INTUITIONISTIC $H$-FUZZY RELATIONS

KUL HUR, SU YOUN JANG, AND HEE WON KANG

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We introduce the category $\text{IRel}(H)$ consisting of intuitionistic fuzzy relational spaces on sets and we study structures of the category $\text{IRel}(H)$ in the viewpoint of the topological universe introduced by Nel. Thus we show that $\text{IRel}(H)$ satisfies all the conditions of a topological universe over $\text{Set}$ except the terminal separator property and $\text{IRel}(H)$ is cartesian closed over $\text{Set}$.

1. Introduction

In 1965, Zadeh [30] introduced a concept of a fuzzy set as the generalization of a crisp set. Also, in 1971, he introduced a fuzzy relation naturally, as a generalization of a crisp relation in [31].

Nel [27] introduced the notion of a topological universe which implies concrete quasitopos [1]. Every topological universe satisfies all the properties of a topos except one condition on the subobject classifier. The notion of a topological universe has already been put to effective use in several areas of mathematics in [24, 25, 28]. In 1980, Cerruti [8] introduced the category of $L$-fuzzy relations and investigated some of its properties. After that time, Hur [14] introduced the category $\text{Rel}(H)$ of the fuzzy relational spaces with a complete Heyting algebra $H$ as a codomain and he studied the category $\text{Rel}(H)$ in the sense of a topological universe.

In 1983, Atanassov [2] introduced the concept of an intuitionistic fuzzy set as the generalization of fuzzy sets and he also investigated many properties of intuitionistic fuzzy sets (cf. [3]). After that time, Banerjee and Basnet [4], Biswas [6], and Hur and his colleagues [15, 16, 17, 20] applied the concept of intuitionistic fuzzy sets to algebra. Also, Çoker [9], Hur and his colleagues [21], and S. J. Lee and E. P. Lee [26] applied one to topology. In particular, Hur and his colleagues [18] applied the notion of intuitionistic fuzzy sets to topological group.

In this paper, we introduce the category $\text{IRel}(H)$ of intuitionistic $H$-fuzzy relational spaces and study the category $\text{IRel}(H)$ in a topological universe viewpoint. In particular, we show that $\text{IRel}(H)$ satisfies all the conditions of a topological universe over $\text{Set}$ except...
the terminal separator property. Also $\textbf{IRel}(H)$ is shown to be cartesian closed over $\textbf{Set}$. For general categorical background, we refer to Herrlich and Strecker [12].

2. Preliminaries

In this section, we will introduce some basic definitions and well-known results which are needed in the next sections.

Let $X$ be a set, let $(X_i)_{i \in I}$ be a family of sets indexed by a class $I$, and let $f_i$ be a mapping with domain $X$ for each $i \in I$. Then a pair $(X,(f_i)_I)$ (simply, $(f_i)_I$) is called a source of mappings. A sink of mappings is the dual notion of a source of mappings.

**Definition 2.1** [12]. Let $\mathbf{A}$ be a concrete category and let $I$ be a class.

1. A source in $\mathbf{A}$ is a pair $(X,(f_i)_I)$ (simply, $(X,f_i)$ or $(f_i)_I$), where $X$ is an $\mathbf{A}$-object and $(f_i : X \to X_i)_I$ is a family of $\mathbf{A}$-morphisms each with domain $X$. In this case, $X$ is called the domain of the source and the family $(X_i)_I$ is called the codomain of the source.

2. A source $(X,f_i)$ is called a monosource provided that the $f_i$ can be simultaneously canceled from the left; that is, provided that for any pair $Y \xrightarrow{r} X$ of morphisms such that $f_i \circ r = f_i \circ s$ for each $i \in I$, it follows that $r = s$.

Dual notions: sink in $\mathbf{A}$ and episink.

**Definition 2.2** [23]. Let $\mathbf{A}$ be a concrete category and let $((Y_i,\xi_i))_I$ be a family of objects in $\mathbf{A}$ indexed by a class $I$. For any set $X$, let $((f_i : X \to Y_i)_I$ be a source of mappings indexed by $I$. An $\mathbf{A}$-structure $\xi$ on $X$ is said to be initial with respect to $(X,(f_i)_I,(Y_i,\xi_i))$ provided that the following conditions hold.

1. For each $i \in I$, $f_i : (X,\xi) \to (Y_i,\xi_i)$ is an $\mathbf{A}$-morphism.

2. If $(Z,\rho)$ is an $\mathbf{A}$-object and $g : Z \to X$ is mapping such that for each $i \in Z$, the mapping $f_i \circ g : (Z,\rho) \to (Y_i,\xi_i)$ is an $\mathbf{A}$-morphism, then $g : (Z,\rho) \to (X,\xi)$ is an $\mathbf{A}$-morphism. In this case, $(f_i : (X,\xi) \to (Y_i,\xi_i))_I$ is called an initial source in $\mathbf{A}$.

Dual notions: final structure and final sink.

**Definition 2.3** [23]. A concrete category $\mathbf{A}$ is said to be topological over $\textbf{Set}$ provided that for each set $X$, for any family $((Y_i,\xi_i))_I$ of $\mathbf{A}$-objects, and for any source $(f_i : X \to Y_i)_I$ of mappings, there exists a unique $\mathbf{A}$-structure $\xi$ on $X$ which is initial with respect to $(X,(f_i)_I,(Y_i,\xi_i))$.

Dual notions: cotopological category.

**Result 2.4** [23, Theorem 1.5]. A concrete category $\mathbf{A}$ is topological if and only if $\mathbf{A}$ is cotopological.

**Result 2.5** [23, Theorem 1.6]. Let $\mathbf{A}$ be a topological category over $\textbf{Set}$. Then $\mathbf{A}$ is complete and cocomplete.

**Definition 2.6** [11]. A category $\mathbf{A}$ is called cartesian closed provided that the following conditions hold.

1. For any $\mathbf{A}$-objects $A$ and $B$, there exists a product $A \times B$ in $\mathbf{A}$.

2. Exponential exists in $\mathbf{A}$, that is, for any $\mathbf{A}$-object $A$, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, that is, for any $\mathbf{A}$-object $B$, there exists an $\mathbf{A}$-object $B^A$ and an $\mathbf{A}$-morphism $e_{A,B} : A \times B^A \to B$ (called the evaluation) such that for any $\mathbf{A}$-object $C$
and any $A$-morphism $f : A \times C \to B$, there exists a unique $A$-morphism $\overline{f} : C \to B^A$ such that the diagram

$$
\begin{array}{ccc}
A \times B^A & \xrightarrow{e_{A,B}} & B \\
\downarrow \exists 1_{A \times B^A} \quad & & \downarrow f \\
A \times C & \xrightarrow{f} & B \\
\end{array}
$$

(2.1)

commutes.

**Definition 2.7** [23]. Let $A$ be a concrete category.

(1) The $A$-fiber of a set $X$ is the class of all $A$-structures on $X$.

(2) $A$ is called properly fibered over $\textbf{Set}$ provided that the following conditions hold.

(i) Fiber-smallness. For each set $X$, the $A$-fiber of $X$ is a set.

(ii) Terminal separator property. For each singleton set $X$, the $A$-fiber of $X$ has precisely one element.

(iii) If $\xi$ and $\eta$ are $A$-structures on a set $X$ such that $1_X : (X, \xi) \to (X, \eta)$ and $1_X : (X, \eta) \to (X, \xi)$ are $A$-morphisms, then $\xi = \eta$.

**Definition 2.8** [27]. A category $A$ is called a topological universe over $\textbf{Set}$ provided that the following conditions hold.

(1) $A$ is well structured over $\textbf{Set}$, that is, (i) $A$ is a concrete category; (ii) $A$ has the fiber-smallness condition; (iii) $A$ has the terminal separator property.

(2) $A$ is cotopological over $\textbf{Set}$.

(3) Final episinks in $A$ are preserved by pullbacks, that is, for any final episink $(g_\lambda : X \to Y)_\Lambda$ and any $A$-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of $f$ and $g_\lambda$ for each $\lambda$, is again a final episink.

**Definition 2.9** [29]. A category $A$ is called a topos provided that the following conditions hold.

(1) There is a terminal object $U$ in $A$, that is, for each $A$-object $A$, there exists one and only one $A$-morphism from $A$ to $U$.

(2) $A$ has equalizers, that is, for any $A$-objects $A$ and $B$ and $A$-morphisms $A \xrightarrow{f} B$ (2.2) there exist an $A$-object $C$ and an $A$-morphism $h : C \to A$ such that

(a) $f \circ h = g \circ h$,

(b) for each $A$-object $C'$ and $A$-morphism $h' : C' \to A$ with $f \circ h' = g \circ h'$, there exists a unique $A$-morphism $\overline{h'} : C' \to C$ such that $h' = h \circ \overline{h'}$, that is, the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{h} & A \xrightarrow{f} B \\
\downarrow \exists h' \quad & & \downarrow g \\
C' & \xrightarrow{h'} & B \\
\end{array}
$$

(2.3)

commutes;
Remark 2.10. Let $A$ be any category with a subobject classifier. If $f$ is any bimorphism in $A$, then $f$ is an isomorphism in $A$ (cf. [7]).

3. The category $\text{IRel}(H)$

First we will list some concepts and one result which are needed in this section and the next section. Next, we introduce the category $\text{IRel}(H)$ of intuitionistic $H$-fuzzy relational spaces and show that it has similar structures as those of $\text{ISet}(H)$.

Definition 3.1 [5, 22]. A lattice $H$ is called a complete Heyting algebra if $H$ satisfies the following conditions:

1. $H$ is a complete lattice;
2. for any $a, b \in H$, the set $\{x \in H : x \land a \leq b\}$ has a greatest element denoted by $a \to b$ (called pseudocomplement of $a$ and $b$), that is, $x \land a \leq b$ if and only if $x \leq (a \to b)$.

In particular, for each $a \in H$, $N(a) = a \to 0$ is called the negation or the pseudocomplement of $a$.

Result 3.2 [5, Example 6, page 46]. Let $H$ be a complete Heyting algebra and let $a, b \in H$. Then

1. if $a \leq b$, then $N(b) \leq N(a)$, that is, $N : H \to H$ is an involutive order-reversing operation in $(H, \leq)$;
2. $a \leq NN(a)$;
3. $N(a) = NNN(a)$;
4. $N(a \lor b) = N(a) \land N(b)$ and $N(a \land b) = N(a) \land N(b)$.

Throughout this paper, we use $H$ as a complete Heyting algebra.

Definition 3.3 [19]. Let $X$ be a set. A triple $(X, \mu, \nu)$ is called an intuitionistic $H$-fuzzy set (in short, $\text{IHFS}$) on $X$ if the following conditions holds:

1. $\mu, \nu \in H^X$, that is, $\mu$ and $\nu$ are $H$-fuzzy sets;
2. $\mu \leq N(\nu)$, that is, $\mu(x) \leq N(\nu(x))$ for each $x \in X$, where $N : H \to H$ is an involutive order-reversing operation in $(H, \leq)$.

Definition 3.4 [19]. Let $(X, \mu_X, \nu_X)$ and $(Y, \mu_Y, \nu_Y)$ be $\text{IHFSs}$. A mapping $f : X \to Y$ is called a morphism if $\mu_X \leq \mu_Y \circ f$ and $\nu_X \geq \nu_Y \circ f$.

From Definitions 3.3 and 3.4, we can form a concrete category $\text{ISet}(H)$ consisting of all $\text{IHFSs}$ and morphisms between them. In this case, each $\text{ISet}(H)$-morphism will be called an $\text{ISet}(H)$-mapping.
It is clear that if \( f : (X, \mu_X, \nu_X) \to (Y, \mu_Y, \nu_Y) \) is an ISet\((H)\)-mapping, then \( f : (X, \mu_X) \to (Y, \mu_Y) \) is a \( \text{Set} \, (H)\)-mapping (cf. [13]).

**Definition 3.5** [14]. (1) Let \( X \) be a set. \( R \) is called an \( H\)-fuzzy relation (or simply, a fuzzy relation) on \( X \) if \( \mu_R : X \times X \to H \) is a mapping. In this case, \((X, R)\) is called an \( H\)-fuzzy relational space (or simply, a fuzzy relational space).

(2) Let \( (X, R_X) \) and \( (Y, R_Y) \) be any fuzzy relational spaces. A map \( f : X \to Y \) is called a relation-preserving map provided that \( \mu_R \leq \mu_R \circ f^2 \), where \( f^2 = f \times f \).

From Definition 3.5, we can form a concrete category \( \text{Rel}(H) \) consisting of all relational spaces and relation preserving mappings between them. Every \( \text{Rel}(H)\)-morphism will be called a \( \text{Rel} \, (H)\)-mapping.

**Definition 3.6.** Let \( X \) be a set. A pair \( R = (\mu_R, \nu_R) \) is called an intuitionistic \( H\)-fuzzy relation (in short, IHFR) on \( X \) if it satisfies the following conditions:

(i) \( \mu_R : X \times X \to H \) and \( \nu_R : X \times X \to H \) are mappings, where \( \mu_R \) and \( \nu_R \) denote the degree of membership (namely, \( \mu_R(x, y) \)) and the degree of nonmembership (namely, \( \nu_R(x, y) \)) of each \( (x, y) \in X \times X \) to \( R \);

(ii) \( \mu_R \leq N(\nu_R) \), that is, \( \mu_R(x, y) \leq N(\nu_R(x, y)) \) for each \( (x, y) \in X \times X \).

In this case, \((X, R)\) or \((X, \mu_R, \nu_R)\) is called an intuitionistic \( H\)-fuzzy relational space (in short, IHFRS).

**Definition 3.7.** Let \( (X, R_X) \) and \( (Y, R_Y) \) be an IHFRSs. A mapping \( f : X \to Y \) is called a relation-preserving mapping if \( \mu_{R_X} \leq \mu_{R_Y} \circ f^2 \) and \( \nu_{R_X} \geq \nu_{R_Y} \circ f^2 \), where \( f^2 = f \times f \).

The following is the immediate result of Definition 3.7.

**Proposition 3.8.** Let \( (X, R_X), (Y, R_Y) \), and \( (Z, R_Z) \) be IHFRSs.

(1) \( 1_X : (X, R_X) \to (X, R_X) \) is a relation-preserving mapping.

(2) If \( f : (X, R_X) \to (Y, R_Y) \) and \( g : (Y, R_Y) \to (Z, R_Z) \) are relation-preserving mappings, then \( g \circ f : (X, R_X) \to (Z, R_Z) \) is a relation-preserving mapping.

From Definitions 3.6 and 3.7, and Proposition 3.8, we can form a concrete category IRel\((H)\) consisting of all IHFRSs and relation-preserving mappings between them. Every IRel\((H)\)-morphism will be called an IRel\((H)\)-mapping. Moreover, it is clear that if \( f : (X, R_X) \to (Y, R_Y) \) is an IRel\((H)\)-mapping, then \( f : (X, \mu_{R_X}) \to (Y, \mu_{R_Y}) \) is a Rel \((H)\)-mapping.

**Theorem 3.9.** IRel\((H)\) is topological over Set.

**Proof.** Let \( X \) be any set and let \(((X_{\alpha}, R_{\alpha}))_{\Gamma}\) be any family of IHFRSs indexed by a class \( \Gamma \). Let \((f_{\alpha} : X \to X_{\alpha})_{\Gamma}\) be any source of mappings. We define two mappings \( \mu_R : X \times X \to H \) and \( \nu_R : X \times X \to H \) by \( \mu_R = \bigwedge_{\Gamma} \mu_{R_{\alpha}} \circ f_{\alpha}^2 \) and \( \nu_R = \bigvee_{\Gamma} \nu_{R_{\alpha}} \circ f_{\alpha}^2 \). Then, by the definition of \( R = (\mu_R, \nu_R) \), \( \mu_R \leq N(\nu_R) \). Thus \((X, R) \in \text{IRel}(H)\). Moreover, \( f_{\alpha} : (X, R) \to (X_{\alpha}, R_{\alpha}) \) is an IRel\((H)\)-mapping for each \( \alpha \in \Gamma \).

For any \((Y, R_Y) \in \text{IRel}(H)\), let \( g : Y \to X \) be any mapping for which \( f_{\alpha} \circ g : (Y, R_Y) \to (X_{\alpha}, R_{\alpha}) \) is an IRel\((H)\)-mapping for each \( \alpha \in \Gamma \). Then we can easily check that \( g : (Y, R_Y) \to (X, R) \) is an IRel\((H)\)-mapping. Hence \( R = (\mu_R, \nu_R) \) is the initial structure on \( X \) with respect to \((X, (f_{\alpha}), ((X_{\alpha}, R_{\alpha}))_{\Gamma})\). This completes the proof. \( \square \)
Example 3.10. (1) Inverse image of an IHFR. Let $X$ be a set, let $(Y, R_Y)$ be an IHFRS, and let $f : X \rightarrow Y$ be any mapping. Then there exists the initial IHFR $R$ on $X$ for which $f : (X, R) \rightarrow (Y, R_Y)$ is an $\text{IRel}(H)$-mapping. In this case, $R$ is called the inverse image of $R_Y$ under $f$. In particular, if $X \subseteq Y$ and $f : X \rightarrow Y$ is the canonical mapping, then $(X, R)$ is called an intuitionistic $H$-fuzzy relational subspace of $(Y, R_Y)$, where $R = (\mu_R, \nu_R)$ is the inverse image of $R_Y$ under $f$. In fact, $\mu_R = \mu_{R_Y}|_{X \times X}$ and $\nu_R = \nu_{R_Y}|_{X \times X}$.

(2) Intuitionistic fuzzy product structure. Let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IHFRSs and let $X = \prod X_\alpha$ be the product set of $(X_\alpha)_\Gamma$. Then there exists the initial IHFR $R$ on $X$ for which each projection $\pi_\alpha : (X, R) \rightarrow (X_\alpha, R_\alpha)$ is an $\text{IRel}(H)$-mapping. In this case, $R$ is called the product of $(R_\alpha)_\Gamma$ and is denoted by $R = \prod R_\alpha$ and $(\prod X_\alpha, \prod R_\alpha)$ is called the intuitionistic $H$-fuzzy product relational space of $((X_\alpha, R_\alpha))_\Gamma$. In fact, $\mu_R = \bigwedge_\Gamma \mu_{R_\alpha} \circ \pi_\alpha^2$ and $\nu_R = \bigvee_\Gamma \nu_{R_\alpha} \circ \pi_\alpha^2$.

In particular, if $H = \{1, 2\}$, then $\mu_{R_1 \times R_2}((x_1, y_1), (x_2, y_2)) = \mu_{R_1}(x_1, x_2) \wedge \mu_{R_2}(y_1, y_2)$ and $\nu_{R_1 \times R_2}((x_1, y_1), (x_2, y_2)) = \nu_{R_1}(x_1, x_2) \vee \nu_{R_2}(y_1, y_2)$ for any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.

Corollary 3.11. $\text{IRel}(H)$ is complete and cocomplete. Moreover, by definition, it is easy to show that $\text{IRel}(H)$ is well powered and co-well-powered.

From Result 2.4 and Theorem 3.9, it is clear that $\text{IRel}(H)$ is cotopological. However, we show directly that $\text{IRel}(H)$ is cotopological.

Theorem 3.12. $\text{IRel}(H)$ is cotopological over Set.

Proof. Let $X$ be any set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IHFRSs indexed by a class $\Gamma$. Let $(f_\alpha : X_\alpha \rightarrow X)_\Gamma$ be any sink of mappings. We define two mappings $\mu_R : X \times X \rightarrow H$ and $\nu_R : X \times X \rightarrow H$ by, for each $(x, y) \in X \times X$,

$$
\mu_R(x, y) = \bigvee_\Gamma \bigvee_{(x_\alpha, y_\alpha) \in f^{-1\circ}_\alpha(x, y)} \mu_{R_\alpha}(x_\alpha, y_\alpha),
$$

$$
\nu_R(x, y) = \bigwedge_\Gamma \bigwedge_{(x_\alpha, y_\alpha) \in f^{-1\circ}_\alpha(x, y)} \nu_{R_\alpha}(x_\alpha, y_\alpha),
$$

where $f^{-1\circ}_\alpha = f^{-1}_\alpha \times f^{-1}_\alpha$. Then clearly $(X, R) \in \text{IRel}(H)$. Moreover, $f_\alpha : (X_\alpha, R_\alpha) \rightarrow (X, R)$ is an $\text{IRel}(H)$-mapping for each $\alpha \in \Gamma$.

For any $(Y, R_Y) \in \text{IRel}(H)$, let $g : X \rightarrow Y$ be any mapping for which $g \circ f_\alpha : (X_\alpha, R_\alpha) \rightarrow (Y, R_Y)$ is an $\text{IRel}(H)$-mapping for each $\alpha \in \Gamma$. Then we can easily check that $g : (X, R) \rightarrow (Y, R_Y)$ is an $\text{IRel}(H)$-mapping. Hence $R = (\mu_R, \nu_R)$ is the final structure on $X$ with respect to $(((X_\alpha, R_\alpha)), (f_\alpha), X)$. This completes the proof.

Example 3.13. (1) Intuitionistic $H$-fuzzy quotient relation. Let $(X, R) \in \text{IRel}(H)$, let $\sim$ be an equivalence relation on $X$, and let $\varphi : X \rightarrow X/R$ the canonical mapping. Then there exists the final intuitionistic $H$-fuzzy relation $(\mu_{X/\sim}, \nu_{X/\sim})$ on $X/\sim$ for which $\varphi : (X, R) \rightarrow (X/\sim, \mu_{X/\sim}, \nu_{X/\sim})$ is an $\text{IRel}(H)$-mapping. In this case, $(\mu_{X/\sim}, \nu_{X/\sim})$ is called the intuitionistic $H$-fuzzy quotient relation of $X$ by $R$.

(2) Sum of intuitionistic $H$-fuzzy relations. Let $((X_\alpha, R_\alpha))_\Gamma$ be a family of IHFRSs, let $X$ be the sum of $(X_\alpha)_\Gamma$ and let $j_\alpha : X_\alpha \rightarrow X$ be the canonical (injection) mapping for

$$
\begin{align*}
\mu_R(x, y) &= \bigvee_\Gamma \bigvee_{(x_\alpha, y_\alpha) \in f^{-1\circ}_\alpha(x, y)} \mu_{R_\alpha}(x_\alpha, y_\alpha), \\
\nu_R(x, y) &= \bigwedge_\Gamma \bigwedge_{(x_\alpha, y_\alpha) \in f^{-1\circ}_\alpha(x, y)} \nu_{R_\alpha}(x_\alpha, y_\alpha),
\end{align*}
$$

where $f^{-1\circ}_\alpha = f^{-1}_\alpha \times f^{-1}_\alpha$. Then clearly $(X, R) \in \text{IRel}(H)$. Moreover, $f_\alpha : (X_\alpha, R_\alpha) \rightarrow (X, R)$ is an $\text{IRel}(H)$-mapping for each $\alpha \in \Gamma$.

For any $(Y, R_Y) \in \text{IRel}(H)$, let $g : X \rightarrow Y$ be any mapping for which $g \circ f_\alpha : (X_\alpha, R_\alpha) \rightarrow (Y, R_Y)$ is an $\text{IRel}(H)$-mapping for each $\alpha \in \Gamma$. Then we can easily check that $g : (X, R) \rightarrow (Y, R_Y)$ is an $\text{IRel}(H)$-mapping. Hence $R = (\mu_R, \nu_R)$ is the final structure on $X$ with respect to $(((X_\alpha, R_\alpha)), (f_\alpha), X)$. This completes the proof.
Then there exists the final IHFR \( R \) on \( X \). In fact, for each \( ((x_\alpha, \alpha), (y_\beta, \beta)) \in X \times X \), \( \mu_R((x_\alpha, \alpha), (y_\beta, \beta)) = \vee_\Gamma \mu_{R_\alpha}(x, y) \) and \( \nu_R((x_\alpha, \alpha), (y_\beta, \beta)) = \bigwedge_\Gamma \nu_{R_\alpha}(x, y) \). In this case, \( R \) is called the sum of \( (R_\alpha)_\Gamma \) and \( (X, R) \) is called the sum of \( ((X_\alpha, R_\alpha))_\Gamma \).

**Theorem 3.14.** Final episinks in \( \text{IRel}(H) \) are preserved by pullbacks.

**Proof.** Let \( (g_\alpha : (X_\alpha, R_\alpha) \to (Y_\alpha, R_\alpha))_\Gamma \) be any final episink in \( \text{IRel}(H) \) and let \( f : (W, R_W) \to (Y, R_Y) \) be any \( \text{IRel}(H) \)-mapping. For each \( \alpha \in \Gamma \), let \( U_\alpha = \{(w, x_\alpha) \in W \times X_\alpha : f(w) = g_\alpha(x_\alpha)\} \) and let us define two mappings \( \mu_{R_{U_\alpha}} : U_\alpha \times U_\alpha \to H \) and \( \nu_{R_{U_\alpha}} : U_\alpha \times U_\alpha \to H \) by for each \( ((w, x_\alpha), (w', x'_\alpha)) \in U_\alpha \times U_\alpha, \)

\[
\begin{align*}
\mu_{R_{U_\alpha}}((w, x_\alpha), (w', x'_\alpha)) &= \mu_{R_{W}}(w, w') \land \mu_{R_\alpha}(x_\alpha, x'_\alpha), \\
\nu_{R_{U_\alpha}}((w, x_\alpha), (w', x'_\alpha)) &= \nu_{R_{W}}(w, w') \lor \nu_{R_\alpha}(x_\alpha, x'_\alpha).
\end{align*}
\]

(3.2)

Let \( e_\alpha : U_\alpha \to W \) and \( p_\alpha : U_\alpha \to X_\alpha \) denote the usual projections of \( U_\alpha \). Then clearly \( (U_\alpha, R_{U_\alpha}) \in \text{IRel}(H) \) for each \( \alpha \in \Gamma \). Moreover, \( e_\alpha : (U_\alpha, R_{U_\alpha}) \to (W, R_W) \) and \( p_\alpha : (U_\alpha, R_{U_\alpha}) \to (X_\alpha, R_\alpha) \) are \( \text{IRel}(H) \)-mappings for each \( \alpha \in \Gamma \). And the following diagram is a pullback square in \( \text{IRel}(H) \):

\[
\begin{array}{ccc}
(U_\alpha, R_{U_\alpha}) & \xrightarrow{p_\alpha} & (X_\alpha, R_\alpha) \\
\downarrow{e_\alpha} & & \downarrow{g_\alpha} \\
(W, R_W) & \xrightarrow{f} & (Y, R_Y)
\end{array}
\]

(3.3)

We will show that \( (e_\alpha : (U_\alpha, R_{U_\alpha}) \to (W, R_W))_\Gamma \) is a final episink in \( \text{IRel}(H) \). By the process of the proof of [14, Theorem 2.5], \( (e_\alpha)_\Gamma \) is an episink in \( \text{IRel}(H) \). Suppose \( R = (\mu_R, \nu_R) \) is another final IHFR on \( W \) with respect to \( (e_\alpha)_\Gamma \). By the process of the proof of [14, Theorem 2.5], \( \mu_R = \mu_{R_W} \). Thus it is sufficient to show that \( \nu_R = \nu_{R_W} \). Let \( (w, w') \in W \times W \). Then

\[
\begin{align*}
\nu_{R_W}(w, w') &= \nu_{R_W}(w, w') \lor \nu_{R_W}(w, w') \\
&\geq \nu_{R_W}(w, w') \lor [\nu_{R_W} \circ f^2(w, w')] \\
&\text{(since } f : (W, R_W) \longrightarrow (Y, R_Y) \text{is an } \text{IRel}(H)-\text{mapping}) \\
&= \nu_{R_W}(w, w') \lor \nu_{R_Y}(f(w), f(w')) \\
&= \nu_{R_W}(w, w') \lor \bigwedge_{(x_\alpha, x'_\alpha) \in g_\alpha^{-1}(f(w), f(w'))} \nu_{R_\alpha}(x_\alpha, x'_\alpha) \\
&\text{(since } (g_\alpha)_\Gamma \text{ is final}) \\
&= \bigwedge_{(x_\alpha, x'_\alpha) \in g_\alpha^{-1}(f(w), f(w'))} [\nu_{R_W}(w, w') \lor \nu_{R_\alpha}(x_\alpha, x'_\alpha)] \\
&= \bigwedge_{(w, x_\alpha), (w', x'_\alpha) \in e_\alpha^{-1}(w, w')} \nu_{R_{U_\alpha}}((w, x_\alpha), (w', x'_\alpha)).
\end{align*}
\]
Thus \( \nu_{R_w}(w,w') \geq \nu_R(w,w') \) for each \((w,w') \in W \times W \). So \( \nu_{R_w} \geq \nu_R \). On the other hand, since \( (e_a : (U_\alpha, R_{U_\alpha}) \to (W, R)_1) \) is final, \( 1_{W} : (W, R) \to (W, R_{W}) \) is an IRel\((H)\)-mapping. Thus \( \nu_R \geq \nu_{R_w} \). So \( \nu_R = \nu_{R_w} \). Hence \( R = R_{W} \). This completes the proof. \( \square \)

For any singleton set \( \{a\} \), since the IHFR \( R \) on \( \{a\} \) is not unique, the category IRel\((H)\) is not properly fibered over Set. Hence, by Theorems 3.12 and 3.14, we obtain the following result.

**Theorem 3.15.** IRel\((H)\) satisfies all the conditions of a topological universe over Set except the terminal separator property.

**Theorem 3.16.** IRel\((H)\) is cartesian closed over Set.

**Proof.** It is clear that IRel\((H)\) has products by Corollary 3.11. We will show that IRel\((H)\) has exponential objects.

For any IHFRs \( X = (X, R_X) \) and \( Y = (Y, R_Y) \), let \( Y^X \) be the set of all mappings from \( X \) into \( Y \). We define two mappings \( \mu_R : Y^X \times Y^X \to H \) and \( \nu_R : Y^X \times Y^X \to H \) as follows: for each \((f,g) \in Y^X \times Y^X\),

\[
\begin{align*}
\mu_R(f,g) &= \bigwedge \{ h \in H : \mu_{R_X}(x,y) \land h \leq \mu_{R_Y}(f(x),g(y)) \text{ for each } (x,y) \in X \times X \}, \\
\nu_R(f,g) &= \bigvee \{ h \in H : \nu_{R_X}(x,y) \lor h \geq \nu_{R_Y}(f(x),g(y)) \text{ for each } (x,y) \in X \times X \}.
\end{align*}
\]

(3.5)

Then clearly \((Y^X, R) \in \text{IRel}(H)\). Let \( Y^X = (Y^X, R) \). Then, by the definition of \( R \),

\[
\begin{align*}
\mu_{R_X}(x,y) \land \mu_R(f,g) &\leq \mu_{R_Y}(f(x),g(y)), \\
\nu_{R_X}(x,y) \lor \nu_R(f,g) &\geq \nu_{R_Y}(f(x),g(y))
\end{align*}
\]

(3.6)

for each \((f,g) \in Y^X \) and \((x,y) \in X \times X \).

Define \( e_{X,Y} : X \times Y^X \to Y \) by \( e_{X,Y}(x, f) = f(x) \) for each \((x, f) \in X \times Y^X \). Let \(((x, f), (y, g)) \in (X \times Y^X) \times (X \times Y^X) \). Then, by the process of the proof of [14, Theorem 2.7], \( \mu_{R_X \times R}((x, f), (y, g)) \leq \mu_{R_Y} \circ e_{X,Y}^2((x, f), (y, g)) \). So \( \mu_{R_X \times R} \leq \mu_{R_Y} \circ e_{X,Y}^2 \). On the other hand,

\[
\nu_{R_X \times R}((x, f), (y, g)) = \nu_{R_X}(x, y) \lor \nu_R(f, g) \\
\geq \nu_{R_Y}(f(x), g(y)) \\
= \nu_{R_Y}(e_{X,Y}(x, f), e_{X,Y}(y, g)) \\
= \nu_{R_Y} \circ e_{X,Y}^2((x, f), (y, g)).
\]

(3.7)

Thus \( \nu_{R_X \times R} \geq \nu_{R_Y} \circ e_{X,Y}^2 \). Hence \( e_{X,Y} : X \times Y^X \to Y \) is an IRel\((H)\)-mapping.

For any \( Z = (Z, R_Z) \in \text{IRel}(H) \), let \( h : X \times Z \to Y \) be an IRel\((H)\)-mapping. We define \( h : Z \to Y^X \) by \([h(z)](x) = h(x, z)\) for each \( z \in Z \) and each \( x \in X \). Let \( z, z' \in Z \) and
let \( x, x' \in X \). Then, by the process of the proof of [14, Theorem 2.7], \( \mu_{RZ}(z, z') \leq \mu_R \circ \overline{h}(z, z') \). So \( \mu_{RZ} \leq \mu_R \circ \overline{h}^2 \). On the other hand,

\[
\nu_{R \times R}(x, x', (x', z')) = \nu_{R_2}(x, x') \lor \nu_{R_2}(z, z')
\]

\[
\geq \nu_{R'} \circ \overline{h}^2((x, z), (x', z'))
\]

(since \( h : X \times Z \to Y \) is an \( \text{IRel}(H) \)-mapping) \hspace{1cm} (3.8)

\[
= \nu_{R'}(h(x, z), h(x', z'))
\]

\[
= \nu_{R'}([\overline{h}(z)](x), [\overline{h}(z')])(x')).
\]

Thus, by the definition of \( R \), \( \nu_{R_2}(z, z') \geq \nu_{R}(\overline{h}(z), \overline{h}(z')) = \nu_{R'} \circ \overline{h}^2(z, z') \). So \( \nu_{R_2} \geq \nu_{R} \circ \overline{h}^2 \). Hence \( \overline{h} : Z \to Y^X \) is an \( \text{IRel}(H) \)-mapping. Moreover, \( \overline{h} \) is the unique \( \text{IRel}(H) \)-mapping such that \( e_{X, Y} \circ (1_{X} \times \overline{h}) = h \). This completes the proof. \( \square \)

**Remark 3.17.** \( \text{IRel}(H) \) has no subobject classifier. Hence \( \text{IRel}(H) \) is not topos.

**Example 3.18.** Let \( H = \{0, 1\} \) be the two points chain and let \( X = \{a\} \). Let \( R_1 \) and \( R_2 \) be the IHFRs on \( X \) given by \( \mu_{R_1}(a, a) = 0 \), \( \nu_{R_1}(a, a) = 1 \) and \( \mu_{R_2}(a, a) = 1 \), \( \nu_{R_2}(a, a) = 0 \). Let \( 1_X : (X, R_1) \to (X, R_2) \) be the identity mapping. Then clearly, \( 1_X \) is both a monomorphism and an epimorphism in \( \text{IRel}(H) \). But, \( 1_X \) is not an isomorphism in \( \text{IRel}(H) \). Hence \( \text{IRel}(H) \) has no subobject classifier (see [7]).

### 4. The relations between \( \text{IRel}(H) \) and \( \text{Rel}(H) \)

**Lemma 4.1.** Define \( \text{IRel}(H) \to \text{Rel}(H) \) by

\[
\begin{align*}
G_1(X, \mu_R, \nu_R) &= (X, \mu_R), \\
G_2(X, \mu_R, \nu_R) &= (X, N(\nu_R)), \\
G_1(f) &= G_2(f) = f.
\end{align*}
\]

Then \( G_1 \) and \( G_2 \) are functors.

**Proof.** Clearly \( G_1(X, \mu_{R_2}, \nu_{R_2}) = (X, \mu_{R_2}) \in \text{Rel}(H) \) for each \( (X, \mu_{R_2}, \nu_{R_2}) \in \text{IRel}(H) \). Let \( (X, \mu_{R_2}, \nu_{R_2}), (Y, \mu_{R_2}, \nu_{R_2}) \in \text{IRel}(H) \) and let \( f : (X, \mu_{R_2}, \nu_{R_2}) \to (Y, \mu_{R_2}, \nu_{R_2}) \) be an \( \text{IRel}(H) \)-mapping. Then \( \mu_{R_2} \leq \mu_{R_2} \circ f^2 \). Thus \( G_1(f) = f : (X, \mu_{R_2}) \to (Y, \mu_{R_2}) \) is a \( \text{Rel}(H) \)-mapping. Hence \( G_1 : \text{IRel}(H) \to \text{Rel}(H) \) is a functor. Also \( G_2(X, \mu_{R_2}, \nu_{R_2}) = (X, N(\nu_{R_2})) \in \text{Rel}(H) \) for each \( (X, \mu_{R_2}, \nu_{R_2}) \in \text{IRel}(H) \). Now let \( (X, \mu_{R_2}, \nu_{R_2}), (Y, \mu_{R_2}, \nu_{R_2}) \in \text{IRel}(H) \) and let \( f : (X, \mu_{R_2}, \nu_{R_2}) \to (Y, \mu_{R_2}, \nu_{R_2}) \) be an \( \text{IRel}(H) \)-mapping. Then \( \nu_{R_2} \geq \nu_{R_2} \circ f^2 \). Thus \( N(\nu_{R_2}) \leq N(\nu_{R_2}) \circ f^2 \). So \( G_2(f) = f : (X, N(\nu_{R_2})) \to (Y, N(\nu_{R_2})) \) is a \( \text{Rel}(H) \)-mapping. Hence \( G_2 : \text{IRel}(H) \to \text{Rel}(H) \) is a functor. \( \square \)

**Lemma 4.2.** Define \( \text{Rel}(H) \to \text{IRel}(H) \) by \( F_1(X, \mu_R) = (X, \mu_R, N(\mu_R)) \) and \( F_1(f) = f \).

Then \( F_1 \) is a functor.
Theorem 4.4. The functor $F_1: \text{Rel}(H) \to \text{IRel}(H)$ is a left adjoint of the functor $G_1: \text{IRel}(H) \to \text{Rel}(H)$.

Proof. For each $(X, \mu_R) \in \text{Rel}(H)$, $1_X: (X, \mu_R) \to G_1F_1(X, \mu_R) = (X, \mu_R)$ is a $\text{Rel}(H)$-morphism. Let $(Y, \mu_R, \nu_R) \in \text{IRel}(H)$ and let $f : (X, \mu_R) \to (Y, \mu_R)$ be an $\text{IRel}(H)$-morphism. We will show that $f : F_1(X, \mu_R) = (X, \mu_R, \nu_R) \to (Y, \mu_R)$ is an $\text{IRel}(H)$-morphism. Since $f : (X, \mu_R) \to (Y, \mu_R)$ is a $\text{Rel}(H)$-morphism, $\mu_R \leq \mu_R \circ f^2$. Thus $NN(\mu_R) \leq NN(\mu_R) \circ f^2$. Moreover $N(\mu_R) \geq N(\mu_R) \circ f^2$. So $F_2(f) = f : F_2(X, \mu_R) \to F_2(Y, \mu_R)$ is an $\text{IRel}(H)$-morphism. Hence $F_2$ is a functor.

Theorem 4.5. Two categories $\text{Rel}(H)$ and $\text{IRel}^*(H)$ are isomorphic.

Proof. It is clear that $F_1: \text{Rel}(H) \to \text{IRel}^*(H)$ is a functor by Lemma 4.2. Consider the restriction $G_1: \text{IRel}^*(H) \to \text{Rel}(H)$ of the functor $G_1$ in Lemma 4.1. Let $(X, \mu_R) \in \text{Rel}(H)$. Then, by Lemma 4.2, $F_1(X, \mu_R) = (X, \mu_R, \nu_R)$. Thus $G_1F_1(X, \mu_R) = G_1(X, \mu_R, \nu_R) = (X, \mu_R)$. So $G_1 \circ F = 1_{\text{Rel}(H)}$. Now let $(X, \mu_R, N(\mu_R)) \in \text{ISet}^*(H)$. Then, by Lemma 4.1, $G_1(X, \mu_R, N(\mu_R)) = (X, \mu_R)$. Thus $FG_1(X, \mu_R, N(\mu_R)) = (X, \mu_R, N(\mu_R))$. So $F \circ G_1 = 1_{\text{ISet}^*(H)}$. Hence $F: \text{Rel}(H) \to \text{ISet}^*(H)$ is an isomorphism. This completes the proof.

Remark 4.6. We are going to investigate “intuitionistic $H$-fuzzy reflexive relations,” “some subcategories of the category $\text{IRelk}(H)$,” and “intuitionistic $H$-fuzzy relations on intuitionistic $H$-fuzzy sets” in the viewpoint of topological universe.

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Intuitionistic $H$-fuzzy relations


Kul Hur: Division of Mathematics and Informational Statistics and Institute of Basic Natural Science, Wonkwang University, Iksan, Chonbuk 579-792, Korea

E-mail address: kulhur@wonkwang.ac.kr

Su Youn Jang: Division of Mathematics and Informational Statistics and Institute of Basic Natural Science, Wonkwang University, Iksan, Chonbuk 579-792, Korea

E-mail address: suyoun123@yahoo.co.kr

Hee Won Kang: Department of Mathematics Education, Woosuk University, Hujong-Ri, Samrae-Eup, Wanju-kun, Chonbuk 565-701, Korea

E-mail address: khwon@woosuk.ac.kr