The purpose of this paper is to study $n$-dimensional compact CR-submanifolds of complex hyperbolic space $\text{CH}^{(n+p)/2}$, and especially to characterize geodesic hypersphere in $\text{CH}^{(n+1)/2}$ by an integral formula.

1. Introduction

Let $\tilde{M}$ be a complex space form of constant holomorphic sectional curvature $c$ and let $M$ be an $n$-dimensional CR-submanifold of $(n - 1)$ CR-dimension in $\tilde{M}$. Then $M$ has an almost contact metric structure $(F, U, \mu, g)$ (see Section 2) induced from the canonical complex structure of $\tilde{M}$. Hence on an $n$-dimensional CR-submanifold of $(n - 1)$ CR-dimension, we can consider two structures, namely, almost contact structure $F$ and a submanifold structure represented by second fundamental form $A$. In this point of view, many differential geometers have classified $M$ under the conditions concerning those structures (cf. [3, 5, 8, 9, 10, 11, 12, 14, 15, 16]). In particular, Montiel and Romero [12] have classified real hypersurfaces $M$ of complex hyperbolic space $\text{CH}^{(n+1)/2}$ which satisfy the commutativity condition

\[(C)\quad FA = AF\]  

by using the $S^1$-fibration $\pi : H^{n+2} \rightarrow \text{CH}^{(n+1)/2}$ of the anti-de Sitter space $H^{n+2}$ over $\text{CH}^{(n+1)/2}$, and obtained Theorem 4.1 stated in Section 2. We notice that among the model spaces in Theorem 4.1, the geodesic hypersphere is only compact.

In this paper, we will investigate $n$-dimensional compact CR-submanifold of $(n - 1)$ CR-dimension in complex hyperbolic space and provide a characterization of the geodesic hypersphere, which is equivalent to condition $(C)$, by using the following integral formula established by Yano [17, 18]:

\[\int_M \text{div} \left\{ \nabla_X X - (\text{div} X)X \right\} \ast 1 = \int_M \left\{ \text{Ric}(X, X) + \frac{1}{2} \| \mathcal{L}_X g \|^2 - \| \nabla X \|^2 - (\text{div} X)^2 \right\} \ast 1 = 0, \]  

(1.2)
where $X$ is an arbitrary vector field tangent to $M$. Our results of the paper are complex hyperbolic versions of those in [6, 15].

2. Preliminaries

Let $M$ be an $n$-dimensional CR-submanifold of $(n - 1)$ CR-dimension isometrically immersed in a complex space form $\overline{M}^{(n+p)/2}(c)$. Denoting by $(J, \overline{g})$ the Kähler structure of $\overline{M}^{(n+p)/2}(c)$, it follows by definition (cf. [5, 6, 8, 9, 13, 16]) that the maximal $J$-invariant subspace

$$\mathcal{D}_x := T_x M \cap JT_x M$$

(2.1)

of the tangent space $T_x M$ of $M$ at each point $x$ in $M$ has constant dimension $(n - 1)$. So there exists a unit vector field $U_1$ tangent to $M$ such that

$$\mathcal{D}^\perp_x := \text{Span}\{U_1\}, \quad \forall x \in M,$n(2.2)

where $\mathcal{D}^\perp_x$ denotes the subspace of $T_x M$ complementary orthogonal to $\mathcal{D}_x$. Moreover, the vector field $\xi_1$ defined by

$$\xi_1 := JU_1$$

(2.3)

is normal to $M$ and satisfies

$$JT_M \subset TM \oplus \text{Span}\{\xi_1\}.$$ (2.4)

Hence we have, for any tangent vector field $X$ and for a local orthonormal basis $\{\xi_1, \xi_\alpha\}_{\alpha=2,\ldots,p}$ of normal vectors to $M$, the following decomposition in tangential and normal components:

$$JX = FX + u^1(X)\xi_1,$$ (2.5)

$$J\xi_\alpha = -U_\alpha + P\xi_\alpha, \quad \alpha = 1, \ldots, p.$$ (2.6)

Since the structure $(J, \overline{g})$ is Hermitian and $J^2 = -I$, we can easily see from (2.5) and (2.6) that $F$ and $P$ are skew-symmetric linear endomorphisms acting on $T_y M$ and $T_x M^\perp$, respectively, and that

$$g(U_\alpha, X) = -u^1(X)\overline{g}(\xi_1, P\xi_\alpha),$$ (2.7)

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - \overline{g}(P\xi_\alpha, P\xi_\beta),$$ (2.8)

where $T_x M^\perp$ denotes the normal space of $M$ at $x$ and $g$ the metric on $M$ induced from $\overline{g}$. Furthermore, we also have

$$g(U_\alpha, X) = u^1(X)\delta_{1\alpha},$$ (2.9)

and consequently,

$$g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2,\ldots,p.$$ (2.10)
Next, applying $J$ to (2.5) and using (2.6) and (2.10), we have
\[ F^2 X = -X + u^1(X)U_1, \quad u^1(X)P\xi_1 = -u^1(FX)\xi_1, \] (2.11)
from which, taking account of the skew-symmetry of $P$ and (2.7),
\[ u^1(FX) = 0, \quad FU_1 = 0, \quad P\xi_1 = 0. \] (2.12)
Thus (2.6) may be written in the form
\[ J\xi_1 = -U_1, \quad J\xi_\alpha = P\xi_\alpha, \quad \alpha = 2, \ldots, p. \] (2.13)

These equations tell us that $(F, g, U_1, u^1)$ defines an almost contact metric structure on $M$ (cf. [5, 6, 8, 9, 16]), and consequently, $n = 2m + 1$ for some integer $m$.

We denote by $\nabla$ and $\nabla$ the Levi-Civita connection on $\bar{M}(n+p)/2(c)$ and $M$, respectively. Then the Gauss and Weingarten formulas are given by
\[ \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \] (2.14)
\[ \bar{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla_\alpha \xi_\alpha, \quad \alpha = 1, \ldots, p, \] (2.15)
for any vector fields $X, Y$ tangent to $M$. Here $\nabla^\perp$ denotes the normal connection induced from $\nabla$ in the normal bundle $TM^\perp$ of $M$, and $h$ and $A_\alpha$ the second fundamental form and the shape operator corresponding to $\xi_\alpha$, respectively. It is clear that $h$ and $A_\alpha$ are related by
\[ h(X, Y) = \sum_{\alpha=1}^p g(A_\alpha X, Y) \xi_\alpha. \] (2.16)
We put
\[ \nabla_\alpha \xi_\beta = \sum_{\beta=1}^p s_{\alpha\beta}(X) \xi_\beta. \] (2.17)
Then $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^\perp$.

Now, using (2.14), (2.15), and (2.17), and taking account of the Kähler condition $\bar{\nabla} J = 0$, we differentiate (2.5) and (2.6) covariantly and compare the tangential and normal parts. Then we can easily find that
\[ (\nabla_X F)Y = u^1(Y)A_1 X - g(A_1 Y, X) U_1, \] (2.18)
\[ (\nabla_X u^1)(Y) = g(FA_1 X, Y), \] (2.19)
\[ \nabla_X U_1 = FA_1 X, \] (2.20)
\[ g(A_\alpha U_1, X) = -\sum_{\beta=2}^p s_{1\beta}(X) \bar{g}(P\xi_\beta, \xi_\alpha), \quad \alpha = 2, \ldots, p, \] (2.21)
for any $X, Y$ tangent to $M$. 
In the rest of this paper, we suppose that the distinguished normal vector field \( \xi_1 \) is parallel with respect to the normal connection \( \nabla^\perp \). Hence (2.17) gives

\[ s_{1\alpha} = 0, \quad \alpha = 2, \ldots, p, \]  

(2.22)

which, together with (2.21), yields

\[ A_\alpha U_1 = 0, \quad \alpha = 2, \ldots, p. \]  

(2.23)

On the other hand, the ambient manifold \( \tilde{M}^{(n+p)/2}(c) \) is of constant holomorphic sectional curvature \( c \) and consequently, its Riemannian curvature tensor \( \tilde{R} \) satisfies

\[
\tilde{R}_{XZ} = \frac{c}{4} \left\{ g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY - 2g(FX, Y)FZ \right\} \\
\left( \nabla_X A_1 \right) Y - \left( \nabla_Y A_1 \right) X = \frac{c}{4} \left\{ g(X, U_1)FY - g(Y, U_1)FX - 2g(FX, Y)U_1 \right\},
\]

(2.25)

(2.24)

for any \( X, Y, Z \) tangent to \( M \) with the aid of (2.22), where \( R \) denotes the Riemannian curvature tensor of \( M \). Moreover, (2.11) and (2.25) yield

\[
\text{Ric}(X, Y) = \frac{c}{4} \left\{ (n + 2)g(X, Y) - 3u^1(X)u^1(Y) \right\} + \sum_\alpha \left\{ (\text{tr} A_\alpha)g(A_\alpha X, Y) - g(A_\alpha^2 X, Y) \right\},
\]

(2.26)

(2.27)

\[
\rho = \frac{c}{4} (n + 3)(n - 1) + n^2 \| \mu \|^2 - \sum_\alpha (\text{tr} A_\alpha^2),
\]

(2.28)

where \( \text{Ric} \) and \( \rho \) denote the Ricci tensor and the scalar curvature, respectively, and

\[
\mu = \frac{1}{n} \sum_\alpha (\text{tr} A_\alpha) \xi_\alpha
\]

(2.29)

is the mean curvature vector (cf. [1, 2, 4, 19]).

3. Codimension reduction of CR-submanifolds of \( \text{CH}^{(n+p)/2} \)

Let \( M \) be an \( n \)-dimensional CR-submanifold of \((n - 1)\) CR-dimension in a complex hyperbolic space \( \text{CH}^{(n+p)/2} \) with constant holomorphic sectional curvature \( c = -4 \).

Applying the integral formula (1.2) to the vector field \( U_1 \), we have

\[
\int_M \left\{ \text{Ric} (U_1, U_1) + \frac{1}{2} \| \nabla U_1 \|^2 - \| \nabla U_1 \|^2 - (\text{div} U_1)^2 \right\} \ast 1 = 0.
\]

(3.1)
Now we take an orthonormal basis \( \{ U_1, e_a, e_a^\ast \} \) of tangent vectors to \( M \) such that
\[
e_{a^\ast} := Fe_a, \quad a = 1, \ldots, \frac{n - 1}{2}.
\]
(3.2)

Then it follows from (2.11) and (2.20) that
\[
div U_1 = tr (FA_1) = \sum_{a=1}^{(n-1)/2} \left\{ g(FA_1 e_a, e_a) + g(FA_1 e_a^\ast, e_a^\ast) \right\} = 0.
\]
(3.3)

Also, using (2.20), we have
\[
\| \nabla U_1 \|^2 = g(FA_1 U_1, FA_1 U_1) + \sum_{a=1}^{(n-1)/2} \left\{ g(FA_1 e_a, FA_1 e_a) + g(FA_1 e_a^\ast, FA_1 e_a^\ast) \right\},
\]
(3.4)
from which, together with (2.11) and (2.12), we can easily obtain
\[
\| \nabla U_1 \|^2 = tr A_1^2 - \| A_1 U_1 \|^2.
\]
(3.5)

Furthermore, (2.20) yields
\[
(\mathcal{L}_{U_1} g)(X, Y) = g(\nabla_X U_1, Y) + g(\nabla_Y U_1, X) = g((FA_1 - A_1 F)X, Y),
\]
(3.6)
and consequently,
\[
\| \mathcal{L}_{U_1} g \|^2 = \| FA_1 - A_1 F \|^2.
\]
(3.7)

On the other hand, (2.27) and (2.28) with \( c = -4 \) yield
\[
Ric (U_1, U_1) = -(n-1) + u^1 (A_1 U_1) (tr A_1) - \| A_1 U_1 \|^2,
\]
(3.8)
\[
tr (A_1^2) = -\rho - (n+3)(n-1) + n^2 \| \mu \|^2 - \sum_{a=2}^{p} tr A_a^2.
\]
(3.9)

Substituting (3.3), (3.5), (3.7), (3.8), and (3.9) into (3.1), we have
\[
\int_M \left\{ \frac{1}{2} \| FA_1 - A_1 F \|^2 + Ric (U_1, U_1) + \rho - n^2 \| \mu \|^2 \right\} \ast 1 = 0,
\]
(3.10)
or equivalently,
\[
\int_M \left\{ \frac{1}{2} \| FA_1 - A_1 F \|^2 + u^1 (A_1 U_1) (tr A_1) - tr A_1^2 - (n - 1) \right\} \ast 1 = 0.
\]
(3.11)

Thus we have the following lemma.
Lemma 3.1. Let $M$ be an $n$-dimensional compact orientable CR-submanifold of $(n - 1)$ CR-dimension in a complex hyperbolic space $\text{CH}^{(n+p)/2}$. If the distinguished normal vector field $\xi_1$ is parallel with respect to the normal connection and if the inequality

$$\text{Ric} (U_1, U_1) + \rho - n^2 \|\mu\|^2 + \|A_1 U_1\|^2 + (n + 3)(n - 1) \geq 0$$

(3.12)

holds on $M$, then

$$A_1 F = FA_1$$

(3.13)

and $A_\alpha = 0$ for $\alpha = 2, \ldots, p$.

Corollary 3.2. Let $M$ be a compact orientable real hypersurface of $\text{CH}^{(n+1)/2}$ over which the inequality

$$\text{Ric} (U_1, U_1) + \rho - n^2 \|\mu\|^2 + \|A_1 U_1\|^2 + (n + 3)(n - 1) \geq 0$$

(3.14)

holds. Then $M$ satisfies the commutativity condition $(C)$.

Combining Lemma 3.1 and the codimension reduction theorem proved in [7, Theorem 3.2, page 126], we have the following theorem.

Theorem 3.3. Let $M$ be an $n$-dimensional compact orientable CR-submanifold of $(n - 1)$ CR-dimension in a complex hyperbolic space $\text{CH}^{(n+p)/2}$. If the distinguished normal vector field $\xi_1$ is parallel with respect to the normal connection and if the inequality

$$\text{Ric} (U_1, U_1) + \rho - n^2 \|\mu\|^2 + \|A_1 U_1\|^2 + (n + 3)(n - 1) \geq 0$$

(3.15)

holds on $M$, then there exists a totally geodesic complex hyperbolic space $\text{CH}^{(n+1)/2}$ immersed in $\text{CH}^{(n+p)/2}$ such that $M \subset \text{CH}^{(n+1)/2}$. Moreover $M$ satisfies the commutativity condition $(C)$ as a real hypersurface of $\text{CH}^{(n+1)/2}$.

Proof. Let

$$N_0(x) := \{ \eta \in T_x M^\perp \mid A_\eta = 0 \}$$

(3.16)

and let $H_0(x)$ be the maximal holomorphic subspace of $N_0(x)$, that is,

$$H_0(x) = N_0(x) \cap JN_0(x).$$

(3.17)

Then, by means of Lemma 3.1,

$$H_0(x) = N_0(x) = \text{Span}\{\xi_2, \ldots, \xi_p\}.$$ 

(3.18)

Hence, the orthogonal complement $H_1(x)$ of $H_0(x)$ in $TM^\perp$ is $\text{Span}\{\xi_1\}$ and so, $H_1(x)$ is invariant under the parallel translation with respect to the normal connection and $\dim H_1(x) = 1$ at any point $x \in M$. Thus, applying the codimension reduction theorem in [4] proved by Kawamoto, we verify that there exists a totally geodesic complex hyperbolic space $\text{CH}^{(n+1)/2}$ immersed in $\text{CH}^{(n+p)/2}$ such that $M \subset \text{CH}^{(n+1)/2}$. Therefore, $M$ can
be regarded as a real hypersurface of \( CH^{(n+1)/2} \) which is totally geodesic in \( CH^{(n+p)/2} \). Tentatively, we denote \( CH^{(n+1)/2} \) by \( M' \), and by \( i_1 \) we denote the immersion of \( M \) into \( M' \), and by \( i_2 \) the totally geodesic immersion of \( M' \) into \( CH^{(n+p)/2} \). Then it is clear from (2.14) that

\[
\nabla'_{i_1 X} i_1 Y = i_1 \nabla_X Y + h'(X,Y) = i_1 \nabla_X Y + g(A'X,Y)\xi',
\]

where \( \nabla' \) is the induced connection on \( M' \) from that of \( CH^{(n+p)/2} \), \( h' \) the second fundamental form of \( M \) in \( M' \), and \( A' \) the corresponding shape operator to a unit normal vector field \( \xi' \) to \( M \) in \( M' \). Since \( i = i_2 \circ i_1 \) and \( M' \) is totally geodesic in \( CH^{(n+p)/2} \), we can easily see that (2.15) and (3.19) imply that

\[
\xi_1 = i_2 \xi', \quad A_1 = A'.
\]

Since \( M' \) is a holomorphic submanifold of \( CH^{(n+p)/2} \), for any \( X \) in \( TM \),

\[
Ji_2 X = i_2 J'X
\]

is valid, where \( J' \) is the induced Kähler structure on \( M' \). Thus it follows from (2.5) that

\[
JiX = Ji_2 \circ i_1 X = i_2 J' i_1 X = i_2 (i_1 F'X + u'(X)\xi')
= iF'X + u'(X)i_2 \xi' = iF'X + u'(X)\xi
\]

for any vector field \( X \) tangent to \( M \). Comparing this equation with (2.5), we have \( F = F' \) and \( u^i = u' \), which, together with Lemma 3.1, implies that

\[
A'F' = F'A'.
\]

\[\square\]

4. An integral formula on the model space \( M^{2p+1,2q+1}_g(r) \)

We first explain the model hypersurfaces of complex hyperbolic space due to Montiel and Romero for later use (for the details, see [12]).

Consider the complex \((n+3)/2\)-space \( C^{1(n+3)/2}_1 \) endowed with the pseudo-Euclidean metric \( g_0 \) given by

\[
g_0 = -dz_0d\bar{z}_0 + \sum_{j=1}^{m} dz_jd\bar{z}_j, \quad (m+1 := \frac{n+3}{2}),
\]

where \( \bar{z}_k \) denotes the complex conjugate of \( z_k \).

On \( C^{1(n+3)/2}_1 \), we define

\[
F(z,w) = -z_0\bar{w}_0 + \sum_{k=1}^{m} z_k\bar{w}_k.
\]

Put

\[
H^{(n+2)}_i = \left\{ z = (z_0, z_1, \ldots, z_m) \in C^{1(n+3)/2}_1 : \langle z, z \rangle = -1 \right\},
\]
where $\langle \cdot, \cdot \rangle$ denotes the inner product on $C_1^{(n+3)/2}$ induced from $g_0$. Then it is known that $H^{n+2}_1$, together with the induced metric, is a pseudo-Riemannian manifold of constant sectional curvature $-1$, which is known as an anti-de Sitter space. Moreover, $H^{n+2}_1$ is a principal $S^1$-bundle over $\text{CH}^{(n+1)}/2$ with projection $\pi: H^{n+2}_1 \to \text{CH}^{(n+1)/2}$ which is a Riemannian submersion with fundamental tensor $f$ and time-like totally geodesic fibers.

Given $p$, $q$ integers with $2p + 2q = n - 1$ and $r \in R$ with $0 < r < 1$, we denote by $M_{2p+1,2q+1}(r)$ the Lorentz hypersurface of $H^{n+2}_1$ defined by the equations

$$- |z_0|^2 + \sum_{k=1}^{m} |z_k|^2 = -1, \quad r \left( - |z_0|^2 + \sum_{k=1}^{p} |z_k|^2 \right) = - \sum_{k=p+1}^{m} |z_k|^2, \quad (4.4)$$

where $z = (z_0, z_1, \ldots, z_m) \in C_1^{(n+3)/2}$. In fact, $M_{2p+1,2q+1}(r)$ is isometric to the product

$$H^{2p+1}_1 \left( \frac{1}{r-1} \right) \times S^{2q+1} \left( \frac{r}{1-r} \right), \quad (4.5)$$

where $1/(r-1)$ and $r/(1-r)$ denote the squares of the radii and each factor is embedded in $H^{n+2}_1$ in a totally umbilical way. Since $M_{2p+1,2q+1}(r)$ is $S^1$-invariant, $M_{2p+1,2q+1}(r) := \pi(M_{2p+1,2q+1}(r))$ is a real hypersurface of $\text{CH}^{(n+1)/2}$ which is complete and satisfies the condition (C).

As already mentioned in Section 1, Montiel and Romero [12] have classified real hypersurfaces $M$ of $\text{CH}^{(n+1)/2}$ which satisfy the condition (C) and obtained the following classification theorem.

**Theorem 4.1.** Let $M$ be a complete real hypersurface of $\text{CH}^{(n+1)/2}$ which satisfies the condition (C). Then there exist the following possibilities.

1. $M$ has three constant principal curvatures $\tanh \theta$, $\coth \theta$, $2 \coth 2 \theta$ with multiplicities $2p$, $2q$, 1, respectively, $2p + 2q = n - 1$. Moreover, $M$ is congruent to $M_{2p+1,2q+1}(\tanh^2 \theta)$.

2. $M$ has two constant principal curvatures $\lambda_1, \lambda_2$ with multiplicities $n - 1$ and 1, respectively. (i) If $\lambda_1 > 1$, then $\lambda_1 = \coth \theta$, $\lambda_2 = 2 \coth 2 \theta$ with $\theta > 0$, and $M$ is congruent to a geodesic hypersphere $M_{1,n}^h(\tanh^2 \theta)$. (ii) If $\lambda_1 < 1$, then $\lambda_1 = \tanh \theta$, $\lambda_2 = 2 \coth 2 \theta$ with $\theta > 0$, and $M$ is congruent to $M_{n,1}^h(\tanh^2 \theta)$. (iii) If $\lambda_1 = 1$, then $\lambda_2 = 2$ and $M$ is congruent to a horosphere.

Combining Corollary 3.2 and Theorem 4.1, we have the following theorem.

**Theorem 4.2.** Let $M$ be a compact orientable real hypersurface of $\text{CH}^{(n+1)/2}$ over which the inequality

$$\text{Ric} \ (U_1, U_1) + \rho - n^2 \|\mu\|^2 + \|A_1 U_1\|^2 + (n + 3)(n - 1) \geq 0 \quad (4.6)$$

holds. Then $M$ is congruent to a geodesic hypersphere $M_{1,n}^h(r)$ in $\text{CH}^{(n+1)/2}$.

Combining Theorems 3.3 and 4.2, we have the following theorem.
Theorem 4.3. Let $M$ be an $n$-dimensional compact orientable CR-submanifold of $(n-1)$ CR-dimension in a complex hyperbolic space $\text{CH}^{(n+p)/2}$. If the distinguished normal vector field $\xi_1$ is parallel with respect to the normal connection and if the inequality

$$\text{Ric} (U_1, U_1) + \rho - n^2 \|\mu\|^2 + \|A_1 U_1\|^2 + (n + 3)(n - 1) \geq 0$$

(4.7)

holds on $M$, then $M$ is congruent to a geodesic hypersphere $M_{h,n}^h(\tanh^2 \theta)$ in $\text{CH}^{(n+1)/2}$.

Remark 4.4. As already shown in (3.10) and (3.11), the equality

$$\text{Ric} (U_1, U_1) + \rho - n^2 \|\mu\|^2 + \|A_1 U_1\|^2 + (n + 3)(n - 1) = u^1 (A_1 U_1) (\text{tr} A_1) - \text{tr} A_1^2 - (n - 1)$$

(4.8)

holds on $M$. On the other hand, the geodesic hypersphere $M_{h,n}^h(\tanh^2 \theta)$ in Theorem 4.1 has constant principal curvatures $\coth \theta$ and $2\coth 2\theta$ with multiplicities $n - 1$ and 1, respectively. Hence we can easily verify the equality

$$u^1 (A_1 U_1) (\text{tr} A_1) - \text{tr} A_1^2 - (n - 1) = 0,$$

(4.9)

and consequently,

$$\text{Ric} (U_1, U_1) + \rho - n^2 \|\mu\|^2 + \|A_1 U_1\|^2 + (n + 3)(n - 1) = 0$$

(4.10)

on $M_{h,n}^h(\tanh^2 \theta)$.

Remark 4.5. If we put $V := \nabla_{U_1} U_1 - (\text{div} U_1) U_1$, then it easily follows from (2.11) that $V = FA_1 U_1$. Taking account of (3.3), (3.5), (3.7), and (3.8), we obtain

$$\text{div} V = \frac{1}{2} \|FA_1 - A_1 F\|^2 + u^1 (A_1 U_1) (\text{tr} A_1) - \text{tr} A_1^2 - (n - 1).$$

(4.11)

Hence if the commutativity condition (C) holds on $M$, then the vector field $V$ is zero since $U_1$ is a principal vector of $A_1$, and consequently,

$$u^1 (A_1 U_1) (\text{tr} A_1) - \text{tr} A_1^2 - (n - 1) = 0.$$ 

(4.12)

Thus, on $n$-dimensional CR-submanifold $M$ of $(n - 1)$ CR-dimension in a complex hyperbolic space $\text{CH}^{(n+p)/2}$ over which the commutativity condition C holds, the function $u^1 (A_1 U_1)$ cannot be zero at any point of $M$. A real hypersurface of a complex hyperbolic space $\text{CH}^{(n+p)/2}$ satisfying the commutativity condition (C) cannot be minimal.

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Application of an integral formula to CR-submanifolds

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