We extend the result in part I, 2003, of certain inequalities among the generalized power means.

1. Introduction

Let \( P_{n,r}(x) \) be the generalized weighted means: \( P_{n,r}(x) = (\sum_{i=1}^{n} q_i x_i^r)^{1/r} \), where \( P_{n,0}(x) \) denotes the limit of \( P_{n,r}(x) \) as \( r \to 0^+ \), \( x = (x_1, x_2, \ldots, x_n) \) and \( q_i > 0 \) (\( 1 \leq i \leq n \)) are positive real numbers with \( \sum_{i=1}^{n} q_i = 1 \). In this paper, we let \( q = \min q_i \) and always assume \( n \geq 2 \).

For mutually distinct numbers \( r, s, t \) and any real numbers \( \alpha, \beta \), we define
\[
\Delta_{r,s,t,\alpha,\beta} = \frac{P_{\alpha n, r} - P_{\alpha n, t}}{P_{\beta n, r} - P_{\beta n, s}},
\]
(1.1)
where we interpret \( P_{\alpha n, r} - P_{\alpha n, s} \) as \( \ln P_{n,r} - \ln P_{n,s} \). When \( \alpha = \beta \), we define \( \Delta_{r,s,t,\alpha} \) to be \( \Delta_{r,s,t,\alpha,\alpha} \). We also define \( \Delta_{r,s,t,1} \) to be \( \Delta_{r,s,t,1,1} \).

Bounds for \( \Delta_{r,s,t,\alpha,\beta} \) have been studied by many mathematicians. For the case \( \alpha \neq \beta \), we refer the reader to the articles [2, 5, 10] for the detailed discussions. In the case \( \alpha = \beta \) and \( r > s > t \), we seek the bound
\[
f_{r,s,t,\alpha}(q) \geq \Delta_{r,s,t,\alpha},
\]
(1.2)
and the bound
\[
\Delta_{r,s,t,\alpha} \geq g_{r,s,t,\alpha}(q),
\]
(1.3)
where \( f_{r,s,t,\alpha}(q) \) is a decreasing function of \( q \) and \( g_{r,s,t,\alpha}(q) \) is an increasing function of \( q \).

For \( r = 1, s = 0, \alpha = 0, t = -1 \), in (1.2) and (1.3), we can take \( f_{1,0,t,0}(q) = 1/q, g_{1,0,t,0}(q) = 1/(1 - q) \). When \( q_i = 1/n, 1 \leq i \leq n \), these are the well-known Sierpiński’s inequalities [12] (see [6] for a refinement of this). If we further require \( t, \alpha > 0 \), then consideration of
the case \( n = 2, x_1 \to 0, x_2 = 1 \) leads to the choice \( f_{t,s,t,a} = C_{r,s,t}((1 - q)^a), g_{s,t,a} = C_{r,s,t}(q^a) \), where
\[
C_{r,s,t}(x) = \frac{1 - x^{1/s - 1/r}}{1 - x^{1/s - 1/r}}, \quad t > 0; \quad C_{r,s,0}(x) = \frac{1}{1 - x^{1/s - 1/r}}. \tag{1.4}
\]
We will show in Lemma 2.1 that \( C_{r,s,t}(x) \) is an increasing function of \( x \) (\( 0 < x < 1 \)), so the above choice for \( f, g \) is plausible. From now on, we will assume \( f, g \) to be so chosen.

Note when \( t > 0 \), the limiting case \( \alpha \to 0 \) in (1.2) leads to Liapunov’s inequality (see [8, page 27]):
\[
\Delta_{r,s,t,0} = \frac{\ln P_{n,r} - \ln P_{n,t}}{\ln P_{n,r} - \ln P_{n,s}} \leq \frac{s(r - t)}{t(r - s)} =: C(r, s, t). \tag{1.5}
\]
From this (or by letting \( q \to 0 \) when \( \alpha = 1 \)), one easily deduces the following result of Hsu [9] (see also [1]): \( \Delta_{r,s,t} \leq C(r, s, t), r > s > t > 0 \).

For \( n = 2 \) and \( r > s \), (1.5) holds for \( \Delta_{r,s,t,0} = (r - t)/(r - s) \) as \( x_2 \to x_1 \). Therefore, the two inequalities (1.2) and (1.3) cannot hold simultaneously in general. Now for any set \( \{a, b, c\} \) with \( a, b, c \) mutually distinct and nonnegative, we let \( r = \max\{a, b, c\}, t = \min\{a, b, c\}, s = \{a, b, c\} \setminus \{r, t\} \). By saying (1.2) (resp. (1.3)) holds for the set \( \{a, b, c\}, \alpha > 0 \), we mean (1.2) (resp. (1.3)) holds for \( r > s > t \geq 0, \alpha > 0 \).

In the case \( \alpha = 1 \), a result of Diananda (see [3, 4]) (see also [1, 11]) shows that (1.2) and (1.3) hold for \( \{1, 1/2, 0\} \) and his result has recently been extended by the author [7] to the cases \( \{r, 1, 0\} \) and \( \{r, 1, 1/2\} \) with \( r \in (0, \infty) \). It is the goal of this paper to further extend the results in [7].

\section*{2. Lemmas}

\textbf{Lemma 2.1.} For \( 0 < x < 1, 0 \leq t < s < r \), \( C_{r,s,t}(x) \) is a strictly increasing function of \( x \). In particular, for \( 0 < q \leq 1/2 \), \( C_{r,s,t}(1 - q) \geq C_{r,s,t}(q) \).

\textit{Proof.} We may assume \( t > 0 \). Note \( C_{r,s,t}(x) = C_{1/s/r,t/r}(x^{1/r}) \), thus it suffices to prove the lemma for \( C_{1/r,s} \) with \( 1 > r \) and \( 0 \leq s < r \). By the Cauchy mean value theorem,
\[
\frac{1/s - 1}{1/r - 1} \cdot \frac{1 - x^{1/r - 1}}{1 - x^{1/s - 1}} = \eta^{1/r - 1/s} < x^{1/r - 1/s} \tag{2.1}
\]
for some \( x < \eta < 1 \) and this implies \( C_{1/r,s}(x) > 0 \) which completes the proof. \hfill \Box

\textbf{Lemma 2.2.} For \( 1/2 < r < 1, C_{1/r,1-r}(1/2) > r/(1 - r) \).

\textit{Proof.} By setting \( x = r/(1 - r) > 1, \) it suffices to show \( f(x) > 0 \) for \( x > 1 \), where \( f(x) = 1 - 2^{-x} - x(1 - 2^{-1/x}) \). Now \( f''(x) = (\ln 2)^2 2^{-2x} x^{-3} (2^{x-1/x} - x^3) \) and let \( g(x) = (x - 1/x) \ln 2 - 3 \ln x \). Note \( g'(x) \) has one root in \( (1, \infty) \) and \( g(1) = 0 \), it follows that \( g(x) \), hence \( f''(x) \), has only one root \( x_0 \) in \( (1, \infty) \). Note when \( f''(x) > 0 \) for \( x > x_0 \), this together with the observation that \( f(1) = 0, f'(1) = \ln 2 - 1/2 > 0, \lim x \to \infty f(x) = 1 - \ln 2 > 0 \) shows \( f(x) > 0 \) for \( x > 1 \). \hfill \Box

\textbf{Lemma 2.3.} Let \( 0 < q \leq 1/2 \). For \( 0 \leq s < r < 1, 1 + s \geq 1, C_{1/r,s}(1 - q) > (1 - s)/(1 - r) \). For \( 0 \leq s < 1 < r, C_{1/s}(1 - q) > (r - s)/(r - 1) \) and for \( 1 < s < r, C_{1/s}(1 - q) > (r - 1)/(r - s) \). For
Proof. We will give a proof for the case $1 > r > s > 0$, $r + s ≥ 1$ here and the proofs for the other cases are similar. We note first that in this case $1/2 < r < 1$. By Lemma 2.1, it suffices to prove $C_{1,r,s}(1/2) > (1 - s)/(1 - r)$. Consider

$$f(s) = (1 - r)\left(1 - \left(\frac{1}{2}\right)^{\frac{1}{r-1}}\right) - (1 - s)\left(1 - \left(\frac{1}{2}\right)^{\frac{1}{s-1}}\right).$$  \hspace{1cm} (2.2)

We have $f(r) = 0$ and Lemma 2.2 implies $f(1 - r) > 0$. Now $f'(r) = 2^{1-1/r}g(1/r)$, where $g(x) = -\ln 2(x^2 - x) + 2x^{-1} - 1$ with $1 < x < 2$. One checks easily $g(1) = g'(1) = 0$, $g''(x) < 0$ which implies $g(x) < 0$. Hence, $f''(r) < 0$, this combined with the observation that

$$f''(s) = (1 - r)\ln 2\left(\frac{1}{2}\right)^{\frac{1}{s-1}}\frac{(2s - \ln 2)}{s^4}$$  \hspace{1cm} (2.3)

has at most one root and $f''(r) > 0$, $f(1 - r) > 0$, $f(r) = 0$ imply that $f(s) > 0$ for $1 - r ≤ s < r$. \hfill \Box

3. The main theorems

Theorem 3.1. Let $α = 1$. Inequality (1.2) holds for the set $\{1, r, s\}$, with $1$, $r$, $s$ mutually distinct and $r > s ≥ 0$, $r + s ≥ 1$. The equality holds if and only if $n = 2$, $x_1 = 0$, $q_1 = q$.

Proof. The case $s = 0$ was treated in [7], so we may assume $s > 0$ here. We will give a proof for the case $1 > r > s > 0$ here and the proofs for the other cases are similar. Define

$$D_n(x) = A_n - P_{n,r} - C(1 - q)(A_n - P_{n,s}), \quad C(x) = \frac{1 - x^{1/r-1}}{1 - x^{1/s-1}}.$$  \hspace{1cm} (3.1)

By Lemma 2.3, we need to show $D_n ≥ 0$ and we have

$$\frac{1}{q_n} \frac{∂D_n}{∂x_n} = 1 - P_{n,r}^{1-r}x_n^{r-1} - C(1 - q)(1 - P_{n,s}^{1-s}x_n^{s-1}).$$  \hspace{1cm} (3.2)

By a change of variables: $x_i/x_n → x_i$, $1 ≤ i ≤ n$, we may assume $0 ≤ x_1 < x_2 < \cdots < x_n = 1$ in (3.2) and rewrite it as

$$g_n(x_1, \ldots, x_{n-1}) := 1 - P_{n,r}^{1-r} - C(1 - q)(1 - P_{n,s}^{1-s}).$$  \hspace{1cm} (3.3)

We want to show $g_n ≥ 0$. Let $a = (a_1, \ldots, a_{n-1}) ∈ [0, 1]^{n-1}$ be the point in which the absolute minimum of $g_n$ is reached. We may assume $a_1 ≤ a_2 ≤ \cdots ≤ a_{n-1}$. If $a_i = a_{i+1}$ for some $1 ≤ i ≤ n - 2$ or $a_{n-1} = 1$, by combing $a_i$ with $a_{i+1}$ and $q_i$ with $q_{i+1}$, or $a_{n-1}$ with 1 and $q_{n-1}$ with $q_n$, it follows from Lemma 2.1 that we can reduce the determination of the absolute minimum of $g_n$ to that of $g_{n-1}$ with different weights. Thus without loss of generality, we may assume $a_1 < a_2 < \cdots < a_{n-1} < 1$.

If $a$ is a boundary point of $[0, 1]^{n-1}$, then $a_1 = 0$, and we can regard $g_n$ as a function of $a_2, \ldots, a_{n-1}$, then we obtain

$$\nabla g_n(a_2, \ldots, a_{n-1}) = 0.$$  \hspace{1cm} (3.4)
otherwise $a_1 > 0$, $a$ is an interior point of $[0,1]^{n-1}$ and
\[ \nabla g_n(a_1, \ldots, a_{n-1}) = 0. \] (3.5)

In either case $a_2, \ldots, a_{n-1}$ solve the equation
\[ (r - 1)P_{n,r} P_{n,s}^{-1} + C(1 - q)(1 - s)P_{n,s}^{-2} = 0. \] (3.6)

The above equation has at most one root (regarding $P_{n,r}, P_{n,s}$ as constants), so we only need to show $g_n \geq 0$ for the case $n = 3$ with $0 = a_1 < a_2 = x < a_3 = 1$ in (3.3). In this case we regard $g_3$ as a function of $x$ and we get
\[ \frac{1}{q_2} g_3'(x) = P_{3,r}^{1-2r} x^{r-1} h(x), \] (3.7)

where
\[ h(x) = r - 1 + (1 - s)C(1 - q)(q_2 x^{s/2} + q_3 x^{-s/2})^{(1-2s)/s} (q_2 x^{r/2} + q_3 x^{-r/2})^{(2r-1)/r}. \] (3.8)

If $q_2 = 0$ (note $q_3 > 0$), then
\[ h(x) = r - 1 + (1 - s)C(1 - q)q_3^{1/s-1/r} x^{s-r}. \] (3.9)

One easily checks that in this case $h(x)$ has exactly one root in $(0,1)$. Now assume $q_2 > 0$, then
\[ h'(x) = (1 - s)C(1 - q)P_{3,r}^{1-3s} P_{3,r}^{r-1} x^{-(r+s)/2} p(x), \] (3.10)

where
\[ p(x) = (r - s)(q_2^2 x^{r+s} - q_2^2) + (r + s - 1)q_2 q_3 (x^r - x^s). \] (3.11)

Now
\[ p'(x) = x^{s-1}((r^2 - s^2)q_2^2 x^r + (r + s - 1)q_2 q_3 (rx^r - s)) := x^{s-1} q(x). \] (3.12)

If $r + s \geq 1$, then $q'(x) > 0$ which implies there can be at most one root for $p'(x) = 0$. Since $p(0) < 0$ and $\lim_{x \to -\infty} p(x) = +\infty$, we conclude that $p(x)$, hence $h'(x)$, has at most one root. Since $h(1) < 0$ by Lemma 2.3 and $\lim_{x \to 0^+} h(x) = +\infty$, this implies $h(x)$ has exactly one root in $(0,1)$.

Thus $g_3'(x)$ has only one root $x_0$ in $(0,1)$. Since $g_3'(1) < 0$, $g_3(x)$ takes its maximum value at $x_0$. Thus $g_3(x) \geq \min\{g_3(0), g_3(1)\} = 0.$
Thus we have shown $g_n \geq 0$, hence $\partial D_n/\partial x_n \geq 0$ with equality holding if and only if $n = 1$ or $n = 2$, $x_1 = 0$, $q_1 = q$. By letting $x_n$ tend to $x_{n-1}$, we have $D_n \geq D_{n-1}$ (with weights $q_1, \ldots, q_{n-2}, q_{n-1} + q_n$). Since $C$ is an increasing function of $q$, it follows by induction that $D_n > D_{n-1} > \cdots > D_2 = 0$ when $x_1 = 0$, $q_1 = q$ in $D_2$. Else $D_n > D_{n-1} > \cdots > D_1 = 0$. Since we assume $n \geq 2$ in this paper, this completes the proof.

The relations between (1.2) and (1.5) seem to suggest that if (1.2) holds for $r > s > t \geq 0$, $\alpha > 0$, then (1.2) also holds for $r > s > t \geq 0$, $k\alpha$ with $k < 1$ and if (1.3) holds for $r > s > t \geq 0$, $\alpha > 0$, then (1.3) also holds for $r > s > t \geq 0$, $k\alpha$ with $k > 1$. We do not know the answer in general but for a special case, we have the following.

**Theorem 3.2.** Let $r > s > 0$. If (1.2) holds for $\{r, s, 0\}$, $\alpha > 0$, then it also holds for $\{r, s, 0\}, k\alpha$ with $0 < k < 1$. If (1.3) holds for $\{r, s, 0\}$, $\alpha > 0$, then it also holds for $\{r, s, 0\}, k\alpha$ with $k > 1$.

**Proof.** We will only prove the first assertion here and the second can be proved similarly. By the assumption, we have

$$P_{n,r}^{\alpha} - G_{n}^{\alpha} \geq \frac{1}{1 - (q^{\alpha})^{1/s - 1/r}} (P_{n,r}^{\alpha} - P_{n,s}^{\alpha}).$$  \hfill (3.13)

We write the above as

$$P_{n,s}^{\alpha} \geq (q^{\alpha})^{1/s - 1/r} P_{n,r}^{\alpha} + (1 - (q^{\alpha})^{1/s - 1/r}) G_{n}^{\alpha}.$$  \hfill (3.14)

We now need to show for $k > 1$,

$$P_{n,s}^{k\alpha} \geq (q^{k\alpha})^{1/s - 1/r} P_{n,r}^{k\alpha} + (1 - (q^{k\alpha})^{1/s - 1/r}) G_{n}^{k\alpha}.$$  \hfill (3.15)

Note by (3.14), via setting $w = (q^{k\alpha})^{1/s - 1/r}$, $x = G_{n}/P_{n,r}$, it suffices to show

$$f(x) := (w + (1 - w)x^k)^{1/k} - w^{1/k} - (1 - w^{1/k})x \leq 0,$$  \hfill (3.16)

for $0 \leq w, x \leq 1$. Note

$$f'(x) = (1 - w)(wx^{-k} + (1 - w))^{1/k-1} - (1 - w^{1/k}),$$  \hfill (3.17)

thus $f'(x)$ can have at most one root in $(0, 1)$, note also $f(0) = f(1) = 0$ and $f'(1) > 0$, we then conclude $f(x) \leq 0$ for $0 \leq x \leq 1$ and this completes the proof.

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On an inequality of Diananda. Part II.

References


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