EXPLICIT GEODESIC FLOW-INVARIANT DISTRIBUTIONS USING $\mathrm{SL}_2(\mathbb{R})$-REPRESENTATION LADDERS

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An explicit construction of a geodesic flow-invariant distribution lying in the discrete series of weight $2k$ isotopic component is found, using techniques from representation theory of $\mathrm{SL}_2(\mathbb{R})$. It is found that the distribution represents an AC measure on the unit tangent bundle of the hyperbolic plane minus an explicit singular set. Finally, via an averaging argument, a geodesic flow-invariant distribution on a closed hyperbolic surface is obtained.

1. Introduction

Flow-invariant quantities play a major role in dynamics, namely, in answering ergodic questions, and are related, via the *cohomological equation*, to the problem of describing time changes for flows (see, e.g., [3, 4, 5, 10] and the references therein).

On the other hand, the study of geometrically invariant objects arising from automorphic forms has been a major player in some questions arising in Riemann surfaces and in physics [1, 6, 7, 8, 9, 13, 15, 17, 22, 20, 21, 25].

In particular, it is well known that geodesic flow-invariant quantities appear quite naturally in quantum chaos: geodesic flow is a model for the time evolution of a classical mechanical system; while the time evolution of the quantum mechanical system is given in terms of the eigenfunctions of a certain selfadjoint operator. The transition between a classical mechanical system and the quantum version of the same system involves a method commonly known as quantization. The inverse process, that of going from a quantum mechanical system to a classical system, involves a limit which the physicists would like to claim as unique. (This is known as the *correspondence principle* or the *semiclassical limit*.) A central problem in physics since the late twenties has been trying to understand this transition. One of the main stumbling blocks seems to stem from the fact that there are many possible quantizations for the same classical system and it is not clear which of these should be the "correct one," furthermore the question of the uniqueness of the limit is also relevant. A mathematical approach to the problem can be stated as the *unique quantum ergodicity* question, where one is interested in finding the weak* limit points of microlocal lifts of eigenfunctions of the hyperbolic Laplacian

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[2, 8, 13, 16, 18, 23, 24, 25]. It is well known that such limits must be geodesic flow-invariant measures on the unit (co-)tangent bundle of the Riemann surface [19], however, it is unknown which invariant measures actually occur.

On this matter, Zelditch, Šnirel'man, and Colin de Verdière [2, 18, 23] independently showed that almost all limits are Liouville measures for a compact hyperbolic surface. There is evidence supporting that this limit is unique for $M = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{R})$ (see [13, 16]), while on the other hand, Jakobson [8] showed that for the flat tori, the possible limits are of the form $\nu = \phi(x)d\mathrm{Vol}$, where $\phi(x)$ is a trigonometric polynomial satisfying a rigid geometric condition.

As a further step in trying to understand the possible measures that could arise from such a limit, we undertake the study of geodesic flow-invariant distributions lying completely in the discrete series of weight $2k$ isotopic component of $\mathrm{SL}_2(\mathbb{R})$ by explicitly constructing them and examining some of their properties. This is done using a "ladder" construction on the (square root of the) unit tangent bundle of the hyperbolic plane, by methods based on the representation theory of $\mathrm{SL}_2(\mathbb{R})$.

The background material is presented in Section 2, while the actual construction is carried out in Section 3 and can be summarized as follows: essentially, a distribution on $T_1^*\mathbb{H}$ that lies completely in the discrete series unitary representation is constructed, then it is made geodesic flow-invariant by requiring it to be in the kernel of the operator corresponding to geodesic flow. The first requirement is met by using a "ladder" construction in which a suitable holomorphic form of a given weight $2k$ is raised to weights $2k+n$, n a positive integer, and these are all summed with a yet unspecified coefficient to form an infinite series. The requirement that this series be geodesic flow-invariant characterizes completely the coefficients (up to a constant) and further analysis shows that this is in fact a distribution of order 0 with singularities of logarithmic type or milder (Theorem 3.8). Furthermore, the singular set has a simple geometrical description (Figures 2.1 and 2.2). Finally, a geodesic flow-invariant distribution is obtained on a closed hyperbolic surface by the usual procedure of averaging over the group, this result is presented as Theorem 3.12.

2. Background

As mentioned in the introduction, we will be using the representation theory of $\mathrm{SL}_2(\mathbb{R})$, hence we introduce some terminology that may not be common to all readers.

2.1. Notation and terminology. Let $G = \mathrm{SL}_2(\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) | ad - bc = 1,\ a,b,c,d \in \mathbb{R}\}$, and let $K = SO(2) = \{\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) | \theta \in [0,2\pi)\} \cong S^1 \subset \mathbb{C}$ be the orthogonal group. Notice that K is the stabilizer of $i \in \mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ is the upper half-plane. The action of G on $\mathbb{H}$ is by fractional linear transformations. We adopt the usual convention that $z = x + iy$, so the action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : z \mapsto (az+b)/(cz+d)$. Notice that this action factors through the center, hence it is an action of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\pm\mathrm{Id}$.

Let $\mu_g(z) = cz + d$; it is called the *automorphy factor* and satisfies the cocycle relation $\mu_{g'g}(z) = \mu_{g'}(gz) \cdot \mu_g(z)$ for all $g', g \in G$. Moreover, $d(gz)/dz = \mu_g(z)^{-2}$ and $\operatorname{Im}(gz) = \operatorname{Im}(z)/|\mu_g(z)|^2$.

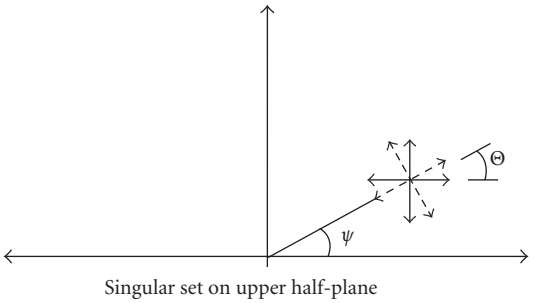

Figure 2.1. Singular set $\mathcal{S}_{k,\gamma_0}$ for t_{1,γ_0} on $T_1^*\mathbb{H}$. Dotted vectors represent $\theta - \psi = j(\pi/2)$.

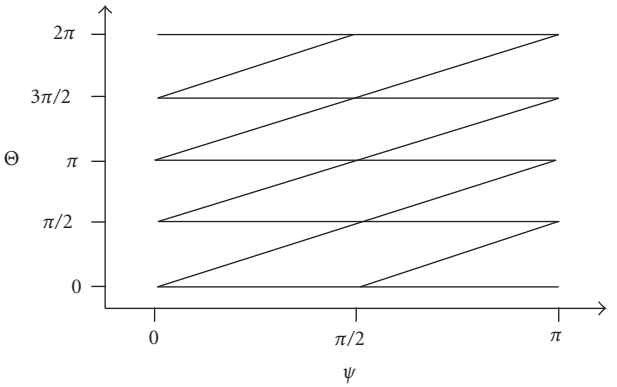

Figure 2.2. Singular set $\mathcal{S}_{k,\gamma_0}$ for t_{k,γ_0}, $k = 1,2$, in the (ψ,θ) coordinates of $\sqrt{T_1^*\mathbb{H}}$.

Since K fixes $i \in \mathbb{H}$, then there is a natural identification between $\mathrm{SL}_2(\mathbb{R})/K$ and $\mathbb{H}$ given by $gK \mapsto gi = (ai+b)/(ci+d)$. In fact we can identify $\mathrm{SL}_2(\mathbb{R})$ with $\sqrt{T_1^*\mathbb{H}}$ the square root (the double cover) of the unit cotangent bundle via the map $g \mapsto (gi, (\mu_g(i)/\mu_g(-i))\alpha)$ for $\alpha \in S^1 \subset \mathbb{C}$. Note that $\mu_g(i)/\mu_g(-i) = (dg/dz)^{1/2}(dg/d\bar{z})^{-1/2} = (d\bar{z}/dz)^{1/2}$. This can also be seen by recalling [12] that an element $g \in \mathrm{SL}_2(\mathbb{R})$ has the unique Iwasawa decomposition

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{2.1}$$

hence the rule

$$x + iy = y^{1/2}\mathrm{e}^{i\theta}(a+ib), \quad y^{-1/2}\mathrm{e}^{i\theta} = d - ic, \tag{2.2}$$

for $z = x + iy \in \mathbb{H}$ and θ the argument for the root cotangent vector, provides the required equivalence between $\mathrm{SL}_2(\mathbb{R})$ and $\sqrt{T_1^*\mathbb{H}}$.

Next, we define an *automorphic form* on $\mathrm{SL}_2(\mathbb{R})$ *for the group* Γ with *weight* $l \in \mathbb{C}$ as a function $f_l : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ satisfying

(i) (left Γ-invariance) $f_l(\gamma g) = f_l(g)$ for all $\gamma \in \Gamma$ and $g \in G$;

(ii) (right action by K) $f_l(gk) = \mu_k(i)^{-l} f_l(g)$ for all $k \in K$. Notice that $\mu_k(i) = \mathrm{e}^{i\theta}$ since $k = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$;

(iii) f_l satisfies an $\mathrm{SL}_2(\mathbb{R})$-invariant differential equation.

With this definition in mind, it is easy to recover the more common definitions of an automorphic form as a function on $\mathbb{H}$. The key lies in the following procedure that takes functions defined on $\mathbb{H}$ and lifts them to functions on $\mathrm{SL}_2(\mathbb{R})$ and vice versa:

$$\begin{gathered} \text{functions on } \mathbb{H} \longrightarrow \text{ weight } l \text{ functions on } \mathrm{SL}_2(\mathbb{R}), \\ f(z) \longmapsto f_l(g) := f(gi)\mu_g(i)^{-l}\mathrm{e}^{il\theta}, \end{gathered} \tag{2.3}$$

where $z = gi$ for some $g \in G$. It is interesting to note that we can rewrite $f_l(g) = f(z)y^{l/2}\mathrm{e}^{il\theta}$, and that f_l transforms on the right according to the character $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto \mathrm{e}^{ik\theta}$ of K. With the procedure described above, we have then that an *automorphic form* on $\mathbb{H}$ for the group Γ with *weight* $l \in \mathbb{C}$ is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying

(i) $f(\gamma z) = \mu_\gamma(z)^l f(z)$ for all $\gamma \in \Gamma$;

(ii) f is the solution to an $\mathrm{SL}_2(\mathbb{R})$-invariant differential equation.

Note that further regularity and growth properties follow from the last condition in each of the two definitions.

As stated, this definition of automorphic form is very wide; by allowing for different growth conditions, we can change the number of automorphic forms that are allowed; the more strict the condition, the less forms satisfying it.

2.2. Some representation theory of $\mathrm{SL}_2(\mathbb{R})$**.** Let $M = \Gamma\backslash\mathbb{H}$, where Γ is a cofinite group of isometries of $\mathbb{H}$. Then the Lie Algebra of M is locally identified with $\mathfrak{p}$, the space of symmetric matrices with trace zero, contained in $\mathfrak{sl}_2(\mathbb{R})$, which in turn enables us to use the techniques of representation theory. In accordance with Bargmann's classification theorem (see, e.g., [12, Theorem IV.6.8]), there are four different types of unitary representations of $\mathrm{SL}_2(\mathbb{R})$: discrete series, mock discrete series, principal series, and complementary series. All of these are infinitesimally isomorphic to some explicit subspaces of functions on $\mathrm{SL}_2(\mathbb{R})$. The explicitness of these subspaces is what will enable us to work with them [11, 12].

Consider the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R}) = ANK$, hence if $g \in \mathrm{SL}_2(\mathbb{R})$, then $g = u\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)r(\theta)$, where $r(\theta) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in K$ and abusing notation $u = \left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right) \in A$, note that $y, u > 0$. Let $s \in \mathbb{C}$. Define $H(s)$ to be the space of functions f on $\mathrm{SL}_2(\mathbb{R})$ such that $f|K \in L_2(K)$, and satisfying the condition $f(g) = f(u\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)r(\theta)) = y^{s+1/2}f(r(\theta))$. In particular, let $\psi_n \in H(s)$ be the function such that $\psi_n(r(\theta)) = \mathrm{e}^{in\theta}$. Let π_s be the representation of $\mathrm{SL}_2(\mathbb{R})$ on $H(s)$ by right translation. By Bargmann's classification theorem, the discrete series representation of weight m corresponds to $\pm s = m - 1$, where m is an integer greater than or equal to 2. Consider the following two subspaces of $H(m-1)$:

$$H^{(m)} = \bigoplus_{\substack{n \geq m \\ n \equiv m \bmod 2}} (\psi_n), \qquad H^{(-m)} = \bigoplus_{\substack{n \leq -m \\ n \equiv m \bmod 2}} (\psi_n). \tag{2.4}$$

The subspace $H_{2k} \equiv H^{(2k)} \oplus H^{(-2k)} \subseteq H(2k-1)$ is called *the discrete series unitary representation of weight* $2k$. Notice that if $f \in H^{(2k)}$, then $\bar{f} \in H^{(-2k)}$. Recall that this is just a *realization* of the discrete series unitary representation.

3. The construction

Let V be the space of functions on $\mathrm{SL}_2(\mathbb{R})$ such that $f|K \in L_2(K)$ and that are also geodesic flow-invariant. Then $\mathrm{SL}_2(\mathbb{R})$ acts on V by right translation. If π is an irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ on V, then $V(\pi) = \bigoplus \mathrm{Image}(\varphi)$, with $\varphi \in \mathrm{Hom}(\pi, V)$, is the π-isotopic component of V and consists of "copies" of the π irreducible subspaces.

Denote by $H(2k-1)(\pi_{2k})$ the *discrete series of weight* $2k$ *isotopic component in* V, and by $\widetilde{H}(2k-1)(\pi_{2k})$ the corresponding distributions. Consider the formal sum $T = \sum_n f_n$, with $f_n \in H(2k-1)(\pi_{2k})$ for all n. By abusing notation, we define T as a distribution on $\mathrm{SL}_2(\mathbb{R})$ associated to the formal sum. We will say that *the distribution* T *lies completely in* $\widetilde{H}(2k-1)(\pi_{2k})$ if the partial sum $T_N = \sum_{n \le N} f_n \in H(2k-1)(\pi_{2k})$ for all $N > 0$.

It is now possible to specifically state the problem addressed in this paper.

Main question. Suppose we have a geodesic flow-invariant distribution lying completely in the discrete series of weight $2k$ isotopic component $\widetilde{H}(2k-1)(\pi_{2k})$. What can we say about the distribution?

3.1. The plan of action. In order to answer the above question, an explicit construction of a geodesic flow-invariant distribution is carried out. The idea behind the construction is as follows: start with a suitable holomorphic form on a hyperbolic manifold M and then via a "ladder" construction obtain the distribution lying in the discrete series of weight $2k$. Since M is a hyperbolic quotient, work on $\mathbb{H}$ by requiring the objects to be automorphic forms of weight $2k$.

The use of relative Poincaré series allows the construction of such automorphic forms; recall that the Petersson series, Θ_{k,γ_n}, are automorphic forms of weight $2k$ associated to primitive hyperbolic elements $\gamma_n \in \Gamma$ (for further details, see [9]). Θ_{k,γ_n} is constructed by summing over the group the function $Q_{\gamma_n}^{-k}(z) = (cz^2 + (d-a)z - b)^{-k}$, associated to the hyperbolic element $\gamma_n = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By conjugating the group with an appropriate element, it is possible to use a primitive hyperbolic element $\gamma_0 \in \Gamma$ whose axis is the imaginary axis (in the upper half-plane model of $\mathbb{H}$). In this case, $Q_{\gamma_0}(z) = (a^{-1} - a)z$. Thus the construction is carried out using Θ_{k,γ_0}.

Since the representation theory of $\mathrm{SL}_2(\mathbb{R})$ is to be used, we use the lifting procedure, outlined in (2.3), to lift forms on $\mathbb{H}$ to functions on $\mathrm{SL}_2(\mathbb{R})$, as well as the identification of $\mathrm{SL}_2(\mathbb{R})$ with $\sqrt{T_1^*\mathbb{H}}$ the square root (double cover) of the unit cotangent bundle of $\mathbb{H}$. This allows us to establish a correspondence between tensors on $\mathbb{H}$ and functions on $\mathrm{SL}_2(\mathbb{R})$: let $f(z)dz^k$ be a symmetric k-tensor on $\mathbb{H}$, consider first the *balanced* tensor $f(z)y^k dz^{k/2} d\bar{z}^{-k/2}$ (recall that the hyperbolic metric is $ds = y^{-1}dz^{1/2}d\bar{z}^{-1/2}$). Then associate, under the identification of $\mathrm{SL}_2(\mathbb{R})$ with $T_1^*\mathbb{H}$, the weight $2k$ function on $\mathrm{SL}_2(\mathbb{R})$ given by $\Psi_k(z,\theta) = f(z)y^k e^{i2k\theta}$. The function $\Psi_k(z,\theta)$ on $\mathrm{SL}_2(\mathbb{R})$ is called the *lift of* f.

Under this procedure, the function $Q_{\gamma_0}^{-k}(z)$ is lifted (up to a constant) to the function $g_{k,\gamma_0}(z,\theta) = y^k e^{i2\theta k}/z^k$. The intention then is to construct a "ladder" in H_{2k} out of the lift

of the Petersson series Θ_{k,γ_0}, and use the techniques of representation theory of $\mathrm{SL}_2(\mathbb{R})$ to make the "ladder" geodesic flow-invariant.

Instead of summing over the group immediately (and obtaining the lift of $\Theta_{k,\gamma_0}(z)$), we opt to first obtain its "ladder" in H_{2k}.

In order to construct the "ladder", we will need to compute the action of the following elements of the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$: $E_+ = \left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right)$, $E_- = \left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right)$, and $W = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The infinitesimal representation $d\pi_s$ acts on elements of $H(s)$ via the Lie derivative, hence abusing notation and denoting by $\mathbf{E}_+$, $\mathbf{E}_-$, and $\mathbf{W}$ the Lie derivatives of E_+, E_-, and W, respectively, we have

$$\begin{aligned} \mathbf{E}_+ &= \mathrm{e}^{i2\theta}\left(2(z-\bar{z})\frac{\partial}{\partial z} + \frac{1}{i}\frac{\partial}{\partial\theta}\right), \\ \mathbf{E}_- &= \mathrm{e}^{-i2\theta}\left(-2(z-\bar{z})\frac{\partial}{\partial \bar{z}} - \frac{1}{i}\frac{\partial}{\partial\theta}\right), \\ \mathbf{W} &= \frac{\partial}{\partial\theta}, \end{aligned} \tag{3.1}$$

these are the "raising," "lowering," and "weight" operators, respectively, (since they raise, lower, and pick out the weights of the ψ_n). Notice that one can think of the operator $\mathbf{E}_+$ as generalizing the complex exterior differential ∂ which maps forms of type dz^k to forms of type dz^{k+1}, also note that formally we have $\mathbf{E}_- = \overline{\mathbf{E}_+}$, and since we are interested in calculating ladders in $H_{2k} = H^{(2k)} \oplus H^{(-2k)}$, then it follows by the above remark and the following standard lemma (whose proof can be found in, say [11, 12]), that it is only necessary to calculate in $H^{(2k)}$.

Lemma 3.1 (discrete series ladder). *Let $u \in H(s)$. Suppose that*

(1) $\mathbf{W}u = i2k\,u$, *for k integer, $k \geq 1$,*

(2) $\mathbf{E}_- u = 0$.

Then, $V = \langle \mathbf{E}_+^n u\rangle \oplus \langle \mathbf{E}_-^n \overline{u}\rangle$, $n = 0,1,2,\ldots$, is a representation of $\mathfrak{sl}_2(\mathbb{R})$ isomorphic to $H_{2k} = H^{(2k)} \oplus H^{(-2k)}$.

With this in mind, the construction should proceed as follows.

(1) We will start with the $\langle\gamma_0\rangle$-invariant function g_{k,γ_0}, and proceed to construct its ladder

$$t_{k,\gamma_0}(z,\theta) = \sum_{m\geq 0} a_m \mathbf{E}_+^m g_{k,\gamma_0}(z,\theta). \tag{3.2}$$

(2) Next we will determine the coefficients a_m which make the ladder geodesic flow-invariant (Section 3.3).

(3) Finally sum over the group to obtain the geodesic flow-invariant distribution on M lying only in H_{2k} (Section 3.4).

3.2. Recursion relations for some polynomials. The following subsection puts together some facts about some polynomials that will be fundamental for the rest of the construction.

Let $k \in \mathbb{Z}^+$ be fixed. Consider the polynomials

$$q_n(x) = \sum_{j=0}^{n} \binom{n-j+k-1}{k-1} \binom{j+k-1}{k-1} x^j. \tag{3.3}$$

Lemma 3.2. *The polynomials defined by (3.3) satisfy the following recursion relation:*

$$\begin{gathered} q_0(x) = 1, \\ (n+1)q_{n+1}(x) = (n+k+kx)q_n(x) - x(1-x)\frac{d}{dx}q_n(x). \end{gathered} \tag{3.4}$$

Proof. The proof is a simple induction argument on n. □

Lemma 3.3 (generating function for polynomials). *The polynomials defined by (3.3) have generating function*

$$(1-xy)^{-k}(1-y)^{-k} = \sum_{n\geq 0} q_n(x)y^n. \tag{3.5}$$

Proof. Expand $(1-y)^{-k}$ and $(1-xy)^{-k}$ in a power series about the origin, and obtain the product. Notice that the radius of convergence of each series is 1, hence the series converges for $|y| < 1$, and $|x| < 1/|y|$. □

Now consider the polynomials

$$p_n(x) = (1-x)^{2k-1}q_n(x). \tag{3.6}$$

We have the following lemma.

Lemma 3.4. *The polynomials $p_n(x) = (1-x)^{2k-1}q_n(x)$ satisfy*

(i) *$p_n(x)$ is a polynomial in x of degree $n+2k-1$,*

(ii) *$p_n(x)$ has exactly $2k$ terms, that is,*

$$\begin{aligned} p_n(x) = b_{n,0} + b_{n,1}x + \cdots + b_{n,k-1}x^{k-1} + b_{n,n+k}x^{n+k} \\ + b_{n,n+k+1}x^{n+k+1} + \cdots + b_{n,n+2k-1}x^{n+2k-1}, \end{aligned} \tag{3.7}$$

(iii) *the coefficients $b_{n,j}$, for $j = 1,2,\ldots,k-1,n+k,n+k+1,\ldots,n+2k-1$, are polynomials in n of degree $\leq k-1$.*

Proof. From (3.6), and since $\deg(p_n) = 2k-1+\deg(q_n)$, (i) follows because $\deg(q_n) = n$.

In the proof of statement (ii), we will drop the index corresponding to n in the coefficients of the polynomial $p_n(x)$, that is, we will write b_j instead of $b_{n,j}$ (since n is fixed).

In order to prove this statement, we first need a recursion relation for the polynomials p_n, we obtain this from (3.4). From the definition of $p_n(x)$, we have that

$$\begin{gathered} q_n(x) = (1-x)^{-(2k-1)}p_n(x), \\ \frac{d}{dx}q_n(x) = (1-x)^{-(2k-1)}\frac{d}{dx}p_n(x) + (2k-1)(1-x)^{-2k}p_n(x), \end{gathered} \tag{3.8}$$

so substituting in (3.4), we obtain the desired recursion relation for the polynomials p_n:

$$\begin{aligned} p_0(x) &= (1-x)^{2k-1}, \\ (n+1)p_{n+1}(x) &= \big(n+k+(1-k)x\big)p_n(x) - x(1-x)\frac{d}{dx}p_n(x). \end{aligned} \tag{3.9}$$

We now proceed by induction on n. For $n=0$, the result follows by the binomial theorem as follows:

$$p_0(x) = (1-x)^{2k-1} = \sum_{j=0}^{2k-1}(-1)^j\binom{2k-1}{j}x^j. \tag{3.10}$$

Next we consider $n = m+1$. By induction hypothesis,

$$\begin{aligned} p_m(x) = b_0 + b_1x + \cdots + b_{k-1}x^{k-1} + b_{m+k}x^{m+k} \\ + b_{m+k+1}x^{m+k+1} + \cdots + b_{m+2k-1}x^{m+2k-1} \end{aligned} \tag{3.11}$$

hence substituting in (3.9) and collecting terms of the same degree,

$$\begin{aligned} (m+1)p_{m+1}(x) = {} & (m+k)b_0 + \big((m+k-1)b_1 - (k-1)b_0\big)x \\ & + \big((m+k-2)b_2 - (k-2)b_1\big)x^2 + \cdots \\ & + \big((m+1)b_{k-1} - b_{k-2}\big)x^{k-1} + \big((m+1)b_{m+k} - b_{m+k+1}\big)x^{m+k+1} \\ & + \big((m+2)b_{m+k+1} - 2b_{m+k+2}\big)x^{m+k+2} + \cdots \\ & + \big((m+k-1)b_{m+2k-2} - (k-1)b_{m+2k-1}\big)x^{m+2k-1} \\ & + (m+k)b_{m+2k-1}x^{m+2k}. \end{aligned} \tag{3.12}$$

So, indeed $p_n(x)$ has exactly $2k$ terms.

Finally to prove statement (iii), we know that $p_n(x) = (1-x)^{2k-1}q_n(x)$, hence by (3.3) and by the binomial theorem, we have

$$\begin{aligned} p_n(x) &= \sum_{i=0}^{2k-1}(-1)^i\binom{2k-1}{i}x^i\sum_{j=0}^{n}\binom{n-j+k-1}{k-1}\binom{j+k-1}{k-1}x^j \\ &= \sum_{m=0}^{n+2k-1}\left[\sum_{i+j=m}(-1)^i\binom{2k-1}{i}\binom{n-j+k-1}{k-1}\binom{j+k-1}{k-1}\right]x^m, \end{aligned} \tag{3.13}$$

so in fact

$$b_{n,m} = \sum_{i+j=m}(-1)^i\binom{2k-1}{i}\binom{n-j+k-1}{k-1}\binom{j+k-1}{k-1}, \tag{3.14}$$

and since

$$\binom{n-j+k-1}{k-1} = \underbrace{(n-j+1)(n-j+2)(n-j+3)\cdots(n-j+k-1)}_{(k-1)\text{-terms}}, \tag{3.15}$$

then indeed $b_{n,m}$ is a polynomial in n of degree $\leq k-1$. □

3.3. Explicit construction on $\mathrm{SL}_2(\mathbb{R})$**.** Let $\gamma_0 \in \Gamma$ be a hyperbolic element whose axis is the imaginary axis on $\mathbb{H}$. Consider the function on $\mathrm{SL}_2(\mathbb{R})$

$$g_{k,\gamma_0}(z,\theta) = \frac{y^k e^{i2\theta k}}{z^k}, \tag{3.16}$$

(recall that this is the lift of $Q_{\gamma_0}^{-k}(z)$), and its ladder

$$t_{k,\gamma_0}(z,\theta) = \sum_{m\geq 0} a_m \mathbf{E}_+^m g_{k,\gamma_0}(z,\theta). \tag{3.17}$$

Notice that, as required by Lemma 3.1, $\mathbf{W} g_{k,\gamma_0} = i2k\, g_{k,\gamma_0}$ and $\mathbf{E}_- g_{k,\gamma_0} = 0$.

To find the coefficients a_m, we will make use of the fact that we require t_{k,γ_0} to be geodesic flow-invariant. The operator generating geodesic flow is given in terms of $\mathbf{E}_+$ and $\mathbf{E}_-$ by $\mathbf{G} = (\mathbf{E}_+ + \mathbf{E}_-)/2$. Hence we require that $\mathbf{G} t_{k,\gamma_0} = 0$, or equivalently,

$$\sum_{n\geq 0} (\mathbf{E}_+ + \mathbf{E}_-) a_n \mathbf{E}_+^n g_{k,\gamma_0} = 0. \tag{3.18}$$

This will enable us to determine explicitly the coefficients that make the ladder (3.17) geodesic flow-invariant.

Remark 3.5. It is to be noted that since $(E_+ + E_-)/2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then the requirement that the ladder (3.17) is to be geodesic flow-invariant can be stated as saying that it must be A-invariant (where $A < \mathrm{SL}_2(\mathbb{R})$ is the subgroup of diagonal matrices).

Lemma 3.6 (geodesic flow-invariant coefficients). *The coefficients that make the ladder (3.17) geodesic flow-invariant are given by*

$$a_n = \begin{cases} \dfrac{(k-1/2)!}{2^{4m}(m+k-1/2)!m!} & \text{if } n = 2m, \\ 0 & \text{if } n = 2m+1, \end{cases} \tag{3.19}$$

for $m \in \mathbb{Z}$. In other words, with the above choice of coefficients (formal) A-invariance of the ladder (3.17) exists (up to the choice of a_0, see proof).

Proof. The proof of the lemma proceeds as follows.

Since $\mathbf{E}_+$ and $\mathbf{E}_-$ are the raising and lowering operators, they satisfy the commutation relation

$$[\mathbf{E}_+, \mathbf{E}_-] = -4i\mathbf{W}, \tag{3.20}$$

where $\mathbf{W} = \partial/\partial\theta$ corresponds to the element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and "picks out the weight" of the function on which it acts (so $\mathbf{W}g_{k,\gamma_0} = i2k\, g_{k,\gamma_0}$).

Thus from (3.18), we obtain the following recursion relation:

$$\begin{aligned} a_0\mathbf{E}_- g_{k,\gamma_0} &= 0, \\ a_1\mathbf{E}_-\mathbf{E}_+ g_{k,\gamma_0} &= 0, \\ a_n\mathbf{E}_+^{n+1} g_{k,\gamma_0} + a_{n+2}\mathbf{E}_-\left(\mathbf{E}_+^{n+2}\right) g_{k,\gamma_0} &= 0, \quad \forall n \geq 0. \end{aligned} \tag{3.21}$$

From this and the fact that $\mathbf{E}_- g_{k,\gamma_0} = 0$, since g_{k,γ_0} is a holomorphic form, we have that a_0 is a free coefficient. Without loss of generality, we let $a_0 = 1$.

By repeated use of (3.20), we have

$$\begin{aligned} \mathbf{E}_-\left(\mathbf{E}_+^{n+2}\right) g_{k,\gamma_0} &= \mathbf{E}_-\mathbf{E}_+\left(\mathbf{E}_+^{n+1}\right) g_{k,\gamma_0} = \left(\mathbf{E}_+\mathbf{E}_- + 4i\mathbf{W}\right)\mathbf{E}_+^{n+1} g_{k,\gamma_0} \\ &= \left[\mathbf{E}_+^2\mathbf{E}_-\mathbf{E}_+^n + 4i\mathbf{E}_+\mathbf{W}\mathbf{E}_+^n + 4i\mathbf{W}\mathbf{E}_+^{n+1}\right] g_{k,\gamma_0} \\ &\vdots \\ &= \left[\mathbf{E}_+^{n+2}\mathbf{E}_- + 4i\sum_{j=0}^{n+1}\mathbf{E}_+^j\mathbf{W}\mathbf{E}_+^{n-j}\right] g_{k,\gamma_0}, \end{aligned} \tag{3.22}$$

with the convention that $\mathbf{E}_+^0 \equiv 1$. But since g_{k,γ_0} is a holomorphic form, then $\mathbf{E}_- g_{k,\gamma_0} = 0$, and since $\mathbf{W}$ "picks out" the weight, $\mathbf{W}\mathbf{E}_+^m g_{k,\gamma_0} = 2i(k+m)g_{k,\gamma_0}$, then

$$\begin{aligned} \mathbf{E}_-\left(\mathbf{E}_+^{n+2}\right) g_{k,\gamma_0} &= 4i\sum_{j=0}^{n+1} 2i(k+j)\mathbf{E}_+^{n+1} g_{k,\gamma_0} \\ &= -2^2\left[(n+2)^2 + (2k-1)(n+2)\right]\mathbf{E}_+^{n+1} g_{k,\gamma_0}. \end{aligned} \tag{3.23}$$

Hence substituting in (3.21), we obtain

$$\left\{a_n - 2^2\left[(n+2)^2 + (2k-1)(n+2)\right]a_{n+2}\right\}\mathbf{E}_+^{n+1} g_{k,\gamma_0} = 0 \quad \forall n \geq 0, \tag{3.24}$$

that is,

$$a_n = \frac{a_{n-2}}{2^2\left[n^2 + (2k-1)n\right]} \quad \forall n \geq 2. \tag{3.25}$$

Since $a_0 = 1$, we obtain that for $n = 2m$,

$$a_n = \prod_{j=1}^{n/2}\left[2^2\left((2j)^2 + 2j(2k-1)\right)\right]^{-1} = \frac{(k-1/2)!}{2^{2n}(k+(n-1)/2)!(n/2)!}. \tag{3.26}$$

On the other hand for n odd, we have from (3.21) that $a_1\mathbf{E}_-\mathbf{E}_+g_{k,y_0} = 0$, and by the same arguments as above, we see that

$$0 = a_1\mathbf{E}_-\mathbf{E}_+g_{k,y_0} = a_1\left(\mathbf{E}_+\mathbf{E}_- + 4iW\right)g_{k,y_0} = 4ia_1Wg_{k,y_0} \tag{3.27}$$

hence $a_1 = 0$, and by (3.25), $a_n = 0$ for n odd.
This proves the lemma. □

Remark 3.7. Notice that in the proof of the above lemma, we have used the facts that g_{k,y_0} is an eigenvector of the Casimir operator ω and that ω commutes with $\mathbf{E}_+$, $\mathbf{E}_-$, and $\mathbf{W}$. Furthermore, the determination of the coefficients (and hence of the ladder (3.17)) is unique up to the choice of the coefficient a_0.

It will now be convenient to introduce the change of variables $\alpha = \bar{z}/z = \mathrm{e}^{-i2\psi}$ and $\beta = \mathrm{e}^{i2\theta}$, where $z = x + iy = re^{i\psi}$, and θ is the fiber variable for $\sqrt{T_1^*\mathbb{H}}$. Recall that since $y > 0$, then $0 < \psi < \pi$, and θ represents the "direction" in the unit cotangent bundle, so $0 \le \theta < \pi$ (see Figure 2.1).

In these new variables, we have

$$g_k(\alpha,\beta) = \frac{(1-\alpha)^k\beta^k}{(2i)^k}, \qquad \mathbf{E}_+ = 2\beta\left(\beta\frac{\partial}{\partial\beta} - \alpha(1-\alpha)\frac{\partial}{\partial\alpha}\right). \tag{3.28}$$

An easy calculation shows that

$$\mathbf{E}_+^n g_k(\alpha,\beta) = 2^n n!\beta^n q_n(\alpha)g_k(\alpha,\beta), \tag{3.29}$$

where $q_n(\alpha)$ is a polynomial in α of degree n satisfying the following recursion relation:

$$\begin{gathered} q_0(\alpha) = 1, \\ (n+1)q_{n+1}(\alpha) = (n+k+k\alpha)q_n(\alpha) - \alpha(1-\alpha)\frac{d}{d\alpha}q_n(\alpha). \end{gathered} \tag{3.30}$$

By Lemma 3.2, we have that

$$q_n(\alpha) = \sum_{j=0}^{n}\binom{n-j+k-1}{k-1}\binom{j+k-1}{k-1}\alpha^j. \tag{3.31}$$

Hence,

$$\begin{aligned} a_{2m}\mathbf{E}_+^{2m}g_k(\alpha,\beta) &= \frac{(2m)!(k-1/2)!}{2^{2m}(m+k-1/2)!m!}\beta^{2m}q_{2m}(\alpha)g_k(\alpha,\beta) \\ &= \frac{(1-\alpha)^k\beta^k}{(2i)^k}C_{2m,k}\beta^{2m}\sum_{j=0}^{2m}\binom{2m-j+k-1}{k-1}\binom{j+k-1}{k-1}\alpha^j, \end{aligned} \tag{3.32}$$

where

$$C_{2m,k} = \frac{(2m)!(k-1/2)!}{2^{2m}(m+k-1/2)!m!} = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{(2m+1)(2m+3)(2m+5)\cdots(2m+2k-1)}, \tag{3.33}$$

so (3.17) becomes

$$t_{k,\gamma_0}(\alpha,\beta) = g_{k,\gamma_0}(\alpha,\beta) \sum_{m\geq 0} C_{2m,k}\beta^{2m} \sum_{j=0}^{2m} \binom{2m-j+k-1}{k-1}\binom{j+k-1}{k-1}\alpha^j, \tag{3.34}$$

$$t_{k,\gamma_0}(z,\theta) = \frac{y^k e^{i2k\theta}}{z^k} \sum_{m\geq 0} \frac{C_{2m,k}e^{i\theta 4m}}{z^{2m}} \sum_{j=0}^{2m} \binom{2m-j+k-1}{k-1}\binom{j+k-1}{k-1}\bar{z}^j z^{2m-j}. \tag{3.35}$$

We can now state the following theorem.

Theorem 3.8 (geodesic flow-invariant ladder on $\mathrm{SL}_2(\mathbb{R})$). *The formal sum on* $\mathrm{SL}_2(\mathbb{R})$ *given explicitly by*

$$t^{\mathrm{Re}}_{k,\gamma_0}(\alpha,\beta) = 2\,\mathrm{Re}\left(g_{k,\gamma_0}(\alpha,\beta) \sum_{m\geq 0} C_{2m,k}\beta^{2m} q_{2m}(\alpha)\right) \tag{3.36}$$

is a distribution of order 0, *with a pole of order* $k-1$ *at* $\alpha = 1$, *and singularities of logarithmic type or milder; otherwise. Furthermore, it is invariant under the right action of geodesic flow and the left action of* γ_0, *and lies completely in* $\widetilde{H}(2k-1)(\pi_{2k})$, *the discrete series of weight* $2k$ *isotopic component.*

Proof. Notice that by construction, the formal sum given by (3.34) (alternatively by (3.35)) lies in $\widetilde{H}(2k-1)(\pi_{2k})$ and it is invariant under geodesic flow and the left action of γ_0.

In order to prove that the formal sum is in fact a distribution as stated in the theorem, we show that it represents a holomorphic function on each of the variables α and β on $S^1 \times S^1$ except on an explicit singular set $\mathcal{S}_{k,\gamma_0}$ (see below).

Lemma 3.9. *The series*

$$t_k(\alpha,\beta) = \sum_{m\geq 0} C_{2m,k}\beta^{2m} q_{2m}(\alpha), \tag{3.37}$$

with $C_{2m,k}$ *given by (3.33) and* $q_n(x)$ *given by (3.3), converges absolutely for* $|\alpha|, |\beta| < 1$ *and* $k \geq 1$.

Proof. Notice that $C_{2m,k} \leq 1$, hence,

$$\left|t_k(\alpha,\beta)\right| \leq \sum_{m\geq 0} C_{2m,k}|\beta|^{2m}\left|q_{2m}(\alpha)\right| \leq \sum_{m\geq 0} |\beta|^{2m}\left|q_{2m}(\alpha)\right| \leq \sum_{m\geq 0} |\beta|^{m}\left|q_{m}(\alpha)\right|. \tag{3.38}$$

But by (3.3) $|q_m(\alpha)| \leq q_m(|\alpha|)$, and using Lemma 3.3, we obtain

$$\left|t_k(\alpha,\beta)\right| \leq \frac{1}{(1-|\alpha\beta|)^k(1-|\beta|)^k} < \infty. \tag{3.39}$$

This proves the lemma. □

Hence, this lemma shows that the series (3.34) defines a holomorphic function on $\{(\alpha,\beta) : |\alpha\beta| < 1,\ |\beta| < 1\} \supseteq \{(\alpha,\beta) : |\alpha| < 1,\ |\beta| < 1\}$.

Now, we notice that in the case $k = 1$, we have $C_{2m,1} = 1/(2m+1)$, so the expression (3.34) for t_{k,γ_0} reduces to

$$\begin{aligned} t_{1,\gamma_0}(\alpha,\beta) &= -\frac{1}{2i}\sum_{m\geq 0}\frac{\beta^{2m+1}(\alpha^{2m+1}-1)}{2m+1} \\ &= \frac{1}{2i}(\operatorname{Arctanh}(\beta)-\operatorname{Arctanh}(\alpha\beta)) \\ &= \frac{1}{4i}\log\left(\frac{1+\beta}{1-\beta}\cdot\frac{1-\alpha\beta}{1+\alpha\beta}\right), \end{aligned} \tag{3.40}$$

for $|\alpha\beta| < 1$, $|\beta| < 1$. By Fatou's theorem (see, e.g., [14, Volume 1, page 404]), the series converges (in fact, uniformly) to the above function on

$$\{(\alpha,\beta) : |\alpha| \leq 1,\ |\beta| \leq 1\} - \{(\alpha,\beta) : \beta = 1,\ \alpha\beta = -1\}, \tag{3.41}$$

so in particular the series defines a distribution of order 0 on $S^1 \times S^1 - \mathcal{S}_{1,\gamma_0}$, where $\mathcal{S}_{1,\gamma_0} = \{(\alpha,\beta) : \beta = 1,\ \alpha\beta = -1\}$ is the singular set, on $S^1 \times S^1$, of $\log((1+\beta)/(1-\beta)\cdot(1-\alpha\beta)/(1+\alpha\beta))$.

For the cases of $k > 1$, we argue as follows. The general term of the series in question is

$$a_{2m}\mathbf{E}_+^{2m}g_{k,\gamma_0}(\alpha,\beta) = \frac{\beta^k}{(2i)^k(1-\alpha)^{k-1}}C_{2m,k}\beta^{2m}p_{2m}(\alpha), \tag{3.42}$$

where $p_n(x) = (1-x)^{2k-1}q_n(x)$. By Lemma 3.4(ii), we obtain that

$$t_{k,\gamma_0}(\alpha,\beta) = \frac{\beta^k}{(2i)^k(1-\alpha)^{k-1}}\sum_{j=0}^{k-1}\left[\sum_{m\geq 0} b_{2m,j}C_{2m,k}\alpha^j\beta^{2m} + \sum_{m\geq 0} b_{2m,2m+k+j}C_{2m,k}\alpha^{2m+k+j}\beta^{2m}\right], \tag{3.43}$$

where the coefficients $b_{2m,j}$ and $b_{2m,2m+k+j}$ are polynomials in $2m$ of degree $k-1$. On the other hand, $C_{2m,k}^{-1}$ is a polynomial in $2m$ of degree k, and in fact for large m, the products $b_{2m,j}C_{2m,k}$ and $b_{2m,2m+k+j}C_{2m,k}$ behave like $1/(2m+r)$, with $r \geq 1$ being an odd positive integer. Hence we obtain $2k$ power series, on each of which we may again apply Fatou's theorem to obtain convergence of t_{k,γ_0} on its regular points on $S^1 \times S^1$. Thus we see that t_{k,γ_0} has a pole of order $k-1$ arising from the factor $(1-\alpha)^{1-k}$ and singularities of logarithmic type or milder arising from the $2k$ terms each of which has logarithmic or milder behavior. □

Remark 3.10. Notice that from Remark 3.7 and the fact that the discrete series representations occur with multiplicity, any other irreducible subrepresentation is a multiple of this explicit one (obtained by changing the coefficient a_0 alluded to in Remark 3.7).

Remark 3.11. In other words, $t_{k,\gamma_0}^{\mathrm{Re}}$ represents an absolutely continuous measure on $T_1^*\mathbb{H} - \{\mathcal{S}_{k,\gamma_0}\}$, where $\mathcal{S}_{k,\gamma_0}$ is the singular set of $t_{k,\gamma_0}^{\mathrm{Re}}$.

From (3.34), for $k = 2$, and the definition of $C_{2m,2}$,

$$t_{2,\gamma_0}(\alpha,\beta) = \frac{3}{4\beta(1-\alpha)} \sum_{m\geq 0} \frac{\beta^{2m+3}(\alpha^{2m+3}-1)}{2m+3} + \frac{3}{8}\frac{\alpha\beta}{1-\alpha} \log\left(\frac{1+\beta}{1-\beta}\cdot\frac{1-\alpha\beta}{1+\alpha\beta}\right), \tag{3.44}$$

or equivalently in terms of Arctanh,

$$t_{2,\gamma_0}(\alpha,\beta) = \frac{3}{4}\left(1 + \frac{1-\alpha\beta^2}{\beta(\alpha-1)}\left(\operatorname{Arctanh}(\beta) - \operatorname{Arctanh}(\alpha\beta)\right)\right). \tag{3.45}$$

In particular, when $k = 1,2$ (weights $2,4$), we observe from (3.40) and (3.45) that the singular set for t_{k,γ_0}

$$\mathcal{S}_{k,\gamma_0} = \left\{(\psi,\theta) \mid \theta = n\frac{\pi}{2}, \theta - \psi = j\frac{\pi}{2}, \text{ for } n = 0,1,2,3, \text{ and } j = -1,0,1,2,3\right\} \tag{3.46}$$

has a simple geometric description (see Figures 2.1 and 2.2).

For $k \geq 3$, we have explicit descriptions of the distributions

$$\begin{aligned}
t_{3,\gamma_0}(\alpha,\beta) &= \frac{3\cdot 5}{2^3 i\beta^2(1-\alpha)^2} \sum_{m\geq 0} \frac{(m+1)\beta^{2m+5}(\alpha^{2m+5}-1)}{(2m+3)(2m+5)} \\
&\quad - \frac{3\cdot 5\cdot \alpha}{2^3 i(1-\alpha)^2} \sum_{m\geq 0} \frac{\beta^{2m+3}(\alpha^{2m+3}-1)}{(2m+3)} \\
&\quad + \frac{3\cdot 5\cdot \alpha^2\beta^2}{2^3 i(1-\alpha)^2} \sum_{m\geq 0} \frac{(m+2)\beta^{2m+1}(\alpha^{2m+1}-1)}{(2m+1)(2m+3)}, \\
t_{4,\gamma_0}(\alpha,\beta) &= -\frac{5\cdot 7}{(2i)^4\beta^3(1-\alpha)^3} \sum_{m\geq 0} \frac{(m+1)\beta^{2m+7}(\alpha^{2m+7}-1)}{(2m+5)(2m+7)} \\
&\quad + \frac{3\cdot 5\cdot 7\cdot \alpha}{(2i)^4\beta(1-\alpha)^3} \sum_{m\geq 0} \frac{(m+1)\beta^{2m+5}(\alpha^{2m+5}-1)}{(2m+3)(2m+5)} \\
&\quad - \frac{3\cdot 5\cdot 7\cdot \alpha^2\beta}{(2i)^4(1-\alpha)^3} \sum_{m\geq 0} \frac{(m+3)\beta^{2m+3}(\alpha^{2m+3}-1)}{(2m+3)(2m+5)} \\
&\quad + \frac{5\cdot 7\cdot \alpha^3\beta^3}{(2i)^4(1-\alpha)^3} \sum_{m\geq 0} \frac{(m+3)\beta^{2m+1}(\alpha^{2m+1}-1)}{(2m+1)(2m+3)},
\end{aligned} \tag{3.47}$$

and so forth. From these, it is clear that the same description of the singular set that was found for $k = 1,2$ works as well for $k \geq 3$.

3.4. Explicit construction of an *automorphic* geodesic flow-invariant distribution. Because the Riemann surfaces in question are quotients of $\mathbb{H}$, it is natural to consider relative Poincaré series of the geodesic flow-invariant distributions: notice that t_{k,γ_0} is not Γ-automorphic, but by summing over the group, one formally obtains the relative Poincaré series

$$T_{k,\gamma_0}(z,\theta) = \sum_{\gamma\in(\gamma_0)\backslash\Gamma} (t_{k,\gamma_0}\circ\gamma)(z,\theta), \tag{3.48}$$

and by switching the order of summation, one can formally obtain that

$$T_{k,\gamma_0} = \sum_{n\geq 0} a_n \mathbf{E}_+^n G_{k,\gamma_0}, \tag{3.49}$$

where

$$G_{k,\gamma_0} = \sum_{\gamma\in(\gamma_0)\backslash\Gamma} g_{k,\gamma_0}\circ\gamma \tag{3.50}$$

is, as mentioned before, the lift to functions on $\mathrm{SL}_2(\mathbb{R})$ of the Petersson series Θ_{k,γ_0}.

Theorem 3.12 (weight $2k$ discrete series). *Assume that $k>1$. Let $T_{k,\gamma_0}^{\mathrm{Re}} = 2\,\mathrm{Re}\left(T_{k,\gamma_0}\right)$. The discrete series component of weight $2k$ of a geodesic flow-invariant distribution on a closed hyperbolic surface M is given by a linear combination of $T_{k,\gamma}^{\mathrm{Re}}$ for different primitive geodesics $[\gamma]$. Furthermore, T_{k,γ_0} is a distribution of order ε with singularities of mild type, and it satisfies*

$$T_{k,\gamma^{-1}\cdot\gamma_0\cdot\gamma}(z,\theta) = \left(T_{k,\gamma_0}\circ\gamma\right)(z,\theta). \tag{3.51}$$

Proof. Notice first that for $\gamma\in\mathrm{SL}_2(\mathbb{R})$, a formal calculation shows that

$$T_{k,\gamma^{-1}\cdot\gamma_0\cdot\gamma}(z,\theta) = \left(T_{k,\gamma_0}\circ\gamma\right)(z,\theta). \tag{3.52}$$

Moreover, observe that every geodesic flow-invariant distribution on the discrete series of weight $2k$ is a linear combination of $T_{k,\gamma}^{\mathrm{Re}}$ for different primitive geodesics $[\gamma]$, this follows directly from the fact that the space $S_{2k}(\Gamma)$ of Γ-automorphic forms of weight $2k$ is generated by $\{\Theta_{k,\gamma} : \gamma\in\mathcal{G}\}$, where $\mathcal{G}$ is a finite set of primitive geodesics on M (see [9]).

Considering that

$$T_{k,\gamma_0}(z,\theta) = \sum_{\gamma\in(\gamma_0)\backslash\Gamma} \left(t_{k,\gamma_0}\circ\gamma\right)(z,\theta), \tag{3.53}$$

and noticing that

$$\int_{\Gamma\backslash\mathrm{SL}_2(\mathbb{R})} T_{k,\gamma_0} = \int_{(\gamma_0)\backslash\mathrm{SL}_2(\mathbb{R})} t_{k,\gamma_0}, \tag{3.54}$$

and since $(\gamma_0)\backslash\mathrm{SL}_2(\mathbb{R}) \cong \{(y,\psi,\theta) : \mathrm{Im}\, z_0 < y < \mathrm{Im}\,\gamma_0(z_0),\ \psi\in(0,\pi),\ \theta\in(0,2\pi)\}$ (see Figure 2.2), for some $z_0\in[\gamma_0] = i\mathbb{R}^+$, it follows by the (local) L_1-boundedness of t_{k,γ_0} (since t_{k,γ_0} is a sum of $2k$ terms each of which has a pole of order $k-1$ or has logarithmic or milder behavior) that T_{k,γ_0} is in fact a distribution of order ε and the singularities are of logarithmic or milder type. □

Remark 3.13. Recall that the choice of holomorphic form of weight $2k$, with which the geodesic flow-invariant distribution on $\mathbb{H}$ is constructed, is chosen such that the *automorphic* geodesic flow-invariant distribution is in fact the ladder of raising of the Petersson

series of weight $2k$. This is of interest since there is a large volume of work on the Petersson series but no one has (to the extent of my knowledge) examined the natural extension of studying the ladder of a Petersson series.

The study of other properties of T_{k,y_0}, as well as extending the method presented in this paper to the other unitary representations given by Bargmann's classification, is underway, and will be presented elsewhere.

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