COMPLETELY REGULAR FUZZIFYING TOPOLOGICAL SPACES

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Received 24 March 2005 and in revised form 22 September 2005

Some of the properties of the completely regular fuzzifying topological spaces are investigated. It is shown that a fuzzifying topology τ is completely regular if and only if it is induced by some fuzzy uniformity or equivalently by some fuzzifying proximity. Also, τ is completely regular if and only if it is generated by a family of probabilistic pseudometrics.

1. Introduction

The concept of a fuzzifying topology was given in [1] under the name L -fuzzy topology. Ying studied in [9, 10, 11] the fuzzifying topologies in the case of $L = [0, 1]$. A classical topology is a special case of a fuzzifying topology. In a fuzzifying topology τ on a set X , every subset A of X has a degree $\tau(A)$ of belonging to τ , $0 \leq \tau(A) \leq 1$. In [4], we defined the degrees of compactness, of local compactness, Hausdorffness, and so forth in a fuzzifying topological space (X, τ) . We also introduced the fuzzifying proximities. Every fuzzifying proximity δ induces a fuzzifying topology τ_δ . In [6], we studied the level classical topologies τ^θ , $0 \leq \theta < 1$, corresponding to a fuzzifying topology τ . In the same paper, we studied connectedness and local connectedness in fuzzifying topological spaces as well as the so-called sequential fuzzifying topologies. In [5], we introduced the fuzzifying syn-topogenous structures. We also proved that every fuzzy uniformity $\mathcal{U}$, as it is defined by Lowen in [7], induces a fuzzifying proximity $\delta_{\mathcal{U}}$, and that for every fuzzifying proximity δ , there exists at least one fuzzy uniformity $\mathcal{U}$ with $\delta = \delta_{\mathcal{U}}$. Some of the results contained in papers [4, 6] are closely related to those which appeared in the papers [12, 13].

In this paper, we continue with the investigation of fuzzifying topologies. In particular, we study the completely regular fuzzifying topologies, that is, those fuzzifying topologies τ for which each level topology τ^θ is completely regular. As in the classical case, we prove that for a fuzzifying topology τ on X , the following properties are equivalent: (1) τ is completely regular; (2) τ is uniformizable, that is, it is induced by some fuzzy uniformity; (3) τ is proximizable, that is, it is induced by some fuzzifying proximity; and (4) τ is generated by a family of so-called probabilistic pseudometrics on X . We also give a characterization of completely regular fuzzifying spaces in terms of continuous functions. Many Theorems on classical topologies follow as special cases of results obtained in the paper.

2. Preliminaries

A fuzzifying topology on a set X (see [1, 9, 10, 11]) is a map $\tau : 2^X \rightarrow [0, 1]$ (where 2^X is the power set of X) satisfying the following conditions:

- (FT1) $\tau(X) = \tau(\emptyset) = 1$;
- (FT2) $\tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$;
- (FT3) $\tau(\bigcup A_i) \geq \inf_i \tau(A_i)$.

If τ is a fuzzifying topology on X and $x \in X$, then the τ -neighborhood system of x is the function

$$N_x = N_x^\tau : 2^X \longrightarrow [0, 1], \quad N_x(A) = \sup \{ \tau(B) : x \in B \subset A \}. \quad (2.1)$$

By [9, Theorem 3.2], we have that $\tau(A) = \inf_{x \in A} N_x(A)$.

The following theorem is contained in [9] (see also [3, 13]).

THEOREM 2.1. *If τ is a fuzzifying topology on a set X , then the map $x \rightarrow N_x = N_x^\tau$, from X to the fuzzy power set $\mathcal{F}(2^X)$ of 2^X , has the following properties:*

- (FN1) $N_x(X) = 1$ and $N_x(A) = 0$ if $x \notin A$;
- (FN2) $N_x(A_1 \cap A_2) = N_x(A_1) \wedge N_x(A_2)$;
- (FN3) $N_x(A) \leq \sup_{x \in D \subset A} \inf_{y \in D} N_y(D)$.

Conversely, if a map $x \rightarrow N_x$, from X to $\mathcal{F}(2^X)$, satisfies (FN1)–(FN3), then the map

$$\tau : 2^X \longrightarrow [0, 1], \quad \tau(A) = \inf_{x \in A} N_x(A), \quad (2.2)$$

is a fuzzifying topology and $N_x = N_x^\tau$ for every $x \in X$.

Let now (X, τ) be a fuzzifying topological space. To every subset A of X corresponds a fuzzy subset $\tilde{A} = \tilde{A}^\tau$ of X defined by $\tilde{A}(x) = 1 - N_x(A^c)$ (see [13, Remark 3.16]). A function f , from a fuzzifying topological space (X, τ_1) to another one (Y, τ_2) , is said to be continuous at some $x \in X$ (see [4, 13]) if $N_x(f^{-1}(A)) \geq N_{f(x)}(A)$ for every subset A of Y . If f is continuous at every point of X , then it is said that (τ_1, τ_2) -continuous. As it is shown in [4], f is continuous if and only if $\tau_2(A) \leq \tau_1(f^{-1}(A))$ for every subset A of Y . For $f : X \rightarrow Y$ a function and τ a fuzzifying topology on Y , $f^{-1}(\tau)$ is defined to be the weakest fuzzifying topology on X for which f is continuous. By [4], $f^{-1}(\tau)$ is given by the neighborhood structure $N_x(A) = N_{f(x)}(Y \setminus f(A^c))$. If $(\tau_i)_{i \in I}$ is a family of fuzzifying topologies on X , we will denote by $\bigvee_{i \in I} \tau_i$, or by $\sup \tau_i$, the weakest of all fuzzifying topologies on X which are finer than each τ_i . As it is proved in [4], $\bigvee_{i \in I} \tau_i$ is given by the neighborhood structure

$$N_x(A) = \sup \left\{ \inf_{i \in J} N_x^{\tau_i}(A_i) : x \in \bigcap_{i \in J} A_i \subset A \right\}, \quad (2.3)$$

where the infimum is taken over the family of all finite subsets J of I and all $A_i \subset X$, $i \in J$. For Y a subset of a fuzzifying topological space (X, τ) , $\tau|_Y$ will be the fuzzifying topology induced on Y by τ , that is, the fuzzifying topology $f^{-1}(\tau)$, where $f : Y \rightarrow X$ is the inclusion map. For a family $(X_i, \tau_i)_{i \in I}$ of fuzzifying topological spaces, the product

fuzzifying topology $\tau = \prod \tau_i$ on $X = \prod X_i$ is the weakest fuzzifying topology on X for which each projection $\pi_i : X \rightarrow X_i$ is continuous. Thus, $\tau = \bigvee_i \pi_i^{-1}(\tau_i)$ and it is given by the neighborhood structure

$$N_x(A) = \sup \left\{ \inf_{i \in J} N_{x_i}(A_i) : x \in \bigcap_{i \in J} \pi_i^{-1}(A_i) \subset A \right\}, \quad (2.4)$$

where the supremum is taken over the family of all finite subsets J of I and $A_i \subset X_i$, for $i \in J$ (see [4]).

The degree of convergence to an $x \in X$, of a net (x_δ) in a fuzzifying topological space (X, τ) , is the number $c(x_\delta \rightarrow x) = c^\tau(x_\delta \rightarrow x)$ defined by

$$c(x_\delta \rightarrow x) = \inf \{1 - N_x(A) : A \subset X, (x_\delta) \text{ frequently in } A^c\}. \quad (2.5)$$

As it is shown in [6], for $A \subset X$ and $x \in X$, we have

$$\bar{A}(x) = \max \{c(x_\delta \rightarrow x) : (x_\delta) \text{ net in } A\}. \quad (2.6)$$

The degree of Hausdorffness of X (see [4]) is defined by

$$T_2(X) = 1 - \sup_{x \neq y} \sup \{c(x_\delta \rightarrow x) \wedge c(x_\delta \rightarrow y) : (x_\delta) \text{ net in } X\}. \quad (2.7)$$

Also, the degree of X being T_1 is defined by

$$T_1(X) = \inf_x \inf_{y \neq x} \sup \{N_x(B) : y \notin B\}. \quad (2.8)$$

Let now (X, τ) be a fuzzifying topological space. For each $0 \leq \theta < 1$, the family $B_\theta^\tau = \{A \subset X : \tau(A) > \theta\}$ is a base for a classical topology τ^θ on X (see [5]). It is easy to see that a subset B of X is a τ^θ -neighborhood of x if and only if $N_x(B) > \theta$. By [6], $T_2(X)$ (resp., $T_1(X)$) is the supremum of all $0 \leq \theta < 1$ for which τ^θ is T_2 (resp., T_1). Also, for $\tau = \bigvee \tau_i$, we have that $\tau^\theta = \sup_i \tau_i^\theta$ (see [6, Theorem 3.5]). If $\tau = \prod \tau_i$ is a product fuzzifying topology, then $\tau^\theta = \prod \tau_i^\theta$ (see [6, Theorem 3.5]). If Y is a subspace of (X, τ) and $\tau_1 = \tau|_Y$, then $\tau_1^\theta = \tau^\theta|_Y$. By [6, Theorem 3.10], for a fuzzifying topological space (X, τ) , $\text{co}(X)$ coincides with the supremum of all $0 < \theta < 1$ for which $\tau^{1-\theta}$ is compact.

Next, we will recall the notion of a fuzzifying proximity given in [4]. A fuzzifying proximity on a set X is a map $\delta : 2^X \times 2^X \rightarrow [0, 1]$ satisfying the following conditions:

- (FP1) $\delta(A, B) = 1$ if the A, B are not disjoint;
- (FP2) $\delta(A, B) = \delta(B, A)$;
- (FP3) $\delta(\emptyset, B) = 0$;
- (FP4) $\delta(A_1 \cup A_2, B) = \delta(A_1, B) \vee \delta(A_2, B)$;
- (FP5) $\delta(A, B) = \inf \{\delta(A, D) \vee \delta(D^c, B) : D \subset X\}$.

Every fuzzifying proximity δ induces a fuzzifying topology τ_δ given by the neighborhood structure $N_x(A) = 1 - \delta(x, A^c)$. A fuzzifying proximity δ_1 is said to be finer than another one δ_2 if $\delta_1(A, B) \leq \delta_2(A, B)$ for all subsets A, B of X . For $f : X \rightarrow Y$ a function and δ a fuzzifying proximity on Y , the function

$$f^{-1}(\delta) : 2^X \times 2^X \longrightarrow [0, 1], \quad f^{-1}(\delta)(A, B) = \delta(f(A), f(B)), \quad (2.9)$$

is a fuzzifying proximity on X (see [4]) and it is the weakest of all fuzzifying proximities δ_1 on X for which f is (δ_1, δ) -proximally continuous, that is, it satisfies $\delta_1(A, B) \leq \delta(f(A), f(B))$ for all subsets A, B of X . As it is shown in [4], $\tau_{f^{-1}(\delta)} = f^{-1}(\tau_\delta)$.

Let now $(\delta_\lambda)_{\lambda \in \Lambda}$ be a family of fuzzifying proximities on a set X . We will denote by $\delta = \bigvee_\lambda \delta_\lambda$, or by $\sup \delta_\lambda$, the weakest fuzzifying proximity on X which is finer than each δ_λ . By [4, Theorem 8.10], δ is given by

$$\delta(A, B) = \inf \left\{ \sup_{i,j} \inf_{\lambda \in \Lambda} \delta_\lambda(A_i, B_j) \right\}, \quad (2.10)$$

where the infimum is taken over all finite collections $(A_i), (B_j)$ of subsets of X with $A = \bigcup A_i, B = \bigcup B_j$. Moreover, $\tau_\delta = \bigvee \tau_{\delta_\lambda}$ (see [4]).

Finally, we will recall the definition of a fuzzy uniformity introduced by Lowen in [7]. For a set X , let Ω_X be the collection of all functions $\alpha : X \times X \rightarrow [0, 1]$ such that $\alpha(x, x) = 1$ for all $x \in X$. For $\alpha, \beta \in \Omega_X$, the $\alpha \wedge \beta, \alpha \circ \beta$ and α^{-1} are defined by $\alpha \wedge \beta(x, y) = \alpha(x, y) \wedge \beta(x, y)$, $\alpha \circ \beta(x, y) = \sup_z \beta(x, z) \wedge \alpha(z, y)$, $\alpha^{-1}(x, y) = \alpha(y, x)$. If $\alpha = \alpha^{-1}$, then α is called symmetric. A fuzzy uniformity on X is a nonempty subset $\mathcal{U}$ of Ω_X satisfying the following conditions.

(FU1) If $\alpha, \beta \in \mathcal{U}$, then $\alpha \wedge \beta \in \mathcal{U}$.

(FU2) If $\alpha \in \mathcal{U}$ is such that, for every $\epsilon > 0$, there exists a $\beta \in \mathcal{U}$ with $\beta \leq \alpha + \epsilon$, then $\alpha \in \mathcal{U}$.

(FU3) For each $\alpha \in \mathcal{U}$ and each $\epsilon > 0$, there exists a $\beta \in \mathcal{U}$ with $\beta \circ \beta \leq \alpha + \epsilon$.

(FU4) If $\alpha \in \mathcal{U}$, then $\alpha^{-1} \in \mathcal{U}$.

A subset $\mathcal{B}$, of a fuzzy uniformity $\mathcal{U}$, is a base for $\mathcal{U}$ if for each $\alpha \in \mathcal{U}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \leq \alpha + \epsilon$. It is easy to see that for a subset $\mathcal{B}$ of Ω_X , the following are equivalent.

(1) $\mathcal{B}$ is a base for a fuzzy uniformity on X .

(2) (a) If $\alpha, \beta \in \mathcal{B}$ and $\epsilon > 0$, then there exists $\gamma \in \mathcal{B}$ with $\gamma \leq \alpha \wedge \beta + \epsilon$.

(b) For each $\alpha \in \mathcal{B}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \circ \beta \leq \alpha + \epsilon$.

(c) For each $\alpha \in \mathcal{B}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \leq \alpha^{-1} + \epsilon$.

In case (2) is satisfied, the fuzzy uniformity $\mathcal{U}$ for which $\mathcal{B}$ is a base consists of all $\alpha \in \Omega_X$ such that for each $\epsilon > 0$, there exists a $\beta \in \mathcal{B}$ with $\beta \leq \alpha + \epsilon$.

By [5], every fuzzy uniformity $\mathcal{U}$ on X induces a fuzzifying proximity $\delta_{\mathcal{U}}$ defined by

$$\delta_{\mathcal{U}}(A, B) = \inf_{\alpha \in \mathcal{U}} \sup_{x \in A, y \in B} \alpha(x, y). \quad (2.11)$$

In case $\mathcal{B}$ is a base for $\mathcal{U}$, then

$$\delta_{\mathcal{U}}(A, B) = \inf_{\alpha \in \mathcal{B}} \sup_{x \in A, y \in B} \alpha(x, y). \quad (2.12)$$

Every fuzzy uniformity $\mathcal{U}$ induces a fuzzifying topology $\tau_{\mathcal{U}}$ given by the neighborhood structure

$$N_x(A) = 1 - \delta_{\mathcal{U}}(x, A^c) = 1 - \inf_{\alpha \in \mathcal{U}} \sup_{y \notin A} \alpha(x, y). \quad (2.13)$$

For every fuzzifying proximity δ , there exists at least one compatible fuzzy uniformity, that is, a fuzzy uniformity $\mathcal{U}$ with $\delta_{\mathcal{U}} = \delta$ (see [5, Theorem 11.4]).

3. Probabilistic pseudometrics

A fuzzy real number is a fuzzy subset u of the real numbers $\mathbb{R}$ which is increasing, left continuous, and such that $\lim_{t \rightarrow +\infty} u(t) = 1$, $\lim_{t \rightarrow -\infty} u(t) = 0$. A fuzzy real number u is said to be nonnegative if $u(t) = 0$ if $t \leq 0$. We will denote by $\mathbb{R}_{\phi}^{+}$ the collection of all nonnegative fuzzy real numbers. To every real number r corresponds a fuzzy real number $\bar{r}$, where $\bar{r}(t) = 0$ if $t \leq r$ and $\bar{r}(t) = 1$ if $t > r$. For $u, v \in \mathbb{R}_{\phi}^{+}$, we define $u \leq v$ if and only if $v(t) \leq u(t)$ for all $t \in \mathbb{R}$. If $\mathcal{A}$ is a nonempty subset of $\mathbb{R}_{\phi}^{+}$ and if $u_o \in \mathbb{R}_{\phi}^{+}$ is defined by $u_o(t) = \sup_{v \in \mathcal{A}} v(t)$, then u_o is the biggest of all $u \in \mathbb{R}_{\phi}^{+}$ with $u \leq v$ for all $v \in \mathcal{A}$. We will denote u_o by $\inf \mathcal{A}$ or by $\bigwedge \mathcal{A}$. For $u_1, u_2 \in \mathbb{R}_{\phi}^{+}$, we define $u = u_1 \oplus u_2 \in \mathbb{R}_{\phi}^{+}$ by $u(t) = \sup\{u_1(t_1) \wedge u_2(t_2) : t = t_1 + t_2\}$. Also, for $u \in \mathbb{R}_{\phi}^{+}$ and $\lambda > 0$, we define λu by $(\lambda u)(t) = u(\lambda^{-1}t)$. It is easy to see that for $u \in \mathbb{R}_{\phi}^{+}$ and $\lambda > 0$, we have $(\bar{\lambda} \oplus u)(t) = u(t - \lambda)$.

Definition 3.1. A probabilistic pseudometric on a set X (see [2]) is a mapping $F : X \times X \rightarrow \mathbb{R}_{\phi}^{+}$ such that for all $x, y, z \in X$,

$$F(x, x) = \bar{0}, \quad F(x, y) = F(y, x), \quad F(x, z) \leq F(x, y) \oplus F(y, z). \quad (3.1)$$

If in addition $F(x, y)(0+) = 0$ when $x \neq y$, then F is called a probabilistic metric.

If r_1, r_2 are nonnegative real numbers, then $\bar{r}_1 \leq \bar{r}_2$ if and only if $r_1 \leq r_2$. Also, for $r = |r_1 - r_2|$, we have that

$$\bar{r} = \bigwedge \{u \in \mathbb{R}_{\phi}^{+} : \bar{r}_2 \leq u \oplus \bar{r}_1, \bar{r}_1 \leq u \oplus \bar{r}_2\}. \quad (3.2)$$

In fact, let $u_o = \bigwedge \{u \in \mathbb{R}_{\phi}^{+} : \bar{r}_2 \leq u \oplus \bar{r}_1 \text{ and } \bar{r}_1 \leq u \oplus \bar{r}_2\}$ and assume that (say) $r_1 \geq r_2$. Let $u \in \mathbb{R}_{\phi}^{+}$ be such that $\bar{r}_2 \leq u \oplus \bar{r}_1, \bar{r}_1 \leq u \oplus \bar{r}_2$. Then $\bar{r}_1(t) \geq (u \oplus \bar{r}_2)(t) = u(t - r_2)$ for all t . If $s < r_1$, then $0 = \bar{r}_1(s) \geq u(s - r_2)$ and so $u(r_1 - r_2) = \sup_{s < r_1} u(s - r_2) = 0$ which implies that $\bar{r} \leq u$. Thus $\bar{r} \leq u_o$. On the other hand, we have $\bar{r} \oplus \bar{r}_2 = \bar{r}_1$ and $\bar{r} \oplus \bar{r}_1 = \bar{r}_2$. Since $\bar{r}_2 \leq \bar{r}_1$, it follows that $u_o \leq \bar{r}$, and hence $\bar{r} = u_o$. Motivated by the above, we define the following distance function on $\mathbb{R}_{\phi}^{+}$:

$$D : \mathbb{R}_{\phi}^{+} \times \mathbb{R}_{\phi}^{+} \longrightarrow \mathbb{R}_{\phi}^{+}, \quad D(u_1, u_2) = \bigwedge \{u \in \mathbb{R}_{\phi}^{+} : u_1 \leq u_2 \oplus u, u_2 \leq u \oplus u_1\}. \quad (3.3)$$

Then D is a probabilistic pseudometric on $\mathcal{R}_\phi^+$. In fact, it is clear that $D(u_1, u_2) = D(u_2, u_1)$. Also, since $u = u \oplus \bar{0}$, when $u \in \mathcal{R}_\phi^+$, we have that $D(u, u) = \bar{0}$. Finally, let $D(u_1, u_2)(t_1) \wedge D(u_2, u_3)(t_2) > \theta > 0$. There are $v_1, v_2 \in \mathcal{R}_\phi^+$ with $u_1 \leq v_1 \oplus u_2$, $u_2 \leq v_1 \oplus u_1$, $u_3 \leq v_2 \oplus u_2$, $u_2 \leq v_2 \oplus u_3$, $v_1(t_1) > \theta$, $v_2(t_2) > \theta$. Now $u_1 \leq v_1 \oplus u_2 \leq v_1 \oplus (v_2 \oplus u_3) = (v_1 \oplus v_2) \oplus u_3$ and $u_3 \leq v_2 \oplus u_2 \leq v_2 \oplus (v_1 \oplus u_1) = (v_1 \oplus v_2) \oplus u_1$. Thus, $D(u_1, u_3) \leq v_1 \oplus v_2$ and $D(u_1, u_3)(t_1 + t_2) \geq v_1(t_1) \wedge v_2(t_2) > \theta$. This proves that $D(u_1, u_3) \leq D(u_1, u_2) \oplus D(u_2, u_3)$ and the claim follows. We will refer to D as the usual probabilistic pseudometric on $\mathcal{R}_\phi^+$.

Let now F be a probabilistic pseudometric on X . For $t > 0$, let $u_{F,t}$ be defined on X^2 by $u_{F,t}(x, y) = F(x, y)(t)$. The family $\mathcal{B}_F = \{u_{F,t} : t > 0\}$ is a base for a fuzzy uniformity $\mathcal{U}_F$ on X . Let τ_F be the fuzzifying topology induced by $\mathcal{U}_F$.

In the rest of the paper, we will consider on $\mathcal{R}_\phi^+$ the fuzzifying topology induced by the usual probabilistic pseudometric D .

THEOREM 3.2. *A probabilistic pseudometric F , on a fuzzifying topological space (X, τ) , is $\tau \times \tau$ continuous if and only if $\tau_F \leq \tau$.*

Proof. Assume that $\tau_F \leq \tau$ and let G be a subset of $\mathcal{R}_\phi^+$ and $u = F(x_o, y_o)$ with $N_u(G) > \theta > 0$. There exists a $t > 0$ such that $1 - \sup_{v \notin G} D(v, u)(t) > \theta$. For $x, y \in X$, we have

$$F(x, y) \leq F(x, x_o) \oplus (x_o, y_o) \oplus F(y_o, y) = [F(x, x_o) \oplus F(y, y_o)] \oplus F(x_o, y_o). \quad (3.4)$$

Similarly, $F(x_o, y_o) \leq [F(x, x_o) \oplus F(y, y_o)] \oplus F(x, y)$. Thus,

$$D(F(x, y), F(x_o, y_o)) \leq F(x, x_o) \oplus F(y, y_o). \quad (3.5)$$

Let

$$A_1 = \left\{ x \in X : F(x, x_o) \left(\frac{t}{2} \right) \geq 1 - \theta \right\}, \quad A_2 = \left\{ x \in X : F(y, y_o) \left(\frac{t}{2} \right) \geq 1 - \theta \right\}. \quad (3.6)$$

If $x \in A_1$, $y \in A_2$, then

$$D(F(x, y), F(x_o, y_o))(t) \geq F(x, x_o) \left(\frac{t}{2} \right) \wedge F(y, y_o) \left(\frac{t}{2} \right) \geq 1 - \theta, \quad (3.7)$$

and so $F(x, y) \in G$. Also, $N_{x_o}^\tau(A_1) \geq N_{x_o}^{\tau_F}(A_1) \geq 1 - \sup_{x \notin A_1} F(x, x_o)(t/2) \geq \theta$ and $N_{y_o}^\tau(A_2) \geq \theta$. Therefore,

$$N_{(x_o, y_o)}^{\tau \times \tau}(F^{-1}(G)) \geq N_{x_o}^\tau(A_1) \wedge N_{y_o}^\tau(A_1) \geq \theta, \quad (3.8)$$

which proves that $N_{(x_o, y_o)}^{\tau \times \tau}(F^{-1}(G)) \geq N_{F(x_o, y_o)}(G)$ and so F is $\tau \times \tau$ continuous. Conversely, assume that F is $\tau \times \tau$ continuous and let $N_{x_o}^{\tau_F}(A) > \theta > 0$. Choose $\epsilon > 0$ such that $N_{x_o}^{\tau_F}(A) > \theta + \epsilon$. There exists a $t > 0$ such that $1 - \sup_{x \notin A} F(x, x_o)(t) > \theta + \epsilon$. If

$$Z = \{u \in \mathcal{R}_\phi^+ : D(u, \bar{0})(t) = u(t) > 1 - \theta - \epsilon\}, \quad (3.9)$$

then

$$N_{\bar{0}}(Z) \geq 1 - \sup_{u \notin Z} D(u, \bar{0})(t) \geq \theta + \epsilon > \theta. \quad (3.10)$$

Since F is $\tau \times \tau$ continuous and $F(x_o, x_o) = \bar{0}$, there exists a subset A_1 of X containing x_o such that $A_1 \times A_1 \subset F^{-1}(Z)$ and $N_{x_o}(A_1) > \theta$. If $x \in A_1$, then $F(x, x_o) \in Z$ and so $F(x, x_o)(t) > 1 - \theta - \epsilon$, which implies that $x \in A$. Thus, $A_1 \subset A$ and so $N_{x_o}(A) \geq N_{x_o}^{\tau_F}(A)$ for every subset A of X and every $x_o \in X$. Hence, $\tau_F \leq \tau$ and the result follows. $\square$

THEOREM 3.3. *Let F be a probabilistic pseudometric on a set X , $\tau = \tau_F$, $(x_\delta)_{\delta \in \Delta}$ a net in X , and $x \in X$. Then*

$$c(x_\delta \rightarrow x) = \inf_{t>0} \liminf_{\delta} F(x_\delta, x)(t). \quad (3.11)$$

Proof. Let $d = \inf_{t>0} \liminf_{\delta} F(x_\delta, x)(t)$ and assume that $d < \theta < 1$. There exists a $t > 0$ such that $\liminf_{\delta} F(x_\delta, x)(t) < \theta$. Let $A = \{y : F(y, x)(t) > \theta\}$. Then (x_δ) is not eventually in A , and so $c(x_\delta \rightarrow x) \leq 1 - N_x(A) \leq \sup_{y \notin A} F(y, x)(t) \leq \theta$, which proves that $c(x_\delta \rightarrow x) \leq d$. On the other hand, let $c(x_\delta \rightarrow x) < r < 1$. There exists a subset B of X such that (x_δ) is not eventually in B and $1 - N_x(B) < r$. Let $s > 0$ be such that $1 - \sup_{y \notin B} F(y, x)(s) > 1 - r$. For each $\delta \in \Delta$, there exists $\delta' \geq \delta$ with $x_{\delta'} \notin B$, and so $F(x_{\delta'}, x)(s) \leq \sup_{y \notin B} F(y, x)(s)$. Thus, $d \leq \liminf_{\delta} F(x_\delta, x)(s) < r$, which proves that $d \leq c(x_\delta \rightarrow x)$ and the result follows. $\square$

THEOREM 3.4. *Let $F_1, F_2, \dots, F_n$ be probabilistic pseudometrics on X and define F by*

$$F(x, y)(t) = \min_{1 \leq k \leq n} F_k(x, y)(t). \quad (3.12)$$

Then F is a probabilistic pseudometric and $\tau_F = \bigvee_{k=1}^n \tau_{F_k}$.

Proof. Using induction on n , it suffices to prove the result in the case of $n = 2$. It follows easily that F is a probabilistic pseudometric. Since $F_1, F_2 \leq F$, it follows that $\tau_{F_1}, \tau_{F_2} \leq \tau_F$ and so $\tau_o = \tau_{F_1} \vee \tau_{F_2} \leq \tau_F$. On the other hand, let $N_x^{\tau_F}(A) > \theta > 0$. There exists a $t > 0$ such that $1 - \sup_{y \notin A} F(y, x)(t) > \theta$. Let $B_i = \{y \in A^c : F_i(y, x)(t) < 1 - \theta\}$, $i = 1, 2$. Then $A^c = B_1 \cup B_2$ and so $A = A_1 \cap A_2$, $A_i = B_i^c$. Moreover $N_x^{\tau_{F_i}}(A_i) \geq 1 - \sup_{y \in B_i} F_i(y, x)(t) \geq \theta$, and thus

$$N_x^{\tau_o}(A) \geq N_x^{\tau_o}(A_1) \bigwedge N_x^{\tau_o}(A_2) \geq N_x^{\tau_{F_1}}(A_1) \bigwedge N_x^{\tau_{F_2}}(A_2) \geq \theta. \quad (3.13)$$

This proves that $N_x^{\tau_o}(A) \geq N_x^{\tau_F}(A)$ and the result follows. $\square$

For $\mathcal{F}$ a family of probabilistic pseudometrics on a set X , we will denote by $\tau_{\mathcal{F}}$ the supremum of the fuzzifying topologies τ_F , $F \in \mathcal{F}$, that is, $\tau_{\mathcal{F}} = \bigvee_{F \in \mathcal{F}} \tau_F$.

THEOREM 3.5. *If $\tau = \tau_{\mathcal{F}}$, where $\mathcal{F}$ is a family of probabilistic pseudometrics on a set X , then $T_2(X) = T_1(X) = 1 - \sup_{y \neq x} \inf_{F \in \mathcal{F}} F(x, y)(0+)$.*

Proof. Let $d = 1 - \sup_{y \neq x} \inf_{F \in \mathcal{F}} F(x, y)(0+)$. It is always true that $T_2(X) \leq T_1(X)$. Suppose that $T_1(X) > r > 0$ and let $x \neq y$. Since τ^r is T_1 , there exists a τ^r -neighborhood A of x not containing y . Now $N_x(A) > r$, and hence there are subsets $A_1, \dots, A_n$ of X and $F_1, \dots, F_n \in \mathcal{F}$ such that $\bigcap A_k \subset A$, $N_x^{\tau_{F_k}}(A_k) > r$. Since y is not in A , there exists a k with

$y \notin A_k$. Let $t > 0$ be such that

$$1 - \sup_{z \notin A_k} F_k(z, x)(t) > r \text{ and so } \inf_{F \in \mathcal{F}} F(x, y)(t)(0+) \leq F_k(x, y)(t) < 1 - r, \quad (3.14)$$

which proves that $d \geq r$. Thus $d \geq T_1(X)$. On the other hand, assume that $d > \theta > 0$ and let $x \neq y$. Choose $\epsilon > 0$ such that $d > \theta + \epsilon$. There exists $F \in \mathcal{F}$ with $F(x, y)(0+) < 1 - \theta - \epsilon$, and hence $F(x, y)(t) < 1 - \theta - \epsilon$ for some $t > 0$. Let

$$A = \left\{ z : F(z, x) \left(\frac{t}{2} \right) > 1 - \theta - \epsilon \right\}, \quad B = \left\{ z : F(z, y) \left(\frac{t}{2} \right) > 1 - \theta - \epsilon \right\}. \quad (3.15)$$

Clearly $x \in A$, $y \in B$. If $z \in A \cap B$, then

$$F(x, y)(t) \geq F(x, z) \left(\frac{t}{2} \right) \wedge F(z, y) \left(\frac{t}{2} \right) > 1 - \theta - \epsilon, \quad (3.16)$$

a contradiction. Thus $A \cap B = \emptyset$. Moreover

$$N_x(A) \geq N_x^{\tau_F}(A) \geq 1 - \sup_{z \notin A} F(x, z) \left(\frac{t}{2} \right) \geq \theta + \epsilon > \theta, \quad N_y(A) > \theta. \quad (3.17)$$

It follows that $T_2(X) \geq d$ and the proof is complete. $\square$

Let us say that a fuzzifying topology τ on a set X is pseudometrizable if there exists a probabilistic pseudometric F on X with $\tau = \tau_F$.

THEOREM 3.6. *A fuzzifying topology τ on X is pseudometrizable if and only if each level topology τ^θ , $0 \leq \theta < 1$, is pseudometrizable.*

Proof. Assume that $\tau = \tau_F$ for some probabilistic pseudometric F and let $0 \leq \theta < 1$. For each positive integer n , with $n > 1/(1 - \theta)$, let

$$A_n = \left\{ (x, y) \in X^2 : F(x, y) \left(\frac{1}{n} \right) > 1 - \theta - \frac{1}{n} \right\}. \quad (3.18)$$

Then $A_{n+1} \subset A_n$ and the family $\mathcal{D} = \{A_n : n \in \mathbb{N}, n > 1/(1 - \theta)\}$ is a base for a uniformity $\mathcal{U}$ on X . The topology σ_θ induced by $\mathcal{U}$ is pseudometrizable since $\mathcal{D}$ is countable. Moreover $\sigma_\theta = \tau^\theta$. Indeed, let A be a σ_θ -neighborhood of x . There exists $n \in \mathbb{N}$, $n > 1/(1 - \theta)$, such that $B = \{y : F(x, y)(1/n) > 1 - \theta - 1/n\} \subset A$. Now

$$N_x^\tau(A) \geq N_x^\tau(B) \geq 1 - \sup_{y \notin B} F(x, y) \left(\frac{1}{n} \right) \geq \theta + \frac{1}{n} > \theta, \quad (3.19)$$

and so A is a τ^θ -neighborhood of x . Conversely, assume that A is a τ^θ -neighborhood of x . There exists $\epsilon > 0$ with $N_x(A) > \theta + \epsilon$. Now there exists a positive integer $n > 1/\epsilon$ such

that $1 - \sup_{y \notin A} F(x, y)(1/n) > \theta + 1/n$. Hence

$$\left\{ y : F(x, y) \left(\frac{1}{n} \right) > 1 - \theta - \frac{1}{n} \right\} \subset A, \quad (3.20)$$

which implies that A is a σ_θ -neighborhood of x . Thus $\tau^\theta = \sigma_\theta$, and therefore each τ^θ is pseudometrizable. Conversely, suppose that each τ^θ is pseudometrizable. By an argument analogous to the one used in the proof of [6, Theorem 3.3], we show that there exists a family $\{d_\theta : 0 \leq \theta < 1\}$ of pseudometrics on X such that $d_\theta = \sup_{\theta_1 > \theta} d_{\theta_1}$, for each $0 \leq \theta < 1$, and τ^θ coincides with the topology induced by the pseudometric d_θ . Now, for $x, y \in X$, define $F(x, y) : \mathbb{R} \rightarrow [0, 1]$ by $F(x, y)(t) = 0$ if $t \leq 0$ and $F(x, y)(t) = \sup\{\theta : 0 < \theta \leq 1, d_{1-\theta}(x, y) < t\}$ if $t > 0$. It is clear that $F(x, y)$ is increasing and left continuous. For $0 < r < 1$ and $t > d_{1-r}(x, y)$, we have that $F(x, y)(t) \geq r$, and so $\lim_{t \rightarrow \infty} F(x, y)(t) = 1$. Also $F(x, x)(t) = 1$ for every x and every $t > 0$. To show that F is a probabilistic pseudometric on X , we must prove that it satisfies the triangle inequality. So, let $F(x, y)(t_1) \wedge F(y, z)(t_2) > \theta > 0$. Then $d_{1-\theta}(x, y) < t_1$, $d_{1-\theta}(y, z) < t_2$, and so $d_{1-\theta}(x, z) < t_1 + t_2$, which implies that $F(x, z)(t_1 + t_2) \geq \theta$. Thus the triangle inequality is satisfied and F is a probabilistic pseudometric. We will finish the proof by showing that $\tau_F = \tau$. So let $N_x^{\tau_F} > \theta > 0$ and choose $t > 0$ such that $1 - \sup_{y \notin A} F(y, x)(t) > \theta$. If now $d_\theta(x, y) < t$, then $F(x, y)(t) \geq 1 - \theta$, and thus $y \in A$, which proves that A is a $\sigma_\theta = \tau^\theta$ neighborhood of x . Hence $\tau \geq \tau_F$. On the other hand, let B be a τ^θ -neighborhood of x . There exists $\theta_1 > \theta$ such that $N_x(B) > \theta_1$. Now B is a τ_{θ_1} -neighborhood of x , and so there exists $t > 0$ such that $\{y : d_{\theta_1}(x, y) < t\} \subset B$. If $F(x, y)(t) > 1 - \theta_1$, then there exists $\alpha > 1 - \theta_1$ such that $d_{1-\alpha}(x, y) < t$ and so $d_{\theta_1}(x, y) < t$. Thus $\{y : F(x, y)(t) > 1 - \theta_1\} \subset B$, and therefore

$$N_x^{\tau_F}(B) \geq 1 - \sup_{y \notin B} F(x, y)(t) \geq \theta_1 > \theta. \quad (3.21)$$

Thus, $\tau_F \geq \tau$ and the result follows. $\square$

THEOREM 3.7. *Let (X, F) be a probabilistic pseudometric space, $A \subset X$, and $x \in X$.*

Let

$$\begin{aligned} \alpha &= \sup \left\{ \inf_{t>0} \liminf_n F(x_n, x)(t) : (x_n) \text{ sequence in } A \right\}, \\ \beta &= \sup \left\{ \liminf_n F(x_n, x)(t_n) : t_n \rightarrow 0+, (x_n) \text{ sequence in } A \right\}, \\ \gamma &= \sup \left\{ \liminf_n F(x_n, x)(1/n) : (x_n) \text{ sequence in } A \right\}. \end{aligned} \quad (3.22)$$

Then $\alpha = \beta = \gamma = \bar{A}(x)$.

Proof. If $(x_n) \subset A$, then

$$\bar{A}(x) \geq c(x_n \rightarrow x) = \inf_{t>0} \liminf_n F(x_n, x)(t), \quad (3.23)$$

and so $\bar{A}(x) \geq \alpha$. Assume that $\beta > \theta > 0$. There exist a sequence $(x_n) \in A$ and a sequence (t_n) of positive real numbers, with $t_n \rightarrow 0+$, such that $\liminf_n F(x_n, x)(t_n) > \theta$. Let $t > 0$

and choose k such that $t_n < t$ when $n \geq k$. For $m \geq k$, we have $\inf_{n \geq m} F(x_n, x)(t) \geq \inf_{n \geq m} F(x_n, x)(t_n) > \theta$. Thus $\liminf_n F(x_n, x)(t) > \theta$ for each $t > 0$ and so $\alpha \geq \theta$, which proves that $\alpha \geq \beta$. Clearly $\beta \geq \gamma$. Finally, $N_x(A^c) \geq 1 - \sup_{y \in A} F(y, x)(1/n)$, and so $\sup_{y \in A} F(y, x)(1/n) \geq 1 - N_x(A^c) = \bar{A}(x) = \bar{A}(x) - 1/n$. Hence, for each $n \in \mathbb{N}$, there exists $x_n \in A$ with $F(x_n, x)(1/n) > \bar{A}(x) - 1/n$. Consequently,

$$\gamma \geq \liminf_n F(x_n, x)\left(\frac{1}{n}\right) \geq \liminf_n \left(\bar{A}(x) - \frac{1}{n}\right) = \bar{A}(x), \quad (3.24)$$

and so $\gamma \geq \bar{A}(x) \geq \alpha \geq \beta \geq \gamma$, which completes the proof. $\square$

In view of [6, Theorem 4.14], we have the following corollary.

COROLLARY 3.8. *Every pseudometrizable fuzzifying topological space is $\mathbb{N}$ -sequential and hence sequential.*

THEOREM 3.9. *If (F_n) is a sequence of probabilistic pseudometrics on a set X , then there exists a probabilistic pseudometric F such that $\tau_F = \bigvee_n \tau_{F_n}$.*

Proof. If F is a probabilistic pseudometric on X and if $\bar{F}$ is defined by $\bar{F}(x, y)(t) = F(x, y)(t)$ if $t \leq 1$ and $\bar{F}(x, y)(t) = 1$ if $t > 1$, then $\bar{F}$ is a probabilistic pseudometric on X and $\tau_{\bar{F}} = \tau_F$. Hence, we may assume that $F_n(x, y)(t) = 1$, for all n , if $t > 1$. For $x, y \in X$, define $F(x, y)$ on $\mathbb{R}$ by $F(x, y)(t) = 0$ if $t \leq 0$ and $F(x, y)(t) = \inf_n [(1/n)F_n(x, y)](t)$ if $t > 0$. Clearly, $F(x, y)$ is increasing and $F(x, y)(t) = 1$ if $t > 1$. Also, $F(x, y)$ is left continuous. In fact, let $F(x, y)(t) > \theta > 0$ and choose n such that $(n+1)t > 1$. There exists $0 < s_1 < t$ such that $F_k(x, y)(ks_1) > \theta$ for $k = 1, \dots, n$. Choose $s_1 < s < t$ such that $(n+1)s > 1$. Now, $F_m(x, y)(ms) = 1$ if $m > n$. Thus

$$F(x, y)(s) = \min_{1 \leq k \leq n} \left[\frac{1}{k} F_k(x, y) \right](s) > \theta, \quad (3.25)$$

which proves that $F(x, y)$ is in $\mathbb{R}_\theta^+$. It is clear that $F(x, x) = \bar{0}$. We need to prove that F satisfies the triangle inequality. So assume that $F(x, y)(t_1) \wedge F(y, z)(t_2) > \theta > 0$. If m is such that $(m+1)(t_1 + t_2) > 1$, then

$$F(x, z)(t_1 + t_2) = \min_{1 \leq k \leq m} F_k(x, z)(k(t_1 + t_2)). \quad (3.26)$$

Since

$$F_k(x, z)(k(t_1 + t_2)) \geq F_k(x, y)(kt_1) \wedge F_k(y, z)(kt_2) > \theta, \quad (3.27)$$

it follows that $F(x, z)(t_1 + t_2) > \theta$, and so F satisfies the triangle inequality. We will finish the proof by showing that $\tau_F = \bigvee \tau_{F_n}$. To see this, we first observe that $(1/n)F_n \leq F$, which implies that $\tau_{F_n} = \tau_{(1/n)F_n} \leq \tau_F$, and so $\tau_o = \bigvee_n \tau_{(1/n)F_n} \leq \tau_F$. On the other hand, let $N_x^{\tau_F}(A) > \theta$ and choose $\epsilon > 0$ such that $N_x^{\tau_F}(A) > \theta + \epsilon$. Let $t > 0$ be such that $1 - \sup_{y \notin A} F(y, x)(t) > \theta + \epsilon$. If $(m+1)t > 1$, then

$$F(y, z)(t) = \min_{1 \leq k \leq m} F_k(y, z)(kt). \quad (3.28)$$

Let $A_k = \{y : F_k(y, x)(kt) \geq 1 - \theta - \epsilon\}$. Then

$$N_x^{\tau_o}(A_k) \geq N_x^{\tau_{F_k}}(A_k) \geq 1 - \sup_{z \notin A_k} F_k(z, x)(kt) \geq \theta + \epsilon > \theta \quad (3.29)$$

and $\bigcap_{k=1}^m A_k \subset A$. Hence, $N_x^{\tau_o}(A) \geq \min_{1 \leq k \leq m} N_x^{\tau_o}(A_k) > \theta$. This proves that $\tau_F \leq \tau_o$ and the result follows. $\square$

THEOREM 3.10. *Let $f : X \rightarrow Y$ be a function and let F be a probabilistic pseudometric on Y . Then the function*

$$f^{-1}(F) : X^2 \longrightarrow \mathbb{R}_\phi^{+1}, \quad f^{-1}(F)(x, y) = F(f(x), f(y)), \quad (3.30)$$

is a probabilistic pseudometric on X and $\tau_{f^{-1}(F)} = f^{-1}(\tau_F)$.

Proof. It follows easily that $f^{-1}(F)$ is a probabilistic pseudometric on X . Let $x \in X$ and $B \subset X$. If $D = Y \setminus f(B^c)$, then

$$\begin{aligned} N_x^{\tau_{f^{-1}(F)}}(B) &= \inf_{t>0} \left[1 - \sup_{y \notin B} F(f(y), f(x))(t) \right] \\ &= \inf_{t>0} \left[1 - \sup_{z \in D^c} F(z, f(x))(t) \right] \\ &= N_{f(x)}^{\tau_F}(D) = N_x^{f^{-1}(\tau_F)}(B), \end{aligned} \quad (3.31)$$

which clearly completes the proof. $\square$

COROLLARY 3.11. *If F is a probabilistic pseudometric on a set X and $Y \subset X$, then $\tau_F|_Y$ is induced by the probabilistic pseudometric $G = F|_{Y \times Y}$, $G(x, y) = F(x, y)$.*

COROLLARY 3.12. *If (X_n, τ_n) is a sequence of pseudometrizable fuzzifying topological spaces, then the Cartesian product $(X, \tau) = (\prod X_n, \prod \tau_n)$ is pseudometrizable.*

Proof. Let F_n be a probabilistic pseudometric on X_n inducing τ_n . If $G_n = \pi_n^{-1}(F_n)$, then $\tau_{G_n} = \pi_n^{-1}(\tau_n)$, and so $\tau = \bigvee_n \pi_n^{-1}(\tau_n)$ is pseudometrizable. $\square$

4. Level proximities

Let δ be a fuzzifying proximity on a set X . For each $0 < d \leq 1$, let δ^d be the binary relation on 2^X defined by $A \delta^d B$ if and only if $\delta(A, B) \geq d$. It is easy to see that δ^d is a classical proximity on X . We will show that the classical topology σ_d induced by δ^d coincides with τ^{1-d} . In fact, let $x \in A \in \sigma_d$. Then, x is not in the σ_d -closure of A^c , which implies that $x \delta^d A^c$, that is, $\delta(x, A^c) < d$, and so $N_x^\tau(A) = 1 - \delta(x, A^c) > 1 - d$. This proves that $A \in \tau^{1-d}$. Conversely, if $x \in B \in \tau^{1-d}$, then $N_x^\tau(A) > 1 - d$, and thus $\delta(x, A^c) < d$, which implies that x is not in the σ_d -closure of B^c . Hence B^c is σ_d -closed, and so B is σ_d -open.

THEOREM 4.1. *If δ is a fuzzifying proximity on a set X and $0 < d \leq 1$, then*

$$\delta^d = \bigvee_{0 < \theta < d} \delta^\theta. \quad (4.1)$$

Proof. If $0 < \theta < d$, then δ^θ is coarser than δ^d , and so $\delta_\theta = \bigvee_{0 < \theta < d} \delta^\theta$ is coarser than δ^d . On the other hand, let $A \delta_\theta B$. Since δ_θ is finer than δ^θ (for $0 < \theta < d$), we have that $A \delta^\theta B$ and so $\delta(A, B) \geq \theta$, for each $0 < \theta < d$, which implies that $\delta(A, B) \geq d$, that is, $A \delta^d B$. So δ_θ is finer than δ^d and the result follows. $\square$

THEOREM 4.2. *For a family $\{\gamma_d : 0 < d \leq 1\}$ of classical proximities on a set X , the following are equivalent.*

- (1) *There exists a fuzzifying proximity δ on X such that $\delta^d = \gamma_d$ for all d .*
- (2) *$\gamma_d = \bigvee_{0 < \theta < d} \gamma_\theta$ for each $0 < d \leq 1$.*

Proof. In view of the preceding theorem, (1) implies (2). Assume now that (2) is satisfied and define δ on $2^X \times 2^X$ by $\delta(A, B) = \sup\{d : A \gamma_d B\}$ (the supremum over the empty family is taken to be zero). It is clear that $\delta(A, B) = 1$ if the A, B are not disjoint. Also, $\delta(A, B) = \delta(B, A)$ and $\delta(A, B) \geq \delta(A_1, B_1)$ if $A_1 \subset A$, $B_1 \subset B$. Let now $\delta(A, B) < d < 1$. Then $A \not\gamma_d B$, and so there exists a subset D of X such that $A \not\gamma_d D$ and $D^c \not\gamma_d B$. Since $A \not\gamma_d D$, we have that $\delta(A, D) \leq d$. Similarly $\delta(D^c, B) \leq d$, and so $\inf\{\delta(A, D) \wedge \delta(D^c, B)\} \leq \delta(A, B)$. On the other hand, if $\delta(A, D) \wedge \delta(D^c, B) < \theta < 1$, then $A \subset D^c$, and so $\delta(A, B) \leq \delta(D^c, B) < \theta$. This proves that δ is a fuzzifying proximity on X . We will finish the proof by showing that $\delta^d = \gamma_d$ for all d . Indeed, if $A \gamma_d B$, then $\delta(A, B) \geq d$, that is, $A \delta^d B$. On the other hand, let $A \delta^d B$ and let $(A_i), (B_j)$ be finite families of subsets of X with $A = \bigcup_i A_i$, $B = \bigcup_j B_j$. Since $\delta(A, B) = \bigvee_{i,j} \delta(A_i, B_j) \geq d$, there exists a pair (i, j) such that $\delta(A_i, B_j) \geq d$. If now $0 < \theta < d$, then there exists $r > \theta$ with $A_i \gamma_r B_j$, and so $A_i \gamma_\theta B_j$. This proves that $A \gamma_d B$ since $\gamma_d = \bigvee_{0 < \theta < d} \gamma_\theta$. This completes the proof. $\square$

THEOREM 4.3. *Let $(X, \delta_1), (Y, \delta_2)$ be fuzzifying proximity spaces and let $f : X \rightarrow Y$ be a function. Then f is proximally continuous if and only if $f : (X, \delta_1^d) \rightarrow (Y, \delta_2^d)$ is proximally continuous for each $0 < d \leq 1$.*

Proof. It follows immediately from the definitions. $\square$

THEOREM 4.4. *Let $(X_\lambda, \delta_\lambda)_{\lambda \in \Lambda}$ be a family of fuzzifying proximity spaces and let $(X, \delta) = (\prod X_\lambda, \prod \delta_\lambda)$ be the product fuzzifying proximity space. Then $\delta^d = \prod \delta_\lambda^d$ for all $0 < d \leq 1$.*

Proof. Since each projection $\pi_\lambda : (X, \delta^d) \rightarrow (X_\lambda, \delta_\lambda^d)$ is proximally continuous, it follows that δ^d is finer than $\sigma = \prod \delta_\lambda^d$. On the other hand, let $A \sigma B$. We need to show that $\delta(A, B) \geq d$. In fact, let $(A_i), (B_j)$ be finite families of subsets of X such that $A = \bigcup A_i$, $B = \bigcup B_j$. Since $A \sigma B$ and $\sigma = \bigvee_\lambda \pi_\lambda^{-1}(\delta_\lambda^d)$, there exists a pair (i, j) such that $A_i \pi_\lambda^{-1}(\delta_\lambda^d) B_j$, that is, $\delta_\lambda(\pi_\lambda(A_i), \pi_\lambda(B_j)) \geq d$. In view of [4, Theorem 8.9], we conclude that $\delta(A, B) \geq d$. Hence, $\sigma = \delta^d$ and the proof is complete. $\square$

We have the following easily established theorem.

THEOREM 4.5. *Let (Y, δ) be a fuzzifying proximity space and let $f : X \rightarrow Y$. Then $f^{-1}(\delta)^d = f^{-1}(\delta^d)$ for each $0 < d \leq 1$.*

THEOREM 4.6. Let $(\delta_\lambda)_{\lambda \in \Lambda}$ be a family of fuzzifying proximities on a set X and $\delta = \vee_\lambda \delta_\lambda$. Then $\delta^d = \vee_\lambda \delta_\lambda^d$ for each $0 < d \leq 1$.

Proof. Let $\sigma = \vee_\lambda \delta_\lambda^d$. Since δ is finer than each δ_λ , it follows that δ^d is finer than each δ_λ^d , and so δ^d is finer than σ . On the other hand, let $A\sigma B$ and let $(A_i), (B_j)$ be finite families of subsets of X such that $A = \bigcup A_i, B = \bigcup B_j$. There exists a pair (i, j) such that $A_i\sigma B_j$. Since σ is finer than each δ_λ^d , we have that $A_i\delta_\lambda^d B_j$, that is, $\delta_\lambda(A_i, B_j) \geq d$. In view of [4, Theorem 8.10], we get that $\delta(A, B) \geq d$, that is, $A\delta^d B$. So σ is finer than δ^d and the proof is complete. $\square$

5. Completely regular fuzzifying spaces

Definition 5.1. A fuzzifying topological space (X, τ) is called completely regular if each of the classical level topologies $\tau^d, 0 \leq d < 1$, is completely regular.

Definition 5.2. A fuzzifying proximity δ on a set X is said to be compatible with a fuzzifying topology τ if τ coincides with the fuzzifying topology τ_δ induced by δ .

We have the following easily established theorem.

THEOREM 5.3. Subspaces and Cartesian products of completely regular fuzzifying spaces are completely regular.

THEOREM 5.4. Let (X, τ) be a completely regular fuzzifying topological space and define $\delta = \delta(\tau) : 2^X \times 2^X \rightarrow [0, 1]$ by

$$\delta(A, B) = 1 - \sup \{d : 0 \leq d < 1, \exists f : (X, \tau^d) \longrightarrow [0, 1] \text{ continuous } f(A) = 0, f(B) = 1\}. \quad (5.1)$$

Then, (1) δ is a fuzzifying proximity on X compatible with τ ;

(2) if δ_1 is any fuzzifying proximity on X compatible with τ , then δ is finer than δ_1 .

Proof. It is easy to see that δ satisfies (FP1), (FP2), (FP3), and (FP5). We will prove that δ satisfies (FP4). Let

$$\alpha = \inf \{\delta(A, D) \vee \delta(D^c, B) : D \subset X\}. \quad (5.2)$$

If $\delta(A, D) \vee \delta(D^c, B) < \theta$, then $A \subset D^c$, and so $\delta(A, B) \leq \delta(D^c, B) < \theta$, which proves that $\delta(A, B) \leq \alpha$. On the other hand, assume that $\delta(A, B) < r < 1$. There exist a $d, 1 - r < d < 1$, and $f : X \rightarrow [0, 1]$ τ^d -continuous such that $f(A) = 0, f(B) = 1$. Let $D = \{x \in X : 1/2 \leq f(x) \leq 1\}$ and define $h_1, h_2 : [0, 1] \rightarrow [0, 1], h_1(t) = 2t, h_2(t) = 0$ if $0 \leq t \leq 1/2$ and $h_1(t) = 1, h_2(t) = 2t - 1$ if $1/2 < t \leq 1$. If $g_i = h_i \circ f, i = 1, 2$, then $g_1(A) = 0, g_1(D) = 1, g_2(D^c) = 0, g_2(B) = 1$. Thus, $\delta(A, D) \leq 1 - d < r, \delta(D^c, B) < r$, which proves that $\alpha \leq \delta(A, B)$. Hence, δ is a fuzzifying proximity on X . We need to show that $\tau = \tau_\delta$. So, let $\tau(A) > \theta > 0$. Since τ^θ is completely regular, given $x \in A$, there exist $f_x : X \rightarrow [0, 1], \tau^\theta$ -continuous, $f_x(x) = 0, f_x(A^c) = 1$. Thus $\delta(x, A^c) \leq 1 - \theta$, and so $N_x^{\tau^\theta}(A) = 1 - \delta(x, A^c) \geq \theta$. It follows that $\tau_\delta(A) = \inf_{x \in A} N_x^{\tau^\theta}(A) \geq \theta$, which proves that $\tau_\delta \geq \tau$. On the other hand, assume that $\tau_\delta(A) > r > 0$. If $x \in A$, then $\delta(x, A^c) = 1 - N_x^{\tau^\theta}(A) < 1 - r$, and therefore there exist a $d, 0 < 1 - d < 1 - r$ and $f : X \rightarrow [0, 1]$ τ^d -continuous such that $f(x) = 0, f(A^c) = 1$. The set

$G = \{y : f(y) < 1/2\}$ is in τ^d and $x \in G \subset A$. Thus,

$$N_x^\tau(A) \geq N_x^\tau(G) \geq d > r. \quad (5.3)$$

This proves that $\tau(A) \geq r$ and so $\tau \geq \tau_\delta$, which completes the proof of (1).

Let δ_1 be a fuzzifying proximity on X compatible with τ and let A, B be subsets of X with $\delta_1(A, B) < \theta < 1$. If $d = 1 - \theta$, then δ_1^θ is compatible with τ^d . Since $A \delta_1^\theta B$, there exists (by [8, Remark 3.15]) an $f : X \rightarrow [0, 1]$ τ_d -continuous, with $f(A) = 0$, $f(B) = 1$, and so $\delta(A, B) \leq 1 - d = \theta$, which proves that $\delta(A, B) \leq \delta_1(A, B)$, and therefore δ is finer than δ_1 . This completes the proof. $\square$

THEOREM 5.5. *For a fuzzifying topological space (X, τ) , the following are equivalent.*

- (1) (X, τ) is completely regular.
- (2) There exists a fuzzifying proximity δ on X compatible with τ .
- (3) (X, τ) is fuzzy uniformizable, that is, there exists a fuzzy uniformity $\mathcal{U}$ on X such that τ coincides with the fuzzifying topology $\tau_{\mathcal{U}}$ induced by $\mathcal{U}$.

Proof. By [5], (2) is equivalent to (3). Also (1) implies (2) in view of the preceding theorem. Assume now that $\tau = \tau_\delta$ for some fuzzifying proximity δ . For each $0 < d \leq 1$, δ^d is a classical proximity compatible with τ^{1-d} , and so τ^{1-d} is completely regular. This completes the proof. $\square$

THEOREM 5.6. *Every pseudometrizable fuzzy topological space (X, τ) is completely regular.*

Proof. If τ is pseudometrizable, then each τ^d , $0 \leq d < 1$, is pseudometrizable, and hence τ^d is completely regular. $\square$

THEOREM 5.7. *For a fuzzifying topological space (X, τ) , the following are equivalent.*

- (1) (X, τ) is completely regular.
- (2) If $\mathcal{F} = \mathcal{F}_\tau$ is the family of all probabilistic pseudometrics on X which are $\tau \times \tau$ continuous as functions from X^2 to $\mathbb{R}_\phi^+$, then $\tau = \tau_{\mathcal{F}}$.
- (3) There exists a family $\mathcal{F}$ of probabilistic pseudometrics on X such that $\tau = \tau_{\mathcal{F}}$.

Proof. (1) $\Rightarrow$ (2). For each $F \in \mathcal{F}_\tau$, we have that $\tau_F \leq \tau$ (by Theorem 3.2), and so $\tau_{\mathcal{F}_\tau} \leq \tau$. Let now $A \subset X$ and $x_0 \in X$ with $N_{x_0}^\tau(A) > \theta > 0$. Since τ^θ is completely regular, there exists a τ^θ -continuous function f from X to $[0, 1]$ such that $f(x_0) = 0$, $f(A^c) = 1$. For $x, y \in X$, define $F(x, y)$ on $\mathbb{R}$ by

$$F(x, y)(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 - \theta & \text{if } |f(x) - f(y)| \geq t > 0, \\ 1 & \text{if } |f(x) - f(y)| < t. \end{cases} \quad (5.4)$$

Clearly, $F(x, y) = F(y, x) \in \mathbb{R}_\phi^+$ and $F(x, x) = \bar{0}$. We will prove that F satisfies the triangle inequality. So, assume that $F(x, y)(t_1) \wedge F(y, z)(t_2) > F(x, z)(t_1 + t_2)$. Then, $t_1, t_2 > 0$, $F(x, z)(t_1 + t_2) = 1 - \theta$, $F(x, y)(t_1) = F(y, z)(t_2) = 1$. Thus, $t_1 > |f(x) - f(y)|$, $t_2 > |f(y) - f(z)|$, and hence $|f(x) - f(z)| < t_1 + t_2$, which implies that $F(x, z)(t_1 + t_2) = 1$, a contradiction. So F is a probabilistic pseudometric on X . Next we show that F is $\tau \times \tau$ continuous, or equivalently that $\tau_F \leq \tau$. So assume that $N_{x_0}^{\tau_F}(B) > r > 0$. Let $\theta_1 > r$ be such

that $N_x^{\tau_F}(B) > \theta_1$. Choose $t > 0$ such that $1 - \sup_{y \notin B} F(x, y)(t) > \theta_1$, and so $F(x, y)(t) = 1 - \theta$ and $|f(x) - f(y)| \geq t$ if $y \notin B$. Thus, $\{y : |f(x) - f(y)| < t\} \subset B$. This shows that B is a τ^θ -neighborhood of x . As $r < \theta$, B is a τ^r -neighborhood of x , that is, $N_x^r(B) > r$, and so $\tau_F \leq \tau$. Finally if $y \notin A$, then $|f(y) - f(x_0)| = 1$, and so $F(y, x_0)(1/2) = 1 - \theta$, which implies that

$$N_{x_0}^{\tau_{\mathcal{F}}}(A) \geq N_{x_0}^{\tau_F}(A) \geq 1 - \sup_{y \notin A} F(y, x_0)\left(\frac{1}{2}\right) \geq \theta. \quad (5.5)$$

This shows that $N_{x_0}^{\tau_{\mathcal{F}}} \geq N_{x_0}^{\tau}$, and so $\tau \leq \tau_{\mathcal{F}}$, which completes the proof of the implication (1) $\Rightarrow$ (2).

(3) $\Rightarrow$ (1). Assume that $\tau = \tau_{\mathcal{F}}$ for some family $\mathcal{F}$ of probabilistic pseudometrics on X . For each $F \in \mathcal{F}$, τ_F is completely regular and so $\tau_{\mathcal{F}}$ is completely regular since $\tau_{\mathcal{F}}^d = \bigvee_{F \in \mathcal{F}} \tau_F^d$ for each $0 \leq d < 1$. Hence the result follows. $\square$

We will denote by $[0, 1]_\phi$ the subspace of $\mathbb{R}_\phi^+$ consisting of all $u \in \mathbb{R}_\phi^+$ with $u(t) = 1$ if $t > 1$.

THEOREM 5.8. *A fuzzifying topological space (X, τ) is completely regular if and only if the following condition is satisfied. If $N_{x_0}(A) > \theta > 0$, then there exists $f : X \rightarrow [0, 1]_\phi$ continuous such that $f(x_0) = \bar{0}$ and $f(y)(t) = 1 - \theta$ if $y \notin A$ and $0 < t < 1$.*

Proof. Assume that (X, τ) is completely regular and let $N_{x_0}(A) > \theta > 0$. Since τ^θ is completely regular, there exists $h : (X, \tau^\theta) \rightarrow [0, 1]$ continuous, $h(x_0) = 0$, $h(y) = 1$ if $y \notin A$. For $x, y \in X$, define $F(x, y)$ on $\mathbb{R}$ by

$$F(x, y)(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 - \theta & \text{if } |h(x) - h(x_0)| \geq t > 0, \\ 1 & \text{if } |h(x) - h(x_0)| < t. \end{cases} \quad (5.6)$$

Clearly, $F(x, y) \in [0, 1]_\phi$. Also, $F(x, z) \leq F(x, y) \oplus F(y, z)$. In fact, assume that $F(x, y)(t_1) \wedge F(y, z)(t_2) > r > F(x, z)(t_1 + t_2)$. Then $t_1, t_2 > 0$, $F(x, y)(t_1) = F(y, z)(t_2) = 1$. Now, $|h(x) - h(y)| < t_1$, $|h(y) - h(z)| < t_2$, and so $|h(x) - h(z)| < t_1 + t_2$, which implies that $F(x, z)(t_1 + t_2) = 1$, a contradiction. So F is a probabilistic pseudometric. Moreover, F is $\tau \times \tau$ continuous, or equivalently $\tau_F \leq \tau$. In fact, let $N_x^{\tau_F}(B) > r > 0$. There exists a $t > 0$ such that $1 - \sup_{z \notin B} F(z, x)(t) > r$. If $z \notin B$, then $F(z, x)(t) < 1 - r < 1$, and so $F(z, x)(t) = 1 - \theta < 1 - r$, that is, $r < \theta$, and $|h(z) - h(x)| \geq t$. Hence

$$M = \{z : |h(z) - h(x)| < t\} \subset B. \quad (5.7)$$

The set M is a τ^θ -neighborhood of x , and hence a τ^r -neighborhood, that is, $N_x^r(B) > r$. Thus $\tau \geq \tau_F$. Finally, define $f : X \rightarrow [0, 1]_\phi$, $f(y) = F(y, x_0)$. Then f is τ -continuous, $f(x_0) = \bar{0}$. For $y \notin A$ and $0 < t < 1$, we have that $f(y)(t) = F(y, x_0)(t) = 1 - \theta$ (since $|h(x) - h(x_0)| = 1 \geq t$). Conversely, assume that the condition is satisfied and let $\mathcal{F}$ be the family of all $\tau \times \tau$ continuous pseudometrics on X . Then $\tau_{\mathcal{F}} \leq \tau$. Let $N_{x_0}^{\tau}(A) > \theta$. There

exists a $\theta_1 > \theta$ such that $N_{x_o}^\tau(A) > \theta_1$. By our hypothesis, there exists $f : X \rightarrow [0, 1]_\phi$ continuous such that $f(x_o) = \bar{0}$ and $f(y)(t) = 1 - \theta_1$ if $y \notin A$ and $0 < t < 1$. Define $F(x, y) = D(f(x), f(y))$. Then F is $\tau \times \tau$ continuous and

$$\begin{aligned} N_{x_o}^{\tau_{\mathcal{F}}}(A) &\geq N_{x_o}^{\tau_F}(A) \geq 1 - \sup_{y \notin A} F(x_o, y)(1) \\ &= 1 - \sup_{y \notin A} D(\bar{0}, f(y))(1) \\ &= 1 - \sup_{y \notin A} f(y)(1) \geq \theta_1 > \theta. \end{aligned} \quad (5.8)$$

Thus $N_{x_o}^{\tau_{\mathcal{F}}}(A) \geq N_{x_o}^\tau(A)$, for every subset A of X , and so $\tau \leq \tau_{\mathcal{F}}$. Therefore, $\tau = \tau_{\mathcal{F}}$, and so τ is completely regular. $\square$

For a fuzzifying topological space X , we will denote by $C(X, [0, 1]_\phi)$ the family of all continuous functions from X to $[0, 1]_\phi$.

THEOREM 5.9. *A fuzzifying topological space (X, τ) is completely regular if and only if τ coincides with the weakest of all fuzzifying topologies τ_1 on X for which each $f \in C(X, [0, 1]_\phi)$ is continuous.*

Proof. Assume that (X, τ) is completely regular and let τ_1 be the weakest of all fuzzifying topologies on X for which each $f \in C(X, [0, 1]_\phi)$ is continuous. Clearly $\tau_1 \leq \tau$. On the other hand, let τ_2 be a fuzzifying topology on X for which each $f \in C(X, [0, 1]_\phi)$ is continuous. Let $N_x^\tau(A) > \theta > 0$. In view of the preceding theorem, there exists an $f \in C(X, [0, 1]_\phi)$ such that $f(x) = \bar{0}$, $f(y)(t) = 1 - \theta$ if $y \notin A$ and $0 < t < 1$. Let

$$G = \left\{ u \in \mathbb{R}_\phi^+ : D(f(x), u)\left(\frac{1}{2}\right) = u\left(\frac{1}{2}\right) > 1 - \theta \right\}. \quad (5.9)$$

Then

$$N_{\bar{0}}(G) \geq 1 - \sup_{u \notin G} D(f(x), u)\left(\frac{1}{2}\right) \geq \theta. \quad (5.10)$$

Since f is τ_2 -continuous, we have that $N_x^{\tau_2}(f^{-1}(G)) \geq \theta$. But $f^{-1}(G) \subset A$ since, for $y \notin A$, we have that $f(y)(1/2) = 1 - \theta$. Thus $N_x^{\tau_2}(A) \geq \theta$. This proves that $N_x^{\tau_2}(A) \geq N_x^\tau(A)$, for every subset A of X , and so $\tau_2 \geq \tau$. This clearly proves that $\tau_1 = \tau$. Conversely, assume that $\tau_1 = \tau$. If σ is the usual fuzzifying topology of $\mathbb{R}_\phi^+$, then

$$\tau = \tau_1 = \bigvee_{f \in C(X, [0, 1]_\phi)} f^{-1}(\sigma). \quad (5.11)$$

Since σ is completely regular, each $f^{-1}(\sigma)$ is completely regular, and so τ is completely regular. This completes the proof. $\square$

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