. 04JJ6028) and the NSF of Education Department of Hunan Province in China (No. 03A002).

References

- [1] D. K. Burke, *Cauchy sequences in semimetric spaces*, Proc. Amer. Math. Soc. **33** (1972), 161–164.
- [2] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. **57** (1965), 107–115.
- [3] G. Gruenhage, E. Michael, and Y. Tanaka, *Spaces determined by point-countable covers*, Pacific J. Math. **113** (1984), no. 2, 303–332.
- [4] R. W. Heath, *On open mappings and certain spaces satisfying the first countability axiom*, Fund. Math. **57** (1965), 91–96.
- [5] Y. Ikeda, C. Liu, and Y. Tanaka, *Quotient compact images of metric spaces, and related matters*, Topology Appl. **122** (2002), no. 1–2, 237–252.
- [6] J. A. Kofner, *A new class of spaces and some problems from symmetrizable theory*, Dokl. Akad. Nauk SSSR **187** (1969), 270–273 (Russian).
- [7] K. B. Lee, *On certain g -first countable spaces*, Pacific J. Math. **65** (1976), no. 1, 113–118.

- [8] S. Lin, *Generalized Metric Spaces and Mappings*, Kexue Chubanshe (Science Press), Beijing, 1995 (Chinese).
- [9] ———, *A note on the Michael-Nagami problem*, Chinese Ann. Math. Ser. A **17** (1996), no. 1, 9–12.
- [10] ———, *Point-countable covers and sequence-covering mappings*, Chinese Distinguished Scholars Foundation Academic Publications, Kexue Chubanshe (Science Press), Beijing, 2002 (Chinese).
- [11] E. Michael, $\aleph_0$ -spaces, J. Math. Mech. **15** (1966), 983–1002.
- [12] E. Michael and K. Nagami, *Compact-covering images of metric spaces*, Proc. Amer. Math. Soc. **37** (1973), 260–266.
- [13] V. I. Ponomarev, *Axioms of countability and continuous mappings*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 127–133.
- [14] F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, General Topology and Appl. **1** (1971), no. 2, 143–154.
- [15] Y. Tanaka, *Point-countable covers and k -networks*, Topology Proc. **12** (1987), no. 2, 327–349.
- [16] P. Yan and S. Lin, *Compact-covering s -mappings on metric spaces*, Acta Math. Sinica **42** (1999), no. 2, 241–244.

Zhaowen Li: Department of Information, Hunan Business College, Changsha, Hunan 410205, China

E-mail address: lizhaowen8846@163.com