RELATIVE NATURAL CLASSES AND RELATIVE INJECTIVITY

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Let τ be a hereditary torsion theory on the category R-Mod of left R-modules over an associative unitary ring R. We introduce the notion of τ-natural class as a class of modules closed under τ-dense submodules, direct sums, and τ-injective hulls. We study connections between certain conditions involving τ-(quasi-)injectivity in the context of τ-natural classes, generalizing results established by S. S. Page and Y. Q. Zhou (1994) for natural classes.

1. Introduction and preliminaries

The language of natural classes of modules appeared in the early 1990s, allowing to unify similar results that hold for some important classes of modules, such as the category R-Mod of left R-modules, any hereditary torsion-free class of modules, or any stable hereditary torsion class of modules. First, Dauns [2, 3] introduced and studied natural classes for nonsingular modules under the name of saturated classes and, afterwards, Page and Zhou [5, 6] began the study of arbitrary natural classes.

In this paper, we are interested in relativizing the notion of a natural class to the torsion-theoretic framework and discussing connections between conditions on τ-injectivity and τ-quasi-injectivity in the context of τ-natural classes, where τ is an arbitrary hereditary torsion theory on R-Mod.

Now we set the notation and terminology. The reference for general module theory is [8], whereas [1] and [4] will be mainly followed for torsion theories topics. Throughout, R will be an associative ring with nonzero identity and all modules will be left unital R-modules. We will denote by τ a hereditary torsion theory on the category R-Mod of left R-modules.

If τ and σ are two hereditary torsion theories such that every τ-torsion module is σ-torsion, then it is said that σ is a *generalization* of τ and it is denoted by $\tau \leq \sigma$. A submodule B of a module A is called *τ-dense in A* if A/B is τ-torsion. A torsion theory τ is called *noetherian* if for every ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of left ideals of R, the union of which is τ-dense in R, there exists a positive integer k such that I_k is τ-dense in

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R. A nonzero module A is called *τ-cocritical* if A is τ-torsion-free and each of its nonzero submodules is τ-dense in A.

A module is said to be *τ-injective* if it is injective with respect to every monomorphism having a τ-torsion cokernel. For any module A, $E(A)$ and $E_\tau(A)$ denote the injective hull of A and the τ-injective hull of A, respectively. A module A is said to be *τ-quasi-injective* if whenever B is a τ-dense submodule of A, every homomorphism $B \to A$ extends to an endomorphism of A. A module is τ-quasi-injective module if and only if it is a fully invariant submodule of its τ-injective hull [7, Theorem 4.4]. A *τ-quasi-injective hull* of a module B is defined as a τ-quasi-injective module A such that B is a τ-dense essential submodule of A. Every module A has a τ-quasi-injective hull, unique up to an isomorphism [1, Propositions 5.1.8]. A module A is called *$\sum$-τ-(quasi-)injective* if every direct sum of copies of A is τ-(quasi-)injective.

A nonempty class $\mathcal{K}$ of modules is called a *natural class* if $\mathcal{K}$ is closed under isomorphic copies, submodules, direct sums, and injective hulls [5, page 2912]. Motivated by the torsion-theoretic context, a nonzero module A is said to be *$\mathcal{K}$-cocritical* if $A \in \mathcal{K}$ and for every nonzero proper submodule B of A, $A/B \notin \mathcal{K}$ [5, page 2913].

2. τ-(quasi)-injectivity conditions for τ-natural classes

Throughout we will denote by $\mathcal{K}$ a nonempty class of modules closed under isomorphic copies. We will give the following definition.

Definition 2.1. The class $\mathcal{K}$ is called a *τ-natural class* if $\mathcal{K}$ is closed under τ-dense submodules, direct sums, and τ-injective hulls.

Clearly, every natural class is a τ-natural class.

Example 2.2. (i) The category R-Mod, any hereditary torsion-free class of modules and any stable hereditary torsion class of modules are natural classes, hence τ-natural classes.

(ii) Let σ be a hereditary torsion theory such that $\tau \leq \sigma$. Then the class of all σ-torsion modules is a τ-natural class, that is, a natural class if and only if σ is stable.

First we establish some necessary or sufficient conditions under which every direct sum of τ-injective modules in $\mathcal{K}$ is τ-injective.

Theorem 2.3. *Let $\mathcal{K}$ be a τ-natural class and suppose that every direct sum of τ-injective modules in $\mathcal{K}$ is τ-injective. Then every ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of τ-dense left ideals of R such that each $I_{j+1}/I_j \in \mathcal{K}$ terminates.*

Proof. Suppose that $I_1 \subset I_2 \subset \cdots$ is a strictly ascending chain of τ-dense left ideals of R such that each $I_{j+1}/I_j \in \mathcal{K}$. By hypothesis, $E = \bigoplus_j E_\tau(I_{j+1}/I_j) \in \mathcal{K}$ is τ-injective. Let $I = \bigcup_{j=1}^{\infty} I_j$, let $p_j : I_{j+1} \to I_{j+1}/I_j$ be the natural homomorphism, and let $\alpha_j : I_{j+1}/I_j \to E_\tau(I_{j+1}/I_j)$ be the inclusion homomorphism for each j. Since I_{j+1} is τ-dense in R, the τ-injectivity of $E_\tau(I_{j+1}/I_j)$ assures the existence of a homomorphism $\beta_j : R \to E_\tau(I_{j+1}/I_j)$

that extends $\alpha_j p_j$. Hence we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & I_{j+1} \arrow[r] \arrow[d, "p_j"'] & R \arrow[d, dashed, "\beta_j"] \\ & I_{j+1}/I_j \arrow[r, "\alpha_j"'] & E_\tau(I_{j+1}/I_j) \end{tikzcd} \tag{2.1}$$

We may define $f : I \to E$ by $f(x) = (\beta_j(x))_j$ for every $x \in I$. It is easy to check that f is a well-defined homomorphism. Since I is τ-dense in R and E is τ-injective, there exists a homomorphism $g : R \to E$ that extends f. Since $g(1) \subseteq \sum_{j=1}^{n} E_\tau(I_{j+1}/I_j)$ for some positive integer n, we have

$$f(I) = g(I) \subseteq \sum_{j=1}^{n} E_\tau\left(\frac{I_{j+1}}{I_j}\right). \tag{2.2}$$

It follows that $\beta_j(x) = 0$ for every $x \in I$ and every $j > n$. If $x \in I_{n+1}$, then $0 = \beta_{n+1}(x) = x + I_n$. Hence $I_{n+1} = I_n$, a contradiction. □

Remark 2.4. Note that in the proof of Theorem 2.3, each I_{j+1}/I_j is τ-torsion. Hence we have used only the fact that every direct sum of τ-torsion τ-injective modules in $\mathcal{K}$ is τ-injective.

Proposition 2.5. *Let $\mathcal{K}$ be a τ-natural class and suppose that every ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of left ideals of R whose union is τ-dense in R such that each $I_{j+1}/I_j \in \mathcal{K}$ terminates. Then every direct sum of τ-injective modules in $\mathcal{K}$ is τ-injective.*

Proof. It is sufficient to prove that every countable direct sum of τ-injective modules in $\mathcal{K}$ is τ-injective (see [4, page 384]). Let $A = \bigoplus_{i=1}^{\infty} A_i$ be a countable direct sum of τ-injective modules in $\mathcal{K}$. Also let I be a τ-dense left ideal of R and let $f : I \to A$ be a homomorphism. For each n, denote

$$I_n = \left\{ x \in I \mid f(x) \in \bigoplus_{i=1}^{n} A_i \right\}. \tag{2.3}$$

Clearly $I_1 \subseteq I_2 \subseteq \cdots$ and $\bigcup_{j=1}^{\infty} I_j = I$. We may consider the monomorphism

$$\alpha_n : \frac{I_{n+1}}{I_n} \longrightarrow \frac{\left(\bigoplus_{i=1}^{n+1} A_i\right)}{\left(\bigoplus_{i=1}^{n} A_i\right)} \tag{2.4}$$

defined by $\alpha_n(x + I_n) = f(x) + (\bigoplus_{i=1}^{n} A_i)$ for every $x \in I_{n+1}$. Since the codomain of α_n is isomorphic to $A_{n+1} \in \mathcal{K}$, we have $I_{n+1}/I_n \in \mathcal{K}$. By hypothesis, there is a positive integer k such that $I_{k+j} = I_k$ for each j. Then $f(I) \subseteq \bigoplus_{i=1}^{k} A_i$. Since $\bigoplus_{i=1}^{k} A_i$ is τ-injective, there exists a homomorphism $g : R \to \bigoplus_{i=1}^{k} A_i \subseteq A$ that extends f. Thus A is τ-injective. □

Corollary 2.6. *Let τ be noetherian and $\mathcal{K}$ a τ-natural class. Then the following statements are equivalent.*

(i) *Every direct sum of τ-injective modules in $\mathcal{K}$ is τ-injective.*

(ii) *Every ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of τ-dense left ideals of R such that each $I_{j+1}/I_j \in \mathcal{K}$ terminates.*

Proof. (i)$\Rightarrow$(ii) by Theorem 2.3.

(ii)$\Rightarrow$(i). Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of left ideals of R whose union is τ-dense in R such that each $I_{j+1}/I_j \in \mathcal{K}$. Since τ is noetherian, there exists a positive integer k such that I_k is τ-dense in R. Then I_n is τ-dense in R for every $n \geq k$. By hypothesis, the chain $I_k \subseteq I_{k+1} \subseteq \cdots$ terminates, hence the chain $I_1 \subseteq I_2 \subseteq \cdots I_k \subseteq I_{k+1} \subseteq \cdots$ terminates. Now use Proposition 2.5. □

Following [5], denote by $H_{\mathcal{K}}(R)$ the set of left ideals I of R such that $R/I \in \mathcal{K}$.

Theorem 2.7. *Let $\mathcal{K}$ be a τ-natural class and suppose that every ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of τ-dense left ideals of R such that each $I_{j+1}/I_j \in \mathcal{K}$ terminates. Then $H_{\mathcal{K}}(R)$ has ACC on τ-dense left ideals.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of τ-dense left ideals in $H_{\mathcal{K}}(R)$. Then each $R/I_j \in \mathcal{K}$. Since $\mathcal{K}$ is closed under τ-dense submodules, each $I_{j+1}/I_j \in \mathcal{K}$. By hypothesis, the above chain terminates. □

In what follows, we will establish connections between some conditions involving τ-quasi-injective modules in the context of τ-natural classes.

We need the following lemma, that generalizes a classical result for quasi-injective modules.

Lemma 2.8. *A module A is τ-injective if and only if $A \oplus E_\tau(A)$ is τ-quasi-injective.*

Proof. The direct implication is obvious. Suppose now that $A \oplus E_\tau(A)$ is τ-quasi-injective. Consider the exact sequence $0 \to A \xrightarrow{i} E_\tau(A) \xrightarrow{p} E_\tau(A)/A \to 0$, where i is the inclusion homomorphism and p is the natural homomorphism. Denote $j = 1_A \oplus i$ and let $\alpha_1 : A \to A \oplus E_\tau(A)$ and $\alpha_2 : E_\tau(A) \to A \oplus E_\tau(A)$ be the inclusions into A and $E_\tau(A)$, respectively, let $\beta : A \to A \oplus A$ be the inclusion into the second summand and let $\sigma : A \oplus A \to A \oplus E_\tau(A)$ be defined by $\sigma(a_1, a_2) = (a_2, a_1)$ for every $(a_1, a_2) \in A \oplus A$. Now consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\alpha_1"'] & A \oplus A \arrow[r, "j"] \arrow[dl, "\sigma"] & A \oplus E_\tau(A) \arrow[dl, dashed, "\gamma"] \\ A \oplus E_\tau(A) & & \end{tikzcd} \tag{2.5}$$

Since $A \oplus A$ is τ-dense in $E_\tau(A \oplus A) = E_\tau(A) \oplus E_\tau(A)$, it follows that $A \oplus A$ is τ-dense in $A \oplus E_\tau(A)$. But $A \oplus E_\tau(A)$ is τ-quasi-injective, hence there exists a homomorphism $\gamma : A \oplus E_\tau(A) \to A \oplus E_\tau(A)$ such that $\gamma j = \sigma$. Now let $\pi : A \oplus E_\tau(A) \to A$ be the projection

and take $\delta = \pi\gamma\alpha_2$. Then we have $\delta i = \pi\gamma\alpha_2 i = \pi\gamma j\beta = \pi\sigma\beta = \pi\alpha_1 = 1_A$, hence A is a direct summand of $E_\tau(A)$. But A is essential in $E_\tau(A)$, hence we must have $A = E_\tau(A)$, showing that A is τ-injective. □

Theorem 2.9. *Let $\mathcal{K}$ be a τ-natural class. The following statements are equivalent.*

(i) *Every direct sum of τ-quasi-injective modules in $\mathcal{K}$ is τ-quasi-injective.*

(ii) *Every direct sum of τ-injective modules in $\mathcal{K}$ is τ-injective and every τ-quasi-injective module in $\mathcal{K}$ is τ-injective.*

Proof. (i)⇒(ii). Let $A \in \mathcal{K}$ be a τ-quasi-injective module. Since $\mathcal{K}$ is a τ-natural class, $E_\tau(A) \in \mathcal{K}$. By hypothesis, $A \oplus E_\tau(A)$ is τ-quasi-injective. Now by Lemma 2.8, A is τ-injective. Therefore every τ-quasi-injective module in $\mathcal{K}$ is τ-injective.

Now let $A = \bigoplus_{i\in I} A_i$, where each A_i is a τ-injective module in $\mathcal{K}$. Hence each A_i is a τ-quasi-injective module in $\mathcal{K}$. By hypothesis, A is a τ-quasi-injective module in $\mathcal{K}$. By the first part, A is a τ-injective module in $\mathcal{K}$. Therefore every direct sum of τ-injective modules in $\mathcal{K}$ is τ-injective.

(ii)⇒(i). Let $A = \bigoplus_{i\in I} A_i$, where each A_i is a τ-quasi-injective module in $\mathcal{K}$. By hypothesis, each A_i is a τ-injective module in $\mathcal{K}$ and A is a τ-injective module in $\mathcal{K}$. Hence A is a τ-quasi-injective module in $\mathcal{K}$. □

Now consider the following condition.

(C) *For every ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of left ideals of $H_{\mathcal{K}}(R)$, $\bigcup_{j=1}^{\infty} I_j \in H_{\mathcal{K}}(R)$* [5].

For the reader's convenience, we recall some preliminary lemmas. We will consider the same definition of a $\mathcal{K}$-cocritical module for a τ-natural class $\mathcal{K}$ as in the case of a natural class $\mathcal{K}$.

Lemma 2.10 (see [5, Lemma 4]). *Let $\mathcal{K}$ be a natural class and let A be a $\mathcal{K}$-cocritical module. Then A is uniform and any nonzero homomorphism from a submodule of A to a module of $\mathcal{K}$ is a monomorphism. In particular, the class of $\mathcal{K}$-cocritical modules is closed under nonzero submodules.*

Lemma 2.11 (see [5, Lemma 6]). *Let $\mathcal{K}$ be a natural class. If condition (C) holds, then every cyclic module in $\mathcal{K}$ has a $\mathcal{K}$-cocritical homomorphic image.*

Lemma 2.12 (see [5, Lemma 7]). *Let A be a module and let $a_1,\dots,a_n \in A$. If all homomorphic images of $Ra_1,\dots,Ra_n$ which are submodules of $E(A)$ have finite uniform dimension, then $E(Ra_1)+\cdots+E(Ra_n)$ has finite uniform dimension.*

Now we are able to prove the following result, but for a natural class $\mathcal{K}$.

Theorem 2.13. *Let $\mathcal{K}$ be a natural class and suppose that condition (C) holds and every τ-quasi-injective module in $\mathcal{K}$ is τ-injective. Then $H_{\mathcal{K}}(R)$ has ACC on τ-dense left ideals.*

Proof. Suppose that $I_1 \subset I_2 \subset \cdots$ is a strictly ascending chain of τ-dense left ideals of $H_{\mathcal{K}}(R)$. Then $I_{j+1}/I_j \in \mathcal{K}$ for each j. By Lemma 2.11, there exist U_j and V_{j+1} such that $I_j \subseteq U_j \subset V_{j+1} \subseteq I_{j+1}$ and V_{j+1}/U_j is a cyclic $\mathcal{K}$-cocritical module. Since I_j is τ-dense in R, V_{j+1} is τ-dense in R, so that V_{j+1}/U_j is τ-dense in R/U_j. Now let $\alpha_j : V_{j+1}/U_j \to E_\tau(V_{j+1}/U_j)$ be the inclusion homomorphism for each j. By the τ-injectivity of $E_\tau(V_{j+1}/U_j)$, there exists a homomorphism $\beta_j : R/U_j \to E_\tau(V_{j+1}/U_j)$ that extends α_j. Denote $I = \bigcup_{j=1}^{\infty} I_j$ and

$A = \bigoplus_j E_\tau(V_{j+1}/U_j)$. Since $E_\tau(V_{j+1}/U_j) \in \mathcal{K}$, we have $A \in \mathcal{K}$. We may define $f : I \to A$ by $f(x) = (\beta_j(x + U_j))_j$ for every $x \in I$. It is easy to check that f is a well-defined homomorphism. Let

$$Q = \sum \{h(A) \mid h \in \mathrm{End}_R(E_\tau(A))\} \tag{2.6}$$

be the τ-quasi-injective hull of A (see [1, Proposition 5.1.7]). We have $E_\tau(A) \in \mathcal{K}$, hence $Q \in \mathcal{K}$. Then Q is τ-injective. Since I is τ-dense in R, there exists a homomorphism $g : R \to Q$ such that the following diagram, where the unspecified homomorphisms are inclusions, is commutative:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] & R \arrow[d, dashed, "g"] \\ & A \arrow[r] & Q \end{tikzcd} \tag{2.7}$$

Then we have $g(1) \in N = \sum_{k=1}^{t} \sum_{j=1}^{s} h_k(E_\tau(V_{j+1}/U_j))$ for some positive integers t and s, whence it follows that $f(I) \subseteq N$. By Lemma 2.10, $h_k(V_{j+1}/U_j) \cong V_{j+1}/U_j$ is a cyclic $\mathcal{K}$-cocritical module. Moreover, $E_\tau(h_k(V_{j+1}/U_j)) = h_k(E_\tau(V_{j+1}/U_j))$. By Lemma 2.12 and again by Lemma 2.10, N has a finite uniform dimension.

On the other hand, $E_\tau(f(V_2)) = E_\tau(V_2/U_1)$ and $f(V_2) \subseteq f(V_3) \subseteq E_\tau(V_2/U_1) \oplus V_3/U_2$. Since $f(V_3) \nsubseteq E_\tau(f(V_2))$ and all V_{j+1}/U_j are uniform, it follows that $E_\tau(f(V_3)) = E_\tau(V_2/U_1) \oplus E_\tau(V_3/U_2)$. Similarly, for each positive integer n, we have

$$E_\tau(f(V_n)) = E_\tau\left(\frac{V_2}{U_1}\right) \oplus E_\tau\left(\frac{V_3}{U_2}\right) \oplus \cdots \oplus E_\tau\left(\frac{V_{n+1}}{U_n}\right). \tag{2.8}$$

But this means that $E(f(I))$ and, consequently, $f(I)$ have infinite uniform dimension, a contradiction. □

Proposition 2.14. *Let σ be a hereditary torsion theory such that $\tau \leq \sigma$ and let $\mathcal{K}$ be the class of all σ-torsion modules. Suppose also that every τ-quasi-injective module in $\mathcal{K}$ is τ-injective. Then $H_{\mathcal{K}}(R)$ has ACC on τ-dense left ideals.*

Proof. Clearly, $\mathcal{K}$ is a τ-natural class. Note also that the set of all σ-dense left ideals of R is exactly $H_{\mathcal{K}}(R)$, hence condition (C) holds for $\mathcal{K}$. Let A be a $\mathcal{K}$-cocritical module. If there exists a nonzero submodule B of A, then A/B is σ-torsion, that is, $A/B \in \mathcal{K}$, a contradiction. Hence A is simple and thus uniform. Therefore every $\mathcal{K}$-cocritical module is simple.

We mention that Lemma 2.11 holds for this particular τ-natural class $\mathcal{K}$, the proofs being identical. Note also that since every τ-torsion module is σ-torsion, the set of τ-dense left ideals of R is contained in $H_{\mathcal{K}}(R)$. Now the result follows by the same arguments as in the proof of Theorem 2.13. □

The following result on τ-quasi-injective modules will be useful.

Lemma 2.15. *Let A be a τ-quasi-injective module. If $(E_\tau(A))^{(I)}$ is τ-injective, then $A^{(I)}$ is τ-quasi-injective for every set I.*

Proof. The proof is immediate, using the fact that if C is a fully invariant submodule of a module B, then $C^{(I)}$ is a fully invariant submodule of $B^{(I)}$ for every set I. □

Theorem 2.16. *Let $\mathcal{K}$ be a τ-natural class and suppose that every τ-quasi-injective module in $\mathcal{K}$ is $\sum$-τ-quasi-injective. Then $H_{\mathcal{K}}(R)$ has ACC on τ-dense left ideals.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of τ-dense left ideals of R such that each $I_j \in \mathcal{H}_{\mathcal{K}}(R)$. Denote $E_j = E_\tau(R/I_j)$ and $A = \bigoplus_{j=1}^{\infty} E_j$. Clearly each $E_j \in \mathcal{K}$, hence $A \in \mathcal{K}$. Let $p_j : A \to E_j$ be the projection and consider the following diagram, where the unspecified homomorphisms are inclusions:

$$\begin{tikzcd} 0 \arrow[r] & E_j \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[dl, "p_j"'] & E_\tau(A) \arrow[dl, dashed, "q_j"] \\ & E_j & & \end{tikzcd} \tag{2.9}$$

Since A is τ-dense in $E_\tau(A)$ and E_j is τ-injective, there exists a homomorphism $q_j : E_\tau(A) \to E_j$ that extends p_j. Then E_j is a direct summand of $E_\tau(A)$, hence $E_\tau(A) = E_j \oplus C_j$ for some submodule C_j of $E_\tau(A)$. We have

$$(E_\tau(A))^{(\mathbb{N})} \cong \bigoplus_{j=1}^{\infty} (E_j \oplus C_j) = \left(\bigoplus_{j=1}^{\infty} E_j\right) \oplus \left(\bigoplus_{j=1}^{\infty} C_j\right) = A \oplus \left(\bigoplus_{j=1}^{\infty} C_j\right). \tag{2.10}$$

By hypothesis, $(E_\tau(A))^{(\mathbb{N})}$ is τ-quasi-injective, hence A is τ-quasi-injective by Lemma 2.15. Denote $I = \bigcup_{j=1}^{\infty} I_j$. For each j, define a homomorphism $f_j : I/I_1 \to E_j$ by $f_j(x + I_1) = x + I_j$ for every $x \in I$. Then we may define a homomorphism $f : I/I_1 \to A$ by $f(x + I_1) = (f_j(x))_j$ for every $x \in I$. It is easy to check that f is well defined. Consider the following diagram, where the unspecified homomorphisms are inclusions:

$$\begin{tikzcd} 0 \arrow[r] & I/I_1 \arrow[r] \arrow[d, "f"'] & R/I_1 \arrow[r] & E_1 \arrow[d] \\ & A & & A \arrow[ll, dashed, "g"] \end{tikzcd} \tag{2.11}$$

Note that I is τ-dense in R, hence I/I_1 is τ-dense in R/I_1. Clearly, R/I_1 is τ-dense in E_1. Further, $A/E_1 \cong \bigoplus_{j=2}^{\infty} E_j$ is τ-torsion because each $E_j = E_\tau(R/I_j)$ is τ-torsion. Hence E_1 is τ-dense in A. It follows that I/I_1 is τ-dense in A. Now since A is τ-quasi-injective, there exists a homomorphism $g : A \to A$ that extends f. It follows that $f(I/I_1) \subseteq g(R/I_1) \subseteq A$. Since $a = g(1 + I_1) \in A$, we have $f(I/I_1) \subseteq Ra \subseteq \bigoplus_{j=1}^{n} E_j$ for some positive integer n. Then $I_{n+1} = I_{n+2} = \cdots = I$. □

Theorem 2.17. *Let $\mathcal{K}$ be a τ-natural class and suppose that every τ-injective module in $\mathcal{K}$ is $\sum$-τ-injective. Then every τ-quasi-injective module in $\mathcal{K}$ is $\sum$-τ-quasi-injective.*

Proof. Let A be a τ-quasi-injective module in $\mathcal{K}$ and let I be a set. By hypothesis, $(E_\tau(A))^{(I)}$ is τ-injective. Then by Lemma 2.15, $A^{(I)}$ is τ-quasi-injective. Hence A is $\sum$-τ-quasi-injective. □

Theorem 2.18. *Let $\mathcal{K}$ be a natural class. The following statements are equivalent.*

(i) *Every τ-injective module in $\mathcal{K}$ is injective.*

(ii) *Every τ-quasi-injective module in $\mathcal{K}$ is quasi-injective.*

Proof. (i)$\Rightarrow$(ii). Let A be a τ-quasi-injective module in $\mathcal{K}$. Then A is a fully invariant submodule of $E_\tau(A)$. But $E_\tau(A) = E(A)$. Hence A is a fully invariant submodule of $E(A)$, that is, A is quasi-injective.

(ii)$\Rightarrow$(i). Let A be a τ-injective module in $\mathcal{K}$. Then A is a τ-quasi-injective module in $\mathcal{K}$, hence A is quasi-injective by hypothesis. Clearly, $A \oplus E(A) \in \mathcal{K}$. Moreover, $A \oplus E(A)$ is τ-injective, hence τ-quasi-injective. By hypothesis, $A \oplus E(A)$ is quasi-injective. Now by Lemma 2.8 applied for the improper torsion theory (i.e., the torsion theory whose torsion class consists of all modules), it follows that A is injective. □

Remark 2.19. If $\mathcal{K}$ is a natural class and τ is the improper torsion theory, Theorems 2.3, 2.7, 2.9, 2.13, 2.16 yield results of Page and Zhou [5, 6].

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