SUBSPACE GAPS AND WEYL'S THEOREM FOR AN ELEMENTARY OPERATOR

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A range-kernel orthogonality property is established for the elementary operators $\mathcal{E}(X) = \sum_{i=1}^n A_i X B_i$ and $\mathcal{E}_*(X) = \sum_{i=1}^n A_i^* X B_i^*$, where $\mathbf{A} = (A_1, A_2, \dots, A_n)$ and $\mathbf{B} = (B_1, B_2, \dots, B_n)$ are n -tuples of mutually commuting scalar operators (in the sense of Dunford) in the algebra $B(H)$ of operators on a Hilbert space H . It is proved that the operator $\mathcal{E}$ satisfies Weyl's theorem in the case in which $\mathbf{A}$ and $\mathbf{B}$ are n -tuples of mutually commuting generalized scalar operators.

1. Introduction

For a Banach space operator T , $T \in B(\mathcal{X})$, the kernel $T^{-1}(0)$ and the range $T(\mathcal{X})$ are said to have a k -gap for some real number $k \geq 1$, denoted $T^{-1}(0) \perp_k T(\mathcal{X})$, if

$$y \in T^{-1}(0) \implies \|y\| \leq k \operatorname{dist}(y, T(\mathcal{X})) \quad (1.1)$$

[8, Definition, page 94]. Recall from [10, page 93] that a subspace $\mathcal{M}$ of the Banach space $\mathcal{X}$ is *orthogonal* to a subspace $\mathcal{N}$ of $\mathcal{X}$ if $\|m\| \leq \|m+n\|$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$. This definition of orthogonality coincides with the usual definition of orthogonality in the case in which $\mathcal{X} = H$ is a Hilbert space. A 1-gap between $T^{-1}(0)$ and $T(\mathcal{X})$ corresponds to the *range-kernel orthogonality* for the operator T (see [1, 2, 8, 14]). The following implications are straightforward to see

$$T^{-1}(0) \perp_k T(\mathcal{X}) \implies T^{-1}(0) \cap \overline{T(\mathcal{X})} = \{0\} \implies T^{-1}(0) \cap T(\mathcal{X}) = \{0\} \implies \operatorname{asc}(T) \leq 1, \quad (1.2)$$

where $\overline{T(\mathcal{X})}$ denotes the closure of $T(\mathcal{X})$ and $\operatorname{asc}(T)$ denotes the *ascent* of T . A k -gap between $T^{-1}(0)$ and $T(\mathcal{X})$ does not imply that $T(\mathcal{X})$ is closed, or even when $T(\mathcal{X})$ is closed that $\mathcal{X} = T^{-1}(0) \oplus T(\mathcal{X})$ (see, e.g., [1, 2, 23]).

The classical Putnam-Fuglede commutativity theorem says that if A and B are normal Hilbert space operators, A and $B \in B(H)$, and if $\delta_{AB} \in B(B(H))$ is the *generalized derivation* $\delta_{AB}(X) := AX - XB$, then $\delta_{AB}^{-1}(0) = \delta_{A^*B^*}^{-1}(0)$. Extant literature contains various

generalizations of the Putnam-Fuglede theorem, amongst them the two n -tuples $\mathbf{A} = (A_1, A_2, \dots, A_n)$ and $\mathbf{B} = (B_1, B_2, \dots, B_n)$ of mutually commuting normal (Hilbert space) operators A_i and B_i , $1 \leq i \leq n$. Let $\mathcal{E} \in B(B(H))$ be the *elementary operator*

$$\mathcal{E}(X) := \sum_{i=1}^n A_i X B_i. \quad (1.3)$$

If $\text{asc}(\mathcal{E}) \leq 1$, then $\mathcal{E}^{-1}(0) = \mathcal{E}_*^{-1}(0)$, where $\mathcal{E}_*(X) := \sum_{i=1}^n A_i^* X B_i^*$ (see [21, 22, 24]). The conclusion $\mathcal{E}^{-1}(0) \subseteq \mathcal{E}_*^{-1}(0)$ fails if $\text{asc}(\mathcal{E}) > 1$ [21]; moreover, in such a case it may happen that $\mathcal{E}^{-1}(0) \cap \mathcal{E}(B(H)) \neq \{0\}$ [22]. For 2-tuples $\mathbf{A}$ and $\mathbf{B}$ of mutually commuting normal operators it is always the case that $\text{asc}(\mathcal{E}) \leq 1$ (see [14] or [8]).

This paper considers n -tuples $\mathbf{A}$ and $\mathbf{B}$ of mutually commuting *scalar operators* (in the sense of Dunford and Schwartz [10]) A_i and B_i , $1 \leq i \leq n$, to prove that the operator $\mathcal{E}_\mu := (\mathcal{E} - \mu I) \in B(B(H))$ satisfies: (i) there exists a complex number $\lambda = \alpha \exp i\theta$, $\alpha > 0$ and $0 \leq \theta < 2\pi$, such that if $\mathcal{E}_\lambda^{-1}(0) \neq \{0\}$, then $\mathcal{E}_\lambda^{-1}(0) \perp_k \mathcal{E}_\lambda(B(H))$ and $\mathcal{E}_{*\lambda}^{-1}(0) \perp_k \mathcal{E}_{*\lambda}(B(H))$, where $\mathcal{E}_{*\lambda} = (\mathcal{E}_* - \bar{\lambda}I)$. Furthermore, if the operators A_i and B_i in the n -tuples $\mathbf{A}$ and $\mathbf{B}$ are normal, then (ii) $\mathcal{E}_\lambda^{-1}(0) = \mathcal{E}_{*\lambda}^{-1}(0)$. This compares with the fact that the operator $\mathcal{E}$ may fail to satisfy the k -gap property of (i) or the Putnam-Fuglede-theorem-type commutativity property of (ii). However, if we restrict the length n of the n -tuples $\mathbf{A}$ and $\mathbf{B}$ to $n = 1$ (resp., $n = 2$), then both (i) and (ii) hold for all complex numbers λ [7, 9] (resp., $\lambda = 0$ and $\lambda = \alpha \exp i\theta$ for some real number $\alpha > 0$; see [7] and Theorem 2.4 infra). Our proof of (i) and (ii) makes explicit the relationship between the existence of a k -gap between the kernel and the range of the operator $\mathcal{E}_\lambda$, and the Putnam-Fuglede commutativity property for n -tuples $\mathbf{A}$ and $\mathbf{B}$ consisting of mutually commuting normal operators. Letting the n -tuples $\mathbf{A}$ and $\mathbf{B}$ consist of mutually commuting *generalized scalar operators* (in the sense of Colojoară and Foiaş [5]), it is proved that (i) a sufficient condition for $\mathcal{E}_\lambda(B(\mathcal{H}))$ to be closed is that the complex number λ is isolated in the spectrum of $\mathcal{E}$; (ii) $f(\mathcal{E})$ and $f(\mathcal{E}^*)$ satisfy Weyl's theorem for every analytic function f defined on a neighborhood of the spectrum of $\mathcal{E}$, and the conjugate operator $\mathcal{E}^*$ satisfies a -Weyl's theorem. These results will be proved in Sections 2 and 3, but before that, we explain our notation and terminology.

The ascent of $T \in B(\mathcal{H})$, $\text{asc}(T)$, is the least nonnegative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the descent of T , $\text{dsc}(T)$, is the least nonnegative integer n such that $T^n(\mathcal{H}) = T^{n+1}(\mathcal{H})$. We say that $T - \lambda$ is of finite ascent (resp., finite descent) if $\text{asc}(T - \lambda I) < \infty$ (resp., $\text{dsc}(T - \lambda I) < \infty$). The *numerical range* of T is the closed convex set

$$W(B(\mathcal{H}), T) = \{f(T) : f \in B(\mathcal{H})^*, \|f\| = \|f(I)\| = 1\} \quad (1.4)$$

of the set $\mathbb{C}$ of complex numbers (see [3]). A *spectral operator* (in the sense of Dunford) is an operator with a countable additive resolution of the identity defined on the Borel sets of $\mathbb{C}$; a spectral operator T is said to be *scalar type* if it satisfies $T = \int \lambda E(d\lambda)$, where E is the resolution of the identity for T [11, page 1938]. If $\mathbf{A} = (A_1, A_2, \dots, A_n)$ is an n -tuple of mutually commuting scalar operators in $B(H)$, then there exists an invertible self-adjoint operator S such that $S^{-1}A_i S = M_i$ is a normal operator for all $i = 1, 2, \dots, n$ [10, page 1947]. $T \in B(\mathcal{H})$ is a *generalized scalar operator* if there exists a continuous

algebra homomorphism $\Phi : C^\infty \rightarrow B(\mathcal{X})$ for which $\Phi(1) = I$ and $\Phi(Z) = T$, where $C^\infty(\mathbb{C})$ is the Fréchet algebra of all infinitely differentiable functions on $\mathbb{C}$ (endowed with its usual topology of uniform convergence on compact sets for the functions and their partial derivatives) and Z is the identity function on $\mathbb{C}$ (see [5, 16]). We will denote the *spectrum* and the *isolated points of the spectrum* of T by $\sigma(T)$ and $\text{iso } \sigma(T)$, respectively. The closed unit disc in $\mathbb{C}$ will be denoted by $\overline{\mathbf{D}}$, and $\partial \mathbf{D}$ will denote the boundary of the unit disc $\mathbf{D}$. The operator of *left multiplication by T* (*right multiplication by T*) will be denoted by L_T (resp., R_T). It is clear that $[L_S, R_T] = 0$ for all $S, T \in B(\mathcal{X})$, where $[L_S, R_T]$ denotes the commutator $L_S R_T - R_T L_S$. We will henceforth shorten $(T - \lambda I)$ to $(T - \lambda)$.

An operator $T \in B(\mathcal{X})$ is said to be Fredholm, $T \in \Phi(\mathcal{X})$, if $T(\mathcal{X})$ is closed and both the *deficiency indices* $\alpha(T) = \dim(T^{-1}(0))$ and $\beta(T) = \dim(\mathcal{X}/T(\mathcal{X}))$ are finite, and then the *index* of T , $\text{ind}(T)$, is defined to be $\text{ind}(T) = \alpha(T) - \beta(T)$. The operator T is *Weyl* if it is Fredholm of index zero. The (Fredholm) essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_w(T)$ of T are the sets

$$\begin{aligned}\sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}, \\ \sigma_w(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}.\end{aligned}\tag{1.5}$$

Let $\pi_0(T)$ denote the set of *Riesz points* of T (i.e., the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is Fredholm of finite ascent and descent [4]), and let $\pi_{00}(T)$ denote the set of isolated eigenvalues of T of finite geometric multiplicity. Also, let $\pi_{a0}(T)$ be the set of $\lambda \in \mathbb{C}$ such that λ is an isolated point of $\sigma_a(T)$ and $0 < \dim \ker(T - \lambda) < \infty$, where $\sigma_a(T)$ denotes the approximate point spectrum of the operator T . Clearly, $\pi_0(T) \subseteq \pi_{00}(T) \subseteq \pi_{a0}(T)$. We say that *Weyl's theorem holds for T* if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),\tag{1.6}$$

and *a-Weyl's theorem holds for T* if

$$\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_{a0}(T),\tag{1.7}$$

where $\sigma_{ea}(T)$ denotes the *essential approximate point spectrum* (i.e., $\sigma_{ea}(T) = \bigcap \{\sigma_a(T + K) : K \in K(\mathcal{X})\}$ with $K(\mathcal{X})$ denoting the ideal of compact operators on $\mathcal{X}$). If we let $\Phi_+(\mathcal{X}) = \{T \in B(\mathcal{X}) : \alpha(T) < \infty \text{ and } T(\mathcal{X}) \text{ is closed}\}$ denote the semigroup of *upper semi-Fredholm operators* in $B(\mathcal{X})$ and let $\Phi_+^-(\mathcal{X}) = \{T \in \Phi_+(\mathcal{X}) : \text{ind}(T) \leq 0\}$, then $\sigma_{ea}(T)$ is the complement in $\mathbb{C}$ of all those λ for which $(T - \lambda) \in \Phi_+^-(\mathcal{X})$. The concept of *a-Weyl's theorem* was introduced by Rakočević: *a-Weyl's theorem for T* implies Weyl's theorem for T , but the converse is generally false [20].

An operator $T \in B(\mathcal{X})$ has the *single-valued extension property* (SVEP) at $\lambda_0 \in \mathbb{C}$ if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 , the only analytic function $f : \mathcal{D}_{\lambda_0} \rightarrow \mathcal{X}$ which satisfies

$$(T - \lambda)f(\lambda) = 0 \quad \forall \lambda \in \mathcal{D}_{\lambda_0}\tag{1.8}$$

is the function $f \equiv 0$. Trivially, every operator T has SVEP at points of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$; also T has SVEP at $\lambda \in \text{iso } \sigma(T)$. We say that *T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$* . The *quasinilpotent part* $H_0(T)$ and the *analytic core* $K(T)$ of $T \in B(\mathcal{X})$ are

defined, respectively, by

$$\begin{aligned} H_0(T) &= \left\{ x \in \mathcal{X} : \lim_{n \rightarrow \infty} \|T^n(x)\|^{1/n} = 0 \right\}, \\ K(T) &= \left\{ x \in \mathcal{X} : \exists \text{ a sequence } \{x_n\} \subset \mathcal{X}, \delta > 0 \right. \\ &\quad \left. \text{for which } x = x_0, T(x_{n+1}) = x_n, \|x_n\| \leq \delta^n \|x\| \ \forall n = 1, 2, \dots \right\}. \end{aligned} \quad (1.9)$$

We note that $H_0(T - \lambda)$ and $K(T - \lambda)$ are (generally) nonclosed hyperinvariant subspaces of $T - \lambda$ such that $(T - \lambda)^{-q}(0) \subseteq H_0(T - \lambda)$ for all $q = 0, 1, 2, \dots$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$ [18]. If T has SVEP at λ , then $H_0(T - \lambda)$ and $K(T - \lambda)$ are closed. The operator $T \in B(\mathcal{X})$ is said to be *semiregular* if $T(\mathcal{X})$ is closed and $T^{-1}(0) \subset T^\infty(\mathcal{X}) = \bigcap_{n \in \mathbb{N}} T^n(\mathcal{X})$; T admits a *generalized Kato decomposition*, (GKD), if there exists a pair of T -invariant closed subspaces $(\mathcal{M}, \mathcal{N})$ such that $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$, the restriction $T|_{\mathcal{M}}$ is quasnilpotent and $T|_{\mathcal{N}}$ is semiregular. An operator $T \in B(\mathcal{X})$ has a (GKD) at every $\lambda \in \text{iso } \sigma(T)$, namely $\mathcal{X} = H_0(T - \lambda) \oplus K(T - \lambda)$. We say that $T - \lambda$ is of *Kato type* if $(T - \lambda)|_{\mathcal{M}}$ is nilpotent in the GKD for $T - \lambda$. Fredholm operators are Kato type [13, Theorem 4], and operators $T \in B(\mathcal{X})$ satisfying the following property:

$$\mathbf{H}(p) \ H_0(T - \lambda) = (T - \lambda)^{-p}(0),$$

for some integer $p \geq 1$, are Kato type at isolated points of $\sigma(T)$ (but not every Kato type operator T satisfies property $\mathbf{H}(p)$).

2. k -gap and the Putnam-Fuglede theorem

Let, as before, $\mathbf{A} = (A_1, A_2, \dots, A_n)$ and $\mathbf{B} = (B_1, B_2, \dots, B_n)$ be n -tuples of mutually commuting scalar operators, and let $\mathcal{E}_\lambda$ and $\mathcal{E}_{*\lambda}$ denote the elementary operators $\mathcal{E}_\lambda(X) = \sum_{i=1}^n A_i X B_i - \lambda X$ and $\mathcal{E}_{*\lambda}(X) = \sum_{i=1}^n A_i^* X B_i^* - \bar{\lambda} X$. We say in the following that the n -tuple $\mathbf{A}$ is *normally constituted* if A_i is normal for all $1 \leq i \leq n$.

The following theorem is the main result of this section.

THEOREM 2.1. (i) *There exists a real number $\alpha > 0$ such that if $(0 \neq) X \in \mathcal{E}_\alpha^{-1}(0)$, then $\|X\| \leq k \|\mathcal{E}_\alpha(Y) + X\|$ and $\|X\| \leq k \|\mathcal{E}_{*\alpha}(Y) + X\|$ for some real number $k \geq 1$ and all $Y \in B(H)$.*

Furthermore, if $\mathbf{A}$ and $\mathbf{B}$ are normally constituted, then

$$(ii) \ \mathcal{E}_\alpha^{-1}(0) = \mathcal{E}_{*\alpha}^{-1}(0).$$

Proof. There exist invertible self-adjoint operators $T_1, T_2 \in B(H)$ and normal operators $M_i, N_i \in B(H)$, $1 \leq i \leq n$, such that $M_i = T_1^{-1} A_i T_1$, $N_i = T_2^{-1} B_i T_2$, and $[M_i, M_j] = 0 = [N_i, N_j]$ for all $1 \leq i, j \leq n$. Define scalars α_1 and α_2 by $\alpha_1 = \|\sum_{i=1}^n M_i^* M_i\|^{1/2}$ and $\alpha_2 = \|\sum_{i=1}^n N_i^* N_i\|^{1/2}$, define the scalar α by $\alpha = \sqrt{\alpha_1 \alpha_2}$, and define the operators C_i and D_i , $1 \leq i \leq n$, by $C_i = (1/\sqrt{\alpha_1}) M_i$ and $D_i = (1/\sqrt{\alpha_2}) N_i$. Then $(C_1, C_2, \dots, C_n)$ and $(D_1, D_2, \dots, D_n)$ are n -tuples of mutually commuting normal operators. Let $E(X) = \sum_{i=1}^n C_i X D_i$. Set

$$U = [C_1 \ C_2 \ \cdots \ C_n], \quad V = [D_1 \ D_2 \ \cdots \ D_n]^t, \quad (2.1)$$

where $[\cdots]^t$ denotes the transpose of the row matrix $[\cdots]$. Then U and V are contractions. Representing $E(X)$ by

$$E(X) := U(X \otimes I_n)V, \quad (2.2)$$

where I_n denotes the identity of $\mathbf{M}_n(\mathbb{C})$, it then follows that E is a contraction. Hence,

$$W(B(B(H)), E) \subseteq \overline{\mathbf{D}}. \quad (2.3)$$

For $\mu \in \mathbb{C}$, let E_μ denote the operator $E_\mu = E - \mu$. Then

$$W(B(B(H)), E_1) \subseteq \{\lambda \in \mathbb{C} : |\lambda + 1| \leq 1\}. \quad (2.4)$$

In particular, $0 \in \partial W(B(B(H)), E_1)$. Notice that if 0 is an eigenvalue of $\mathcal{E}_\alpha$, then 0 is an eigenvalue of E_1 . It follows from Sinclair [23, Proposition 1] that

$$\|X\| \leq \|E_1(Y) + X\| \quad (2.5)$$

for all $X \in E_1^{-1}(0)$ and $Y \in B(H)$. In particular, $\text{asc}(E_1) \leq 1$, which by a result of Shulman [21] implies that $E_1^{-1}(0) \subseteq E_{*1}^{-1}(0)$ (where $E_{*1} \in B(B(H))$ is the operator $E_{*1} = E_* - 1 : X \rightarrow \sum_{i=1}^n C_i^* X D_i^* - X$). Representing E_* by

$$E_*(X) = U_1(X \otimes I_n)V_1, \quad (2.6)$$

where

$$U_1 = [C_1^* \quad C_2^* \quad \cdots \quad C_n^*], \quad V_1 = [D_1^* \quad D_2^* \quad \cdots \quad D_n^*]^t, \quad (2.7)$$

it follows that E_* is a contraction and 0 is an eigenvalue of E_{*1} in $\partial W(B(B(H)), E_{*1})$. Hence,

$$\|X\| \leq \|E_{*1}(Y) + X\| \quad (2.8)$$

for all $X \in E_{*1}^{-1}(0)$ and $Y \in B(H)$, which implies that $E_{*1}^{-1}(0) \subseteq E_1^{-1}(0)$. Hence, $E_1^{-1}(0) = E_{*1}^{-1}(0)$. The proof of (ii) is now a consequence of the observation that

$$X \in E_1^{-1}(0) \iff \sum_{i=1}^n M_i X N_i - \alpha X = 0 \iff X \in E_{*1}^{-1}(0) \iff \sum_{i=1}^n M_i^* X N_i^* - \alpha X = 0. \quad (2.9)$$

To prove (i), we let $\|T_1\| \|T_1^{-1}\| \|T_2\| \|T_2^{-1}\| = k$. Since

$$\begin{aligned} \alpha \|X\| &\leq \alpha \|E_1(Y) + X\| \\ &= \left\| T_1^{-1} \left\{ \sum_{i=1}^n A_i (T_1 Y T_2^{-1}) B_i - \alpha T_1 Y T_2^{-1} + \alpha T_1 X T_2^{-1} \right\} T_2 \right\| \\ &\implies \|\alpha T_1 X T_2^{-1}\| \leq \alpha \|T_1\| \|T_2^{-1}\| \|X\| \\ &\leq k \|\mathcal{E}_\alpha(T_1 Y T_2^{-1}) + \alpha T_1 X T_2^{-1}\| \end{aligned} \quad (2.10)$$

for all $X \in E_1^{-1}(0)$ and $Y \in B(H)$ (equivalently, all $T_1XT_2^{-1} \in \mathcal{E}_\alpha^{-1}(0)$ and $T_1YT_2^{-1} \in B(H)$), it follows that $\mathcal{E}_\alpha^{-1}(0) \perp_k \mathcal{E}_\alpha(B(H))$. A similar argument, applied this time to $\|X\| \leq \|E_{*1}(Y) + X\|$, implies that $\mathcal{E}_{*1}^{-1}(0) \perp_k \mathcal{E}_{*1}(B(H))$. This completes the proof of the theorem. $\square$

The following corollary is an immediate consequence of the fact that $\text{asc}(\mathcal{E}_\alpha) \leq 1$.

COROLLARY 2.2. *The range of $\mathcal{E}_\alpha$ is closed if and only if $\mathcal{E}_\alpha^{-1}(0) + \mathcal{E}_\alpha(B(H))$ is closed.*

For the proof see [16, Proposition 4.10.4].

The point α of Theorem 2.1 is not unique. Since $0 \in \partial W(B(B(H)), E_\mu)$ for every $\mu \in \mathbb{C}$ such that $|\mu| = 1$, the argument of the proof of Theorem 2.1 implies the following theorem.

THEOREM 2.3. (i) *There exists a complex number $\lambda = \alpha \exp i\theta$, $\alpha > 0$ and $0 \leq \theta < 2\pi$, such that if $(0 \neq) X \in \mathcal{E}_\lambda^{-1}(0)$, then $\|X\| \leq k\|\mathcal{E}_\lambda(Y) + X\|$ and $\|X\| \leq k\|\mathcal{E}_{*\lambda}(Y) + X\|$ for some real number $k \geq 1$ and all $Y \in B(H)$.*

Furthermore, if $\mathbf{A}$ and $\mathbf{B}$ are normally constituted, then

(ii) $\mathcal{E}_\lambda^{-1}(0) = \mathcal{E}_{*\lambda}^{-1}(0)$ for all λ as in part (i).

Let the Hilbert space H be separable, and let $\mathcal{C}_p$ denote the von Neumann-Schatten p -class, $1 \leq p < \infty$, with norm $\|\cdot\|_p$. Then Theorem 2.3 has the following $\mathcal{C}_p$ version.

THEOREM 2.4. (i) *There exists a complex number $\lambda = \alpha \exp i\theta$, $\alpha > 0$ and $0 \leq \theta < 2\pi$, such that if $(0 \neq) X \in \mathcal{E}_\lambda^{-1}(0) \cap \mathcal{C}_p$, then $\|X\|_p \leq k\|\mathcal{E}_\lambda(Y) + X\|_p$ and $\|X\|_p \leq k\|\mathcal{E}_{*\lambda}(Y) + X\|_p$ for some real number $k \geq 1$ and all $Y \in \mathcal{C}_p$.*

Furthermore, if $\mathbf{A}$ and $\mathbf{B}$ are normally constituted, then

(ii) $\mathcal{E}_\lambda^{-1}(0) \cap \mathcal{C}_p = \mathcal{E}_{*\lambda}^{-1}(0) \cap \mathcal{C}_p$ for all λ as in part (i).

Proof. Define the real numbers α_i , $i = 1, 2$, as in the proof of Theorem 2.1, define the normal operators C_i and D_i by $C_i = (1/\sqrt{\alpha_1 n^{1/2p}})M_i$ and $D_i = (1/\sqrt{\alpha_2 n^{1/2p}})N_i$. Let $\alpha = \sqrt{\alpha_1 \alpha_2} n^{1/p}$. Then $E \in B(\mathcal{C}_p)$ is a contraction. Now argue as in the proof of Theorem 2.1. $\square$

As we will see in the following section, $H_0(\mathcal{E}_\lambda) = \mathcal{E}_\lambda^{-p}(0)$ for all $\lambda \in \mathbb{C}$ and some integer $p \geq 1$ (i.e., $\mathcal{E}_\lambda$ satisfies property $\mathbf{H}(p)$), which implies that $\text{asc}(\mathcal{E}_\lambda) \geq 1$ for all $\lambda \in \mathbb{C}$. (Here, as also elsewhere, the statement $\text{asc}(T) \geq 1$ is to be taken to subsume the hypothesis that T is not injective.) However, if the n -tuples $\mathbf{A}$ and $\mathbf{B}$ are of length $n = 1$, then $\text{asc}(\mathcal{E}_\lambda) \leq 1$ for all $\lambda \in \mathbb{C}$ and for a number of classes of not necessarily scalar or normal operators A_1 and B_1 (see [7, 9]). If $n = 2$ and $B_1 = A_2 = I$, then $\text{asc}(\mathcal{E}_\lambda) \leq 1$ (once again for A_1 and B_2 belonging to a number of classes of operators more general than the class of scalar operators [7]). Again, if $n = 2$, then $\text{asc}(\mathcal{E}_\lambda) \leq 1$ for $\lambda = 0$ and $\lambda = \alpha \exp i\theta$, as follows from Theorem 2.1 and the following argument. Define the normal operators M_i and N_i , $i = 1, 2$, as in the proof of Theorem 2.1. Then $[M_1, M_2] = [N_1, N_2] = 0$. Define $\phi \in B(B(H))$ by $\phi(X) = M_1 X N_1 + M_2 X N_2$. Then $\phi^{-1}(0) \perp_k \phi(B(H))$ (see [14] or [8]), which implies that $\text{asc}(\phi) = \text{asc}(\mathcal{E}) \leq 1$. The following corollary, which generalizes [14, Theorem 2], is now obvious.

COROLLARY 2.5. *If $\mathbf{A} = (A_1, A_2)$ and $\mathbf{B} = (B_1, B_2)$ are 2-tuples of commuting scalar operators ($\in B(H)$), if $\mathcal{E} \in B(B(H))$ is defined by $\mathcal{E}(X) = A_1 X B_1 + A_2 X B_2$ and if the complex*

number λ is as in Theorem 2.3, then $\text{asc}(\mathcal{E}_\mu) \leq 1$, and $\mathcal{E}_\mu^{-1}(0) \perp_k \mathcal{E}_\mu(B(H))$ for $\mu = 0, \lambda$. Furthermore, if $\mathbf{A}$ and $\mathbf{B}$ are normally constituted, then $\mathcal{E}_\mu^{-1}(0) = \mathcal{E}_{*\mu}^{-1}(0)$ for $\mu = 0, \lambda$.

Perturbation by quasinilpotents. Recall that every spectral operator $T \in B(\mathcal{H})$ is the sum $T = S + Q$ of a scalar type operator S and a quasinilpotent operator Q such that $[S, Q] = 0$ [11]. Let $\mathbf{A} = (J_1, J_2)$ and $\mathbf{B} = (K_1, K_2)$ be tuples of operators in $B(H)$ such that $J_i = A_i + Q_i$ and $K_i = B_i + R_i$, $i = 1, 2$, for some scalar operators A_i, B_i and quasinilpotent operators Q_i, R_i . If we define $\mathbf{E} \in B(B(H))$ by $\mathbf{E}(X) = J_1 X K_1 + J_2 X K_2$, then $\mathbf{E}(X) = \mathcal{E}(X) + \phi(X)$, where $\mathcal{E}(X)$ is defined as in Corollary 2.5 and $\phi(X) = A_1 X R_1 + A_2 X R_2 + Q_1 X B_1 + Q_2 X B_2 + Q_1 X R_1 + Q_2 X R_2$. Recall that the sum of two commuting quasinilpotent operators, as well as the product of two commuting operators one of which is quasinilpotent, is quasinilpotent [5, Lemma 3.8, Chapter 4]. Representing the operator $X \rightarrow SXT$ by $X \rightarrow L_S R_T(X)$, where (S, T) denotes any of the operator pairs (A_i, R_i) , (Q_i, B_i) , or (Q_i, R_i) , $i = 1, 2$, and assuming that the operators in the sets $\{A_1, A_2, Q_1, Q_2\}$ and $\{R_1, R_2, B_1, B_2\}$ mutually commute, it follows that the operator ϕ is quasinilpotent.

THEOREM 2.6. *Let the operator $\mathbf{E}$ be defined as above. If the operators in the sets $\{A_1, A_2, Q_1, Q_2\}$ and $\{R_1, R_2, B_1, B_2\}$ mutually commute, then $X \in \mathbf{E}^{-1}(0) \Rightarrow X \in \mathcal{E}^{-1}(0)$.*

Proof. Let $X \in \mathbf{E}^{-1}(0)$. The hypothesis that the operators in the sets $\{A_1, A_2, Q_1, Q_2\}$ and $\{R_1, R_2, B_1, B_2\}$ mutually commute then implies that

$$-\phi(X) = \mathbf{E}(X) = T_1 \{M_1(T_1^{-1} X T_2) N_1 + M_2(T_1^{-1} X T_2) N_2\} T_2^{-1}, \quad (2.11)$$

where the operator ϕ is quasinilpotent, and where the normal operators M_i, N_i , $[M_1, M_2] = 0 = [N_1, N_2]$, and the invertible operators T_i , $i = 1, 2$, are defined as in the proof of Theorem 2.1. Define $\Phi \in B(B(H))$ by $\Phi(Y) = M_1 Y N_1 + M_2 Y N_2$. Since the operator ϕ is quasinilpotent,

$$\lim_{n \rightarrow \infty} \|\Phi^n(T_1^{-1} X T_2)\|^{1/n} \leq \|T_1^{-1}\| \|T_2\| \lim_{n \rightarrow \infty} \|\phi^n(X)\|^{1/n} = 0. \quad (2.12)$$

As earlier remarked upon, $H_0(\Phi) = \Phi^{-p}(0)$ for some integer $p \geq 1$. Since $\text{asc}(\Phi) \leq 1$ (by Corollary 2.5), it follows that $\Phi(T_1^{-1} X T_2) = 0$. Hence $X \in \mathcal{E}^{-1}(0)$. $\square$

3. Weyl's theorem

If $A, B \in B(\mathcal{H})$ are generalized scalar operators, then $L_A, R_B \in B(B(\mathcal{H}))$ are commuting generalized scalar operators with two commuting spectral distributions, which implies that $L_A R_B$ and $L_A + R_B$ are generalized scalar operators (see [5, Theorem 3.3, Proposition 4.2, Theorem 4.3, Chapter 4]). Let $\mathbf{A} = (A_1, A_2, \dots, A_n)$ and $\mathbf{B} = (B_1, B_2, \dots, B_n)$ be n -tuples of mutually commuting generalized scalar operators in $B(\mathcal{H})$, and let the elementary operator $\mathbf{E}_\lambda \in B(B(\mathcal{H}))$ be defined by $\mathbf{E}_\lambda(X) = \sum_{i=1}^n A_i X B_i - \lambda X$. Since $[L_{A_i}, R_{B_j}] = 0$ for all $1 \leq i, j \leq n$, the mutual commutativity of the n -tuples implies that $[L_{A_i} R_{B_j}, L_{A_j} R_{B_i}] = 0$ for all $1 \leq i, j \leq n$, the generalized scalar operators $L_{A_i} R_{B_i}$ and $L_{A_j} R_{B_j}$ have two commuting spectral distributions, and (hence that) $L_{A_i} R_{B_i} + L_{A_j} R_{B_j}$ is a generalized

scalar operator. A finitely repeated application of this argument implies that $\mathbf{E}_\lambda$ is a generalized scalar operator for all $\lambda \in \mathbb{C}$. Thus

$$H_0(\mathbf{E}_\lambda) = \mathbf{E}_\lambda^{-p}(0) \quad (3.1)$$

for some integer $p \geq 1$ and all $\lambda \in \mathbb{C}$ see [5, Theorem 4.5, Chapter 4]. In particular, $\text{asc}(\mathbf{E}_\lambda) \leq p < \infty$ for all $\lambda \in \mathbb{C}$ and $\mathbf{E}(=E_0)$ has SVEP.

The following proposition will be required in the proof of our main result.

PROPOSITION 3.1. (a) *The following conditions are equivalent:*

- (i) $\lambda \in \text{iso } \sigma(\mathbf{E})$;
- (ii) λ is a pole of order p of the resolvent of $\mathbf{E}$;
- (iii) $\text{dsc}(\mathbf{E}_\lambda) < \infty$;
- (iv) $\mathbf{E}_\lambda$ is Kato type and (in the definition of Kato type) the subspace $\mathcal{N} \subseteq \mathbf{E}_\lambda(B(\mathcal{X}))$.

(b) *If $\mathbf{E}^*$ denotes the conjugate operator of $\mathbf{E}$, then $\sigma_w(\mathbf{E}^*) = \sigma_w(\mathbf{E})$, $\pi_{00}(\mathbf{E}^*) = \pi_{00}(\mathbf{E}) = \pi_0(\mathbf{E}) = \pi_0(\mathbf{E}^*)$, and $\lambda \in \pi_{00}(\mathbf{E}) \Rightarrow \mathbf{E}_\lambda \in \Phi(B(\mathcal{X}))$, and $\text{ind}(\mathbf{E}_\lambda) = 0$.*

Proof. (a) (i) $\Rightarrow$ (ii). If $\lambda \in \text{iso } \sigma(\mathbf{E})$, then $B(\mathcal{X}) = H_0(\mathbf{E}_\lambda) \oplus K(\mathbf{E}_\lambda) = \mathbf{E}_\lambda^{-p}(0) \oplus K(\mathbf{E}_\lambda)$ for some integer $p \geq 1$. But then $\mathbf{E}_\lambda^{-p}(0)$ is complemented by the closed subspace $K(\mathbf{E}_\lambda) \subseteq \mathbf{E}_\lambda(B(\mathcal{X})) \Rightarrow K(\mathbf{E}_\lambda) = \mathbf{E}_\lambda^p(B(\mathcal{X}))$ [15, Theorem 3.4]. Hence λ is a pole of the resolvent of $\mathbf{E}$.

(ii) $\Rightarrow$ (iii). The implication is obvious.

(iii) $\Rightarrow$ (iv). If $\text{dsc}(\mathbf{E}_\lambda) < \infty$, then we have the following implications:

$$\begin{aligned} H_0(\mathbf{E}_\lambda) &= \mathbf{E}_\lambda^{-p}(0), \quad \forall \lambda \in \mathbb{C}, \\ &\Rightarrow \text{asc}(\mathbf{E}_\lambda) = \text{dsc}(\mathbf{E}_\lambda) \leq p < \infty, \quad [16, \text{Proposition 4.10.6}], \\ &\Rightarrow B(\mathcal{X}) = \mathbf{E}_\lambda^{-p}(0) \oplus \mathbf{E}_\lambda^p(B(\mathcal{X})) = \mathcal{M} \oplus \mathcal{N} \\ &\Rightarrow \mathbf{E}_\lambda \text{ is Kato type, } \mathcal{N} \subseteq \mathbf{E}_\lambda(B(\mathcal{X})). \end{aligned} \quad (3.2)$$

(iv) $\Rightarrow$ (i). If $\mathbf{E}_\lambda$ is Kato type, then $B(\mathcal{X}) = \mathcal{M} \oplus \mathcal{N}$, where $\mathbf{E}_\lambda|_{\mathcal{M}}$ is nilpotent and $\mathbf{E}_\lambda|_{\mathcal{N}}$ is semiregular. Since $\mathbf{E}_\lambda^{-n}(0) \subseteq \mathcal{M} \subseteq H_0(\mathbf{E}_\lambda) = \mathbf{E}_\lambda^{-p}(0)$ for all nonnegative integers n , and the closed subspace $\mathcal{N} \subseteq \mathbf{E}_\lambda(B(\mathcal{X}))$, $\lambda \in \text{iso } \sigma(\mathbf{E})$ [15, Theorem 3.2].

(b) The following implications hold:

$$\begin{aligned} \lambda \notin \sigma_w(\mathbf{E}^*) &\iff \mathbf{E}_\lambda^* \in \Phi(B(\mathcal{X})^*), \quad \text{ind}(\mathbf{E}_\lambda^*) = 0, \\ &\iff \mathbf{E}_\lambda \in \Phi(B(\mathcal{X})), \quad \text{ind}(\mathbf{E}_\lambda) = 0, \\ &\iff \lambda \notin \sigma_w(\mathbf{E}). \end{aligned} \quad (3.3)$$

Hence $\sigma_w(\mathbf{E}) = \sigma_w(\mathbf{E}^*)$. Again,

$$\begin{aligned} \lambda \in \text{iso } \sigma(\mathbf{E}^*) &\iff \lambda \in \text{iso } \sigma(\mathbf{E}) \\ &\iff B(\mathcal{X}) = \mathbf{E}_\lambda^{-p}(0) \oplus \mathbf{E}_\lambda^p(B(\mathcal{X})) \iff \lambda \in \pi_0(\mathbf{E}) \\ &\iff B(\mathcal{X})^* = \mathbf{E}_\lambda^{*-p}(0) \oplus \mathbf{E}_\lambda^{*p}(B(\mathcal{X})^*) \\ &\iff \lambda \in \pi_0(\mathbf{E}^*). \end{aligned} \quad (3.4)$$

Recall that if the ascent and the descent of an operator T are finite, and either $0 < \alpha(T) < \infty$ or $0 < \beta(T) < \infty$, then $\text{asc}(T) = \text{dsc}(T) < \infty$ and $0 < \alpha(T) = \beta(T) < \infty$ [12, Proposition 38.6]. Hence $\pi_{00}(\mathbf{E}^*) = \pi_{00}(\mathbf{E}) = \pi_0(\mathbf{E}) = \pi_0(\mathbf{E}^*)$, and $\lambda \in \pi_{00}(\mathbf{E}) \Rightarrow \mathbf{E}_\lambda \in \Phi(B(\mathcal{X}))$ with $\text{ind}(\mathbf{E}_\lambda) = 0$. $\square$

It is evident from Proposition 3.1(a) that a sufficient condition for $\mathbf{E}_\lambda$ to have closed range is that $\lambda \in \text{iso } \sigma(\mathbf{E})$. Proposition 3.1(b) implies that both $\mathbf{E}$ and $\mathbf{E}^*$ satisfy Weyl's theorem: more is true. Let $\mathbf{H}(\sigma(\mathbf{E}))$ denote the set of functions f which are defined and analytic on an open neighborhood of $\sigma(\mathbf{E})$.

THEOREM 3.2. (a) $f(\mathbf{E})$ and $f(\mathbf{E}^*)$ satisfy Weyl's theorem for every $f \in \mathbf{H}(\sigma(\mathbf{E}))$.

(b) $\mathbf{E}^*$ satisfies a -Weyl's theorem.

Proof. (a) A proof follows from [19, Theorem 3.1]. Alternatively, one argues as follows. If we let $\mathbf{E}'$ denote either of $\mathbf{E}$ or $\mathbf{E}^*$, then $\sigma(f(\mathbf{E}')) = \sigma(f(\mathbf{E}'))$ and $\sigma_w(f(\mathbf{E}')) = \sigma_w(f(\mathbf{E}'))$. Since $\mathbf{E}'$ is isoloid (i.e., isolated points of $\mathbf{E}'$ are eigenvalues of $\mathbf{E}'$) and Weyl's theorem holds for $\mathbf{E}'$ (by Proposition 3.1), $f(\sigma_w(\mathbf{E}')) = f(\sigma(\mathbf{E}') \setminus \pi_{00}(\mathbf{E}')) = \sigma(f(\mathbf{E}')) \setminus \pi_{00}(f(\mathbf{E}'))$ [17, lemma] and $f(\sigma_w(\mathbf{E}')) = \sigma_w(f(\mathbf{E}'))$ [6, Corollary 2.6]. (We note here that although [17, lemma] is stated for a Hilbert space, it equally holds in the setting of a Banach space.) Hence, since $f(\mathbf{E})$ satisfies property $\mathbf{H}(p)$, then [19, Theorem 3.4] implies (by Proposition 3.1) that Weyl's theorem holds for $f(\mathbf{E}')$, $\sigma(f(\mathbf{E}')) \setminus \pi_{00}(f(\mathbf{E}')) = \pi_{00}(f(\mathbf{E}'))$.

(b) The operator $\mathbf{E}$ has SVEP and the operator $\mathbf{E}^*$ satisfies Weyl's theorem; hence $\sigma(\mathbf{E}^*) = \sigma_a(\mathbf{E}^*)$ [16, page 35] and $\sigma_a(\mathbf{E}^*) \setminus \sigma_w(\mathbf{E}^*) = \pi_{a0}(\mathbf{E}^*)$. We prove that $\sigma_{ea}(\mathbf{E}^*) \supseteq \sigma_w(\mathbf{E}^*)$: since $\sigma_{ea}(\mathbf{E}^*) \subseteq \sigma_w(\mathbf{E}^*)$ always, this would complete the proof. If $\lambda \notin \sigma_{ea}(\mathbf{E}^*)$, then $\mathbf{E}_\lambda^* \in \Phi_+(B(\mathcal{X})^*)$ and $\text{ind}(\mathbf{E}_\lambda^*) \leq 0 \Leftrightarrow \mathbf{E}_\lambda \in \Phi_-(B(\mathcal{X}))$ and $\text{ind}(\mathbf{E}_\lambda) \geq 0$, where $\Phi_-(B(\mathcal{X})) = \{T \in B(B(\mathcal{X})) : \beta(T) < \infty\}$. Since $\text{asc}(\mathbf{E}_\lambda) < \infty$, $\text{ind}(\mathbf{E}_\lambda) \leq 0$. Hence $\alpha(\mathbf{E}_\lambda) = \beta(\mathbf{E}_\lambda) < \infty$ and $\text{asc}(\mathbf{E}_\lambda) = \text{dsc}(\mathbf{E}_\lambda) < \infty$ [12, Proposition 38.6], which implies that $\lambda \notin \sigma_w(\mathbf{E}^*)$. $\square$

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