ANOTHER SIMPLE PROOF OF THE QUINTUPLE PRODUCT IDENTITY

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We give a simple proof of the well-known quintuple product identity. The strategy of our proof is similar to a proof of Jacobi (ascribed to him by Enneper) for the triple product identity.

1. Introduction

The well-known quintuple product identity can be stated as follows. For $z \neq 0$ and $|q| < 1$,

$$\begin{aligned} f(z, q) &:= \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 - zq^{2n+1}) \left(1 - \frac{1}{z}q^{2n+1}\right) (1 - z^2q^{4n}) \left(1 - \frac{1}{z^2}q^{4n+4}\right) \\ &= \sum_{n=-\infty}^{\infty} q^{3n^2+n} (z^{3n}q^{-3n} - z^{-3n-1}q^{3n+1}). \end{aligned} \tag{1.1}$$

The quintuple identity has a long history and, as Berndt [5] points out, it is difficult to assign priority to it. It seems that a proof of the identity was first published in H. A. Schwartz's book in 1893 [19]. Watson gave a proof in 1929 in his work on the Rogers-Ramanujan continued fractions [20]. Since then, various proofs have appeared. To name a few, Carlitz and Subbarao gave a simple proof in [8]; Andrews [2] gave a proof involving basic hypergeometric functions; Blecksmith, Brillhart, and Gerst [7] pointed out that the quintuple identity is a special case of their theorem; and Evans [11] gave a short and elegant proof by using complex function theory. For updated history up to the late 80s and early 90s, see Hirschhorn [15] (in which the author also gave a beautiful generalization of the quintuple identity) and Berndt [5] (in which the author also gave a proof that ties the quintuple identity to the larger framework of the work of Ramanujan on q -series and theta functions; see also [1]). Since the early 90s, several authors gave different new proofs of the quintuple identity; see [6, 13, 12, 17]. See also Cooper's papers [9, 10] for the connections between the quintuple product identity and Macdonald identities [18]. Quite recently, Kongsiriwong and Liu [16] gave an interesting proof that makes use of the cube root of unity.

Our proof below is similar to the proof of the triple product identity by Jacobi (ascribed to him by Enneper; see the book by Hardy and Wright [14]). First, we set $f(z, q) = \sum a_n z^n$. Then, by considering the symmetry of $f(z, q)$ as an infinite product, we relate all a_n to a single coefficient a_0 . All we need to do is to evaluate a_0 . This is achieved by comparing $f(i, q)$ and $f(-q^4, q^4)$.

2. Proof of the identity

The first step of our proof is pretty standard, for example, see [16] or [4]. Set

$$f(z, q) = \sum_{n=-\infty}^{\infty} a_n z^n. \quad (2.1)$$

From the definition of $f(z, q)$, one can show that

$$f(z, q) = qz^3 f(zq^2, q), \quad f(z, q) = -z^2 f\left(\frac{1}{z}, q\right). \quad (2.2)$$

The first equality implies that for each n ,

$$a_{3n} = a_0 q^{3n^2-2n}, \quad a_{3n+1} = a_1 q^{3n^2}, \quad a_{3n+2} = a_2 q^{3n^2+2n}, \quad (2.3)$$

whereas the second equality implies that $a_2 = -a_0$ and $a_1 = 0$. By putting all these together, we have

$$f(z, q) = a_0(q) \sum_{n=-\infty}^{\infty} q^{3n^2+n} (z^{3n} q^{-3n} - z^{-3n-1} q^{3n+1}). \quad (2.4)$$

Comparing (2.4) to (1.1) shows that all we need to do is to prove that $a_0(q) = 1$. Note that $a_0(0) = 1$.

We can also write (2.4) in the following forms (which will be useful later):

$$f(z, q) = a_0(q) \sum_{n=-\infty}^{\infty} q^{3n^2-2n} \left(z^{3n} - \frac{1}{z^{3n-2}} \right) \quad (2.5a)$$

$$= a_0(q) \sum_{n=-\infty}^{\infty} q^{3n^2+n} \left(\left(\frac{z}{q} \right)^{3n} - \left(\frac{q}{z} \right)^{3n+1} \right). \quad (2.5b)$$

To obtain (2.5a), we let $n \rightarrow n-1$ in the *second* sum on the right-hand side of (2.4). Equation (2.5b) is simply another way of writing (2.4).

By putting $z = i$ in (2.5a), we have, on the one hand,

$$f(i, q) = a_0(q) \sum_{n=-\infty}^{\infty} q^{3n^2-2n} \left(i^{3n} - \frac{1}{i^{3n-2}} \right) = 2a_0(q) \sum_{n=-\infty}^{\infty} q^{12n^2-4n} (-1)^n. \quad (2.6)$$

Note that, in the second equality, we have used the fact that

$$i^{3n} - \frac{1}{i^{3n-2}} = 2 \cos \frac{3n}{2} \pi = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2(-1)^{n/2}, & \text{if } n \text{ is even.} \end{cases} \quad (2.7)$$

On the other hand, let us evaluate $f(i, q)$ as an infinite product:

$$\begin{aligned}
 f(i, q) &= 2 \prod_{n=1}^{\infty} (1 - q^{2n})(1 - iq^{2n-1})(1 + iq^{2n-1})(1 + q^{4n})^2 \\
 &= 2 \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{4n-2})(1 + q^{4n})^2 \\
 &= 2 \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})(1 + q^{4n}) \\
 &= 2 \prod_{n=1}^{\infty} (1 - q^{4n})(1 + q^{4n}) \\
 &= 2 \prod_{n=1}^{\infty} (1 - q^{8n}).
 \end{aligned} \tag{2.8}$$

Note that we have used the fact that $\prod(1 + q^{4n-2})(1 + q^{4n}) = \prod(1 + q^{2n})$ to derive the third equality.

By putting (2.6) and (2.8) together, we arrive at

$$\prod_{n=1}^{\infty} (1 - q^{8n}) = a_0(q) \sum_{n=-\infty}^{\infty} q^{12n^2-4n}(-1)^n. \tag{2.9}$$

Note that, at this stage, if we appeal to Euler's pentagonal number theorem (with q replaced by q^8) [4], we have

$$\prod_{n=1}^{\infty} (1 - q^{8n}) = \sum_{n=-\infty}^{\infty} q^{12n^2-4n}(-1)^n. \tag{2.10}$$

Compared with (2.9), we see that $a_0(q) = 1$. Alternatively, we can find $a_0(q)$ by evaluating $f(z, q)$ in a different way.

Precisely, let us evaluate $f(-q^4, q^4)$. By (2.5b), we have

$$\begin{aligned}
 f(-q^4, q^4) &= a_0(q^4) \sum_{n=-\infty}^{\infty} q^{12n^2+4n}((-1)^{3n} - (-1)^{3n+1}) \\
 &= 2a_0(q^4) \sum_{n=-\infty}^{\infty} q^{12n^2+4n}(-1)^n \\
 &= 2a_0(q^4) \sum_{n=-\infty}^{\infty} q^{12n^2-4n}(-1)^n.
 \end{aligned} \tag{2.11}$$

For the second equality, we have used the fact that $(-1)^{3n} - (-1)^{3n+1} = 2(-1)^n$. For the last equality, we let $n \rightarrow -n$ in the second line.

Again, evaluating $f(-q^4, q^4)$ as an infinite product gives

$$f(-q^4, q^4) = 2 \prod_{n=1}^{\infty} (1 - q^{8n}) \left(\prod_{k=1}^{\infty} (1 + q^{8k})(1 - q^{16k-8}) \right)^2 = 2 \prod_{n=1}^{\infty} (1 - q^{8n}). \tag{2.12}$$

The second equality is obtained by direct computation, similar to the derivation of (2.8). Alternatively, it follows from an identity due to Euler (e.g., see [3, page 60]) that

$$\prod_{k=1}^{\infty} (1 + q^{8k})(1 - q^{16k-8}) = 1. \quad (2.13)$$

By putting together (2.11) and (2.12), we have

$$\prod_{n=1}^{\infty} (1 - q^{8n}) = a_0(q^4) \sum_{n=-\infty}^{\infty} q^{12n^2-4n}(-1)^n. \quad (2.14)$$

Finally, by comparing (2.9) and (2.14), we conclude that $a_0(q) = a_0(q^4)$. This implies that

$$a_0(q) = a_0(q^4) = a_0(q^{16}) = \cdots = a_0(q^{4^k}) = \cdots = a_0(0) = 1 \quad (2.15)$$

and (1.1) is proven.

We remark that the evaluation of $f(-q^4, q^4)$ above also gives a simple proof of Euler's pentagonal number theorem.

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