COPULA AND SEMICOPULA TRANSFORMS

FABRIZIO DURANTE AND CARLO SEMPI

Received 22 April 2004

In memory of Bruno Bassan, friend and colleague

We characterize the transformation, defined for every copula C, by $C_h(x,y) := h^{[-1]}(C(h(x),h(y)))$, where x and y belong to $[0,1]$ and h is a strictly increasing and continuous function on $[0,1]$. We study this transformation also in the class of quasi-copulas and semicopulas.

1. Introduction

The notion of copula was introduced by Sklar [24] who proved the theorem that now bears his name; it is commonly used in probability and statistics (see, for instance, [19, 22, 23]). Later, in order to characterize a class of operations on distribution functions that derive from operations on random variables defined on the *same* probability space, Alsina et al. [1] introduced the notion of quasi-copula (see also [12, 20, 27]). On the contrary, the notion of semicopula is recent [3, 8] and arises from a statistical application: the study of multivariate aging through the analysis of the Schur concavity of the survival function (see [2, 25]). Semicopulas generalize triangular norms (briefly t-norms), introduced by K. Menger in order to extend the triangle inequality from the setting of metric spaces to probabilistic metric spaces, and successfully used in probability theory, mathematical statistics, and fuzzy logic [15, 22]. We refer to our paper [8] for the properties of semicopulas. Here we recall that a semicopula is a function $S:[0,1]^2 \to [0,1]$ that satisfies the following two conditions:

$$\begin{aligned} &\forall x \text{ in } [0,1] \quad S(x,1) = S(1,x) = x, \\ &S(x,y) \text{ is increasing in each place.} \end{aligned} \tag{1.1}$$

As a consequence of (1.1), given a semicopula S, one has, for all x and y in $[0,1]$,

$$Z(x,y) \le S(x,y) \le M(x,y), \tag{1.2}$$

where $M(x,y) = \min\{x,y\}$ and

$$Z(x,y) = \begin{cases} 0, & (x,y) \in [0,1[^2, \\ \min\{x,y\}, & \text{elsewhere.} \end{cases} \tag{1.3}$$

International Journal of Mathematics and Mathematical Sciences 2005:4 (2005) 645–655
DOI: 10.1155/IJMMS.2005.645

If a semicopula C is 2-increasing, namely, for all x, x', y, y' in $]0,1]$ with $x \le x'$ and $y \le y'$, C satisfies the inequality

$$C(x', y') - C(x, y') - C(x', y) + C(x, y) \ge 0, \tag{1.4}$$

then it is a copula (see [19]).

If a semicopula Q satisfies the 1-Lipschitz condition, namely,

$$\forall x, x', y, y' \in [0,1], \quad |Q(x,y) - Q(x',y')| \le |x - x'| + |y - y'|, \tag{1.5}$$

then it is a quasi-copula.

If a semicopula T is both commutative

$$\forall x, y \text{ in } [0,1], \quad T(x,y) = T(y,x), \tag{1.6}$$

and associative

$$\forall x, y, z \text{ in } [0,1], \quad T(T(x,y),z) = T(x, T(y,z)), \tag{1.7}$$

then it is a t-norm (see [15, 22]).

The class $\mathcal{S}$ of semicopulas strictly includes the class $\mathcal{Q}$ of quasi-copulas, which, in its turn, strictly includes the class $\mathcal{C}$ of copulas, $\mathcal{C} \subset \mathcal{Q} \subset \mathcal{S}$. Moreover, we will denote by $\mathcal{S}_E$ and $\mathcal{S}_C$, respectively, the subsets of commutative (i.e., exchangeable) and continuous semicopulas. The class $\mathcal{S}_C$ strictly includes $\mathcal{Q}$ and $\mathcal{S}_E$ strictly includes the set $\mathcal{T}$ of t-norms (see [8, 9]).

Notice that the notion of semicopula is new in a statistical context but is not new in general, since it has appeared in other contexts several times.

The first appearance of which we are aware is in [22, Definition 7.1.5], where the authors introduce the set $\mathcal{I}$ of binary operations on $[0,1]$ that are nondecreasing in each place and have 1 as the neutral element. By the way, at the same time, they also introduce the subset $\mathcal{I}_C$ of all the functions $T \in \mathcal{I}$ that satisfy (1.5), namely, the set of quasi-copulas!

Then it was again "introduced" in [26] under the name of *t-seminorm*. Finally, in other words, a semicopula is a *binary aggregation operator* with neutral element 1 [4] or a *conjunctor* [14].

In Section 2, we will study transformations of semicopulas via a continuous and strictly increasing function on $[0,1]$. In Sections 3 and 4, these transformations will be characterized, respectively, on the class of copulas and quasi-copulas.

2. The transform of semicopulas

Given a function $h : [0,1] \to [0,1]$ that is continuous and strictly increasing with $h(1) = 1$, its *pseudoinverse* is the function $h^{[-1]} : [0,1] \to [0,1]$ defined for all $t \in [0,1]$ by

$$h^{[-1]}(t) := \begin{cases} h^{-1}(t), & h(0) \le t \le 1, \\ 0, & 0 \le t \le h(0). \end{cases} \tag{2.1}$$

We denote by Θ the set of all the functions h so defined and we will also consider the subset Θ_i of Θ defined by those $h \in \Theta$ for which $h(0) = 0$; the functions in Θ_i are invertible and the pseudoinverse coincides with the inverse of h, $h^{[-1]} = h^{-1}$.

Proposition 2.1. *For all h and g in Θ,*

(a) *$h^{[-1]}$ is continuous and strictly increasing in $[h(0),1]$;*
(b) *for all $t \in [0,1]$, $h^{[-1]}(h(t)) = t$ and $h(h^{[-1]}(t)) = \max\{t, h(0)\}$;*
(c) *$(h \circ g)^{[-1]} = g^{[-1]} \circ h^{[-1]}$.*

Proof. Statements (a) and (b) are easily proved. In order to prove (c), let h and g be in Θ. Then, for all $t \in [0,1]$, one has

$$\begin{aligned}
(h \circ g)^{[-1]}(t) &= \begin{cases} (h \circ g)^{-1}(t), & t \in [(h \circ g)(0), 1], \\ 0, & \text{otherwise}, \end{cases} \\
g^{[-1]}(h^{[-1]}(t)) &= \begin{cases} g^{-1}(h^{[-1]}(t)), & g(0) \le h^{[-1]}(t) \le 1, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} g^{-1}(h^{-1}(t)), & t \in D, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned} \tag{2.2}$$

where

$$D := \{t \in [h(0),1] : g(0) \le h^{[-1]}(t) \le 1\} = [(h \circ g)(0), 1], \tag{2.3}$$

which proves assertion (c). □

More details on pseudoinverses can be found in [15, Chapter 3]. The following theorem is basic for what follows and for the applications.

Theorem 2.2. *For all $h \in \Theta$ and $S \in \mathcal{S}$, the function $S_h : [0,1]^2 \to [0,1]$, defined, for all x and y in $[0,1]$, by*

$$S_h(x,y) := h^{[-1]}(S(h(x), h(y))), \tag{2.4}$$

is a semicopula. Moreover, if S is continuous, also its transform S_h is continuous.

Proof. If t is in $[0,1]$, then

$$S_h(t,1) = h^{[-1]}(S(h(t), h(1))) = h^{[-1]}(h(t)) = t = S_h(1,t). \tag{2.5}$$

Let x, x', y be in $[0,1]$ with $x \le x'$. Then

$$\begin{aligned}
h(x) \le h(x') &\Longrightarrow S(h(x), h(y)) \le S(h(x'), h(y)) \\
&\Longrightarrow h^{[-1]}(S(h(x), h(y))) \le h^{[-1]}(S(h(x'), h(y))),
\end{aligned} \tag{2.6}$$

namely, $x \mapsto S_h(x,y)$ is increasing; similarly, one proves that $y \mapsto S_h(x,y)$ is increasing. □

Theorem 2.2 introduces a mapping $\Psi : \mathcal{S} \times \Theta \to \mathcal{S}$ defined, for all x and y in $[0,1]$, by

$$\Psi(S,h)(x,y) := h^{[-1]}(S(h(x),h(y))). \tag{2.7}$$

We will often set

$$\Psi_h S := \Psi(S,h). \tag{2.8}$$

The set $\{\Psi_h, h \in \Theta\}$ is closed with respect to the composition operator $\circ$. Moreover, given $h,g \in \Theta$, for all $S \in \mathcal{S}$, one has

$$\begin{aligned}(\Psi_g \circ \Psi_h)(S(x,y)) &= \Psi(\Psi(S,h),g)(x,y) = g^{[-1]}(\Psi_h S(g(x),g(y))) \\ &= g^{[-1]}(h^{[-1]}S((h\circ g)(x),(h\circ g)(y))) \\ &= (h\circ g)^{[-1]}(S((h\circ g)(x),(h\circ g)(y))) = \Psi_{h\circ g}S(x,y).\end{aligned} \tag{2.9}$$

The identity mapping in $\mathcal{S}$, which coincides with $\Psi_{id_{[0,1]}}$, is, obviously, the neutral element of the composition operator $\circ$ in $\{\Psi_h, h \in \Theta\}$. Notice that only if $h \in \Theta_i$, does Ψ_h admit an inverse function given by $\Psi_h^{-1} = \Psi_{h^{-1}}$. Notice also that the mapping $\Psi : \mathcal{S} \times \Theta_i \to \mathcal{S}$ is the *action* of the group Θ_i on $\mathcal{S}$. Moreover, for all $h \in \Theta$, one has $\Psi_h M = M$ and $\Psi_h Z = Z$.

Remark 2.3. If $\Pi(x,y) = xy$ is the copula of independence, then, for all $h \in \Theta$, $\Psi_h\Pi$ is an Archimedean and continuous t-norm; moreover, the operation Ψ gives rise to the whole family $\mathcal{A}_C$ of continuous Archimedean t-norms (written with a multiplicative generator),

$$\mathcal{A}_C = \{\Psi_h\Pi : h \in \Theta\}. \tag{2.10}$$

We recall that an Archimedean t-norm T can be represented in the form

$$T(x,y) = g^{[-1]}(g(x)+g(y)), \tag{2.11}$$

where g is an additive generator, or in the form

$$T(x,y) = h^{[-1]}(h(x)h(y)), \tag{2.12}$$

where h is a multiplicative generator.

In the class $\mathcal{S}$ of semicopulas, one can introduce the usual pointwise order: for all $S, S' \in \mathcal{S}$, one puts $S \prec S'$ if $S(x,y) \le S'(x,y)$, for all $x,y \in [0,1]$.

Proposition 2.4. *Given S and S' in $\mathcal{S}$, and h in Θ,*

(a) *the operation Ψ is order-preserving in the first place, that is, if $S \prec S'$, then $\Psi_h S \prec \Psi_h S'$;*

(b) *if $\Psi_h S \prec \Psi_h S'$, then $S(x,y) \le S'(x,y)$ for all $(x,y) \in [h(0),1]^2$.*

Definition 2.5. A subset $\mathcal{B}$ of $\mathcal{S}$ is said to be *stable* (or *closed*) with respect to (or under) Ψ if the image of $\mathcal{B} \times \Theta$ under Ψ is contained in $\mathcal{B}$, $\Psi_h\mathcal{B} \subset \mathcal{B}$ for every $h \in \Theta$.

It is easily proved that the subsets $\mathcal{S}_E$ and $\mathcal{S}_C$ are closed under Ψ. Moreover, the following result can be proved (see also [15, 22]).

Proposition 2.6. *The class $\mathcal{T}$ of all t-norms is closed under Ψ.*

Proof. For each $h \in \Theta$ and $T \in \mathcal{T}$, it suffices to show that the function $T_h := \Psi_h T$, defined by

$$\forall x, y \in [0,1] \quad T_h(x,y) := h^{[-1]}(T(h(x),h(y))), \tag{2.13}$$

is associative, namely, it satisfies (1.7). Set $\delta := h(0) \geq 0$. Then, if s, t, and u all belong to $[0,1]$, simple calculations lead to the following two expressions:

$$\begin{aligned} T_h[T_h(s,t),u] &= h^{[-1]}T[T(h(s),h(t)) \vee \delta, h(u)], \\ T_h[s,T_h(t,u)] &= h^{[-1]}T[h(s),T(h(t),h(u)) \vee \delta]. \end{aligned} \tag{2.14}$$

If $T(h(s),h(t)) \leq \delta$, then one has

$$T_h[T_h(s,t),u] = h^{[-1]}(T(\delta,h(u))) \leq h^{[-1]}(\delta) = 0, \tag{2.15}$$

and either

$$\begin{aligned} T_h[s,T_h(t,u)] &= h^{[-1]}(T(h(s),T(h(t),h(u)))) \\ &= h^{[-1]}(T(T(h(s),h(t)),h(u))) \\ &\leq h^{[-1]}(T(\delta,h(u))) \leq h^{[-1]}(\delta) = 0 \end{aligned} \tag{2.16}$$

or

$$T_h[s,T_h(t,u)] = h^{[-1]}(T(h(s),\delta)) \leq h^{[-1]}(\delta) = 0. \tag{2.17}$$

Therefore the associativity equation holds.

If $T[h(s),h(t)] > \delta$, the considerations are analogous. □

The proof of the following proposition is immediate and will therefore not be reproduced here.

Proposition 2.7. *The class $\mathcal{A}_C$ is closed under Ψ. In particular, if g is a multiplicative generator of the Archimedean and continuous t-norm A, then $g \circ h$ is a multiplicative generator of $\Psi_h A$.*

It follows from the definition of the operator Ψ that $\Psi_h C$ is a semicopula for all $h \in \Theta$ and for every copula $C \in \mathcal{C}$. However, it is easily checked that $\Psi_h C$ need not be a copula. In order to see this, take $C = \Pi$ so that Remark 2.3 ensures that $\Psi_h \Pi$ is an Archimedean and continuous t-norm for every $h \in \Theta$. Now it suffices to recall that a t-norm is a copula if, and only if, its additive generator is convex [22, Theorem 6.3.3] and, then, to choose h in such a way that the corresponding additive generator $t \mapsto \varphi(t) = -\ln h(t)$ is *not* convex; thus $\Psi_h \Pi$ is not a copula. For example, let h be in Θ defined by $h(t) := t^2$ for all $t \in [0,1]$. Let W be the lower Fréchet bound defined by $W(x,y) := \max\{x+y-1,0\}$ for all x, y in $[0,1]$. Then

$$W_h(x,y) = h^{-1}(W(h(x),h(y))) = \sqrt{\max\{x^2+y^2-1,0\}}, \tag{2.18}$$

namely,

$$W_h(x,y) = \begin{cases} 0, & x^2+y^2 \le 1, \\ \sqrt{x^2+y^2-1}, & \text{otherwise.} \end{cases} \tag{2.19}$$

The function W_h is one of a family of t-norms [22, page 72]. One has

$$\begin{aligned} W_h\left(\left[\frac{6}{10},1\right]^2\right) &= W_h(1,1) - W_h\left(1,\frac{6}{10}\right) - W_h\left(\frac{6}{10},1\right) + W_h\left(\frac{6}{10},\frac{6}{10}\right) \\ &= -\frac{2}{10} < 0, \end{aligned} \tag{2.20}$$

then, in view of [12, Proposition 3], W_h is not a quasi-copula.

So, the image $\Psi_h C$ of a copula should be neither a copula nor a quasi-copula, so that neither the family $\mathscr{C}$ of all copulas nor that $\mathfrak{Q}$ of all quasi-copulas are stable under Ψ.

3. The transform of copulas

Given a copula C and a function $h \in \Theta$, the *transform* of C is defined on $[0,1]^2$ by

$$C_h(x,y) := h^{[-1]}(C(h(x),h(y))). \tag{3.1}$$

Theorem 3.1. *For each $h \in \Theta$, the following statements are equivalent:*

(a) *h is concave;*
(b) *for every copula C, the transform (3.1) is a copula.*

Proof. (a)⇒(b). It suffices to show that C_h satisfies inequality (1.4). To this end, let x_1, y_1, x_2, y_2 be points of $[0,1]$ such that $x_1 \le x_2$ and $y_1 \le y_2$. Then the points s_i $(i = 1,2,3,4)$, defined by

$$\begin{aligned} s_1 &= C(h(x_1),h(y_1)), & s_2 &= C(h(x_1),h(y_2)), \\ s_3 &= C(h(x_2),h(y_1)), & s_4 &= C(h(x_2),h(y_2)), \end{aligned} \tag{3.2}$$

satisfy

$$s_1 \le \min\{s_2,s_3\} \le \max\{s_2,s_3\} \le s_4, \quad s_1 - s_2 - s_3 + s_4 \ge 0. \tag{3.3}$$

By using the notations of [17], one has $(s_3,s_2) \prec_w (s_4,s_1)$, where $\prec_w$ is the *weak majorization ordering*. Because $h^{[-1]}$ is convex, continuous, and increasing, it follows from Tomic's theorem (see [17, (4.B.2)]) that

$$h^{[-1]}(s_3) + h^{[-1]}(s_2) \le h^{[-1]}(s_4) + h^{[-1]}(s_1), \tag{3.4}$$

namely, inequality (1.4) holds.

(b)⇒(a). It suffices to show that $h^{[-1]}$ is Jensen-convex, that is,

$$\forall s,t \in [0,1] \quad h^{[-1]}\left(\frac{s+t}{2}\right) \leq \frac{h^{[-1]}(s)+h^{[-1]}(t)}{2}, \tag{3.5}$$

because, then, $h^{[-1]}$ is convex and, hence, h is concave.

Without loss of generality consider the copula W and points s and t in $[0,1]$ with $s \leq t$. If $(s+t)/2$ is in $[0,h(0)]$, then (3.5) is immediate. If $(s+t)/2$ is in $]h(0),1]$, then one has

$$\begin{gathered} W\left(\frac{s+1}{2},\frac{s+1}{2}\right) = s, \qquad W\left(\frac{t+1}{2},\frac{t+1}{2}\right) = t, \\ W\left(\frac{s+1}{2},\frac{t+1}{2}\right) = \frac{s+t}{2} = W\left(\frac{t+1}{2},\frac{s+1}{2}\right). \end{gathered} \tag{3.6}$$

There are points x_1 and x_2 in $[0,1]$ such that

$$h(x_1) = \frac{1+s}{2}, \qquad h(x_2) = \frac{1+t}{2}. \tag{3.7}$$

Since W_h is a copula, it satisfies inequality (1.4):

$$W_h(x_1,x_1) - W_h(x_1,x_2) - W_h(x_2,x_1) + W_h(x_2,x_2) \geq 0; \tag{3.8}$$

as a consequence, one has

$$h^{[-1]}(s) - h^{[-1]}\left(\frac{s+t}{2}\right) - h^{[-1]}\left(\frac{s+t}{2}\right) + h^{[-1]}(t) \geq 0, \tag{3.9}$$

which is the desired conclusion. □

The set of concave functions in Θ will be denoted by Θ_C. It is easy to prove that, for all $h,g \in \Theta_C$, $\lambda h + (1-\lambda)g$ ($\lambda \in [0,1]$) and $h \circ g$ are in Θ_C. Moreover, if h is in Θ_C, then $h(t^\alpha)$ and $(h(t))^\alpha$ are in Θ_C for all $\alpha \in]0,1[$. For instance, the following functions are in Θ_C:

(a) $h(x) = x^{1/\alpha}$ and $h^{-1}(x) = x^\alpha$ with $\alpha \geq 1$;
(b) $h(x) = \sin(\pi x/2)$ and $h^{-1}(x) = (2/\pi)\arcsin x$;
(c) $h(x) = (4/\pi)\arctan x$ and $h^{-1}(x) = \tan(\pi x/4)$.

Theorem 3.1 introduces, for all $h \in \Theta_C$, a mapping

$$\Psi_h : \mathscr{C} \longrightarrow \mathscr{C}, \qquad C \longmapsto \Psi_h C := C_h, \tag{3.10}$$

which verifies the properties given in the proposition below.

Proposition 3.2. *The following propertis hold:*

(a) *for every h and g in Θ_C, $\Psi_h \circ \Psi_g = \Psi_{g\circ h}$;*
(b) *if $\{C^n\}$ is a sequence of copulas that converges pointwise to a copula C and $h \in \Theta_C$, then $\{C_h^n\}$ converges pointwise to C_h;*

(c) Ψ_h *is continuous, in the sense that, for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that, for* $A, B \in \mathscr{C}$, $\|A - B\|_\infty < \delta$ *implies* $\|\Psi_h A - \Psi_h B\|_\infty < \epsilon$; *here*

$$\|A - B\|_\infty := \max\{\,|A(x,y) - B(x,y)| : (x,y) \in [0,1]^2\}; \tag{3.11}$$

(d) Ψ_h *is convex, in the sense that, for every copulas* A *and* B *and for* $\lambda \in [0,1]$, $\Psi_h(\lambda A + (1-\lambda)B) \prec \lambda \Psi_h A + (1-\lambda)\Psi_h B$.

As in Section 2, a subset $\mathscr{B}$ of $\mathscr{C}$ is said to be *stable* with respect to Ψ if the image of $\mathscr{B} \times \Theta_C$ under Ψ is contained in $\mathscr{B}$, $\Psi(\mathscr{B} \times \Theta_C) \subset \mathscr{B}$. By using the properties of their generators, it is easily proved that the class of Archimedean and Archimax copulas are stable (for these notions, see [5, 11]).

Example 3.3. Let C be a copula and let r be a function defined on $[0,1]$ by $r(t) = at + b$, with $a, b \in]0,1[$, $a + b = 1$. Then $r^{[-1]}(t) = \max\{0, (t-b)/a\}$ and one has

$$C_r(x,y) = \begin{cases} \dfrac{1}{a}[C(ax+b, ay+b) - b], & C(ax+b, ay+b) \geq b, \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

The copula C_r is said to be *linear transform of* C.

Remark 3.4. An interesting probabilistic interpretation of formula (3.1) was presented in [13]: if $h(t) = t^{1/n}$ for some $n \geq 1$, then C_h is the copula associated with componentwise maxima, $X = \max(X_1, \ldots, X_n)$ and $Y = \max(Y_1, \ldots, Y_n)$ of a random sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ from some arbitrary distribution with underlying copula C.

Power transformation of copulas was introduced in the theory of extreme value distributions [5, 6, 18]; recently Klement et al. [16] have studied the copulas that are invariant under power transformations and under increasing bijections.

Remark 3.5. Let H be a bivariate distribution function with unidimensional marginals F and G and let h be a strictly increasing function in Θ_C. From the proof of Theorem 3.1, it is easily proved that the function

$$\widetilde{H}(x,y) = h(H(x,y)), \quad (x,y) \in \overline{\mathbb{R}}^2, \tag{3.13}$$

is a bivariate distribution function with marginals $h(F)$ and $h(G)$ and with copula $C_{h^{-1}}$. Transformations of type (3.13) were used in the field of insurance pricing [10, 28] and they are also called *distorted probability measures* in the context of nonadditive probabilities [7].

We conclude this section with an open problem. Let C be a fixed copula. What is the subset $\Theta(C)$ of Θ, depending on C, that ensures that C_h is still a copula for all $h \in \Theta(C)$? For example, if C is an Archimedean copula with additive generator φ, it is easily shown that C_h is a copula if, and only if, $\varphi \circ h$ is convex. In this way, the two following remarks can be useful.

Remark 3.6. For a given copula C, its transform C_h may be a copula even though h is not concave. For instance, let h be the function defined on $[0,1]$ by $h(t) = t^2$. Then h is not concave, but $\Pi_h = \Pi$ is obviously a copula.

Remark 3.7. For a given copula C, the transforms C_h and C_g may be equal, $C_h = C_g$, even though the functions h and g are not equal, $h \neq g$. For instance, we consider the copula W and let h be the function defined on $[0,1]$ by $h(t) = (t+1)/2$. Then $W_h = W$ and $W_{id} = W$, but $id \neq h$.

4. The transform of quasi-copulas

Given a quasi-copula $(z_1,z_2) \mapsto Q(z_1,z_2)$ and a function $h \in \Theta$, the *transform* of Q is defined on $[0,1]^2$ by

$$Q_h(x_1,x_2) := h^{[-1]}(Q(h(x_1),h(x_2))). \tag{4.1}$$

Lemma 4.1. *Under the above assumptions, Q_h is a quasi-copula if, and only if, for almost all (x_1,x_2) in $[0,1]^2$ and for $i = 1,2$,*

$$h'(x_i) \cdot D_iQ(h(x_1),h(x_2)) \leq h'(h^{[-1]}(Q(h(x_1),h(x_2)))), \tag{4.2}$$

where $D_iQ = \partial Q/\partial z_i$ $(i = 1,2)$ exist a.e. on $[0,1]$.

Proof. For almost all (x_1,x_2) in $[0,1]^2$ and for $i = 1,2$, one has

$$D_iQ_h(x_1,x_2) = \frac{h'(x_i) \cdot D_iQ(h(x_1),h(x_2))}{h'(h^{[-1]}(Q(h(x_1),h(x_2))))}. \tag{4.3}$$

Since Q_h satisfies the boundary conditions and is increasing in each place, in view of [21, Theorem 2.1], Q_h is a quasi-copula if, and only if, $|D_iQ_h| \leq 1$, namely, if, and only if, the condition (4.2) holds. □

Lemma 4.2. *If h is in Θ_C, then Q_h is a quasi-copula.*

Proof. For all x, y in $[0,1]$, one has

$$x = h^{[-1]}(Q(h(x),1)) \geq h^{[-1]}(Q(h(x),h(y))), \tag{4.4}$$

then, since h' is decreasing a.e. on $[0,1]$, and since the partial derivatives of Q are smaller than, or equal to, 1,

$$h'(x) \cdot D_iQ(h(x),h(y)) \leq h'(x) \leq h'(h^{[-1]}(Q(h(x),h(y)))) \quad (i = 1,2), \tag{4.5}$$

that is, the condition (4.2). □

Connecting the above lemma and the proof of Theorem 3.1(part (b)⇒(a)), one has the following theorem.

Theorem 4.3. *For each $h \in \Theta$, the following statements are equivalent:*

(a) *h is concave;*
(b) *for every quasi-copula Q, Q_h is a quasi-copula, namely, $\Psi_h : \mathfrak{Q} \to \mathfrak{Q}$.*

Acknowledgments

We wish to thank Professor Radko Mesiar who generously showed us the connection of our work with his ongoing research. The topic of the research reported here arose from conversations with B. Bassan and F. Spizzichino; we regret that the untimely death of Bruno Bassan prevents us from thanking him as we should have wished.

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Fabrizio Durante: Dipartimento di Matematica "Ennio De Giorgi," Università di Lecce, 73100 Lecce, Italy

E-mail address: fabrizio.durante@unile.it

Carlo Sempi: Dipartimento di Matematica "Ennio De Giorgi," Università di Lecce, 73100 Lecce, Italy

E-mail address: carlo.sempi@unile.it