We define the power-commutative kernel of a group. In particular, we describe the power-commutative kernel of locally nilpotent groups, and of finite groups having a nontrivial center.

A group $G$ is called \emph{power commutative}, or a \emph{PC-group}, if $[x^m, y^n] = 1$ implies $[x, y] = 1$ for all $x, y \in G$ such that $x^m \neq 1$, $y^n \neq 1$. So power-commutative groups are those groups in which commutativity of nontrivial powers of two elements implies commutativity of the two elements. Clearly, $G$ is a \emph{PC-group} if and only if $C_G(x^n) = C_G(x^n)$ for all $x \in G$ and all integers $n$ such that $x^n \neq 1$. Obvious examples of \emph{PC-groups} are groups in which commutativity is a transitive relation on the set of nontrivial elements (\emph{CT-groups}) and groups of prime exponent.

Recall that a group $G$ is called an \emph{R-group} if $x^n = y^n$ implies $x = y$ for all $x, y \in G$ and for all positive integers $n$. In other words, \emph{R-groups} are groups in which the extraction of roots is unique. A result due to Mal’cev and Cernikov (see, e.g., [3]) states that every nilpotent torsion-free group is an \emph{R-group}. There is a natural connection between \emph{PC-groups} and \emph{R-groups}. For, as pointed out in [3], a torsion-free group is a \emph{PC-group} if and only if it is an \emph{R-group}.

In [5], Wu gave the classification of locally finite \emph{PC-groups}. In particular, she proved that a finite group is a \emph{PC-group} if and only if the centralizer of each nontrivial element is abelian or of prime exponent. This result implies that a finite group having a nontrivial center is a \emph{PC-group} if and only if it is abelian or it has prime exponent. Moreover, the class of \emph{PC-groups} is contained in the class of groups in which the centralizer of each nontrivial element is nilpotent. This class of groups was investigated by many authors (see, e.g., [1, 4]).

In analogy to what is done in [2] to define the commutative-transitive kernel of a group, we introduce an ascending series

$$\{1\} = P_0(G) \leq P_1(G) \leq \cdots \leq P_t(G) \leq \cdots$$

(1)

of characteristic subgroups of $G$ contained in the derived subgroup $G'$. We define $P_t(G)$ as
the subgroup of $G'$ generated by those commutators $[x,y]$ such that there exist positive integers $n$, $m$ with $x^n \neq 1$, $y^m \neq 1$, and $[x^n, y^m] = 1$. If $t > 1$ then $P_t(G)$ is defined by $P_t(G) / P_{t-1}(G) = P_1(G / P_{t-1}(G))$. Finally, the $PC$-kernel of $G$ is the subgroup $P(G)$ of $G'$ defined by

$$P(G) = \bigcup_{t \in \mathbb{N}} P_t(G).$$

Obviously, for any group $G$, the $PC$-kernel $P(G)$ is characteristic in $G$, $G / P(G)$ is a $PC$-group, and $G$ is a $PC$-group if and only if $P(G) = \{1\}$.

Let $\mathcal{X}$ be a class of groups. Then one can ask whether there exists a nonnegative integer $n$ such that $P_n(G) = P(G)$ for all $G \in \mathcal{X}$. Of course $P(G) = P_n(G)$ if and only if $G / P_n(G)$ is a $PC$-group.

In this paper, we give affirmative answers to the previous question when $\mathcal{X}$ is the class of locally nilpotent groups, or the class of finite groups having a nontrivial center. In both cases, we prove that $P(G) = P_1(G)$ for all $G \in \mathcal{X}$.

Our first results are concerned with the power-commutative kernel of finite nilpotent groups.

**Proposition 1.** Let $p$ be a prime and $G$ a finite $p$-group. Then $G / P_1(G)$ is a $PC$-group.

**Proof.** Notice that $P_1(G) \leq M$ for every maximal subgroup $M$ of $G$ since $P_1(G) \leq \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of $G$. This implies that $M / P_1(G)$ is a maximal subgroup of $G / P_1(G)$ if and only if $M$ is a maximal subgroup of $G$.

Let $G$ be a counterexample of least order. For any maximal subgroup $M$ of $G$ we obtain $M / P_1(G) \cong (M / P_1(M)) / (P_1(G) / P_1(M))$. Hence $M / P_1(G)$ is a $PC$-group since it is a quotient of a finite $PC$-group (see [5]). It follows that a maximal subgroup of $G / P_1(G)$ is abelian or it has exponent $p$.

Put $G = G / P_1(G)$ and $H = H / P_1(G)$ for all $P_1(G) \leq H \leq G$. If every maximal subgroup $\overline{M}$ of $G$ has exponent $p$, then $G$ is cyclic or of exponent $p$. In any case $\overline{G}$ is a $PC$-group, that is a contradiction. So we may assume that $\overline{G}$ has a maximal subgroup $\overline{M}$ such that $\overline{M}$ is abelian and $\overline{M}^p \neq 1$. Consider $g \in \overline{G} \setminus \overline{M}$, so $\overline{G} = \langle \overline{M}, g \rangle$. Moreover $|\overline{G} : \overline{M}| = p$.

If there exists $a \in \overline{M}$ such that $(ga)^p \neq 1$, then $(ga)^p \in \overline{M} \setminus \{1\}$. So, for all $y \in \overline{M}$ we get $[y, (ga)^p] = 1$, hence $[y,g] = [y, ga] = 1$. It follows that $\overline{G}$ is abelian, a contradiction. Thus $(ga)^p = 1$ for all $a \in \overline{M}$, and in particular $g^p = 1$. It follows that $a^{g^p+\cdots+g+1} = (ga)^p = 1$ for all $a \in \overline{M}$. This implies $a^p = 1$ for all $a \in C_{\overline{G}}(g)$, so $(C_{\overline{G}}(g))^p = C_{\overline{G}}^p(g) = 1$. But $\overline{M}^p \cap Z(\overline{G}) \neq 1$ since $\overline{M}^p \neq 1$, that is a contradiction.

**Proposition 2.** Let $G$ be a finite nilpotent group of order $n = p_1^{a_1} \cdots p_l^{a_l}$ ($p_1, \ldots, p_l$ distinct primes). If $t > 1$ then $G / P_1(G)$ is abelian.

**Proof.** Let $G_{p_i}$ be the Sylow $p_i$-subgroup of $G$ for all $i \in \{1, \ldots, t\}$; we will prove that $G_{p_i}^{p_i} \leq P_1(G)$ for all $i \in \{1, \ldots, t\}$. Let $x, y \in G_{p_i} \setminus \{1\}$, $a \in G_{p_1} \times \cdots \times G_{p_{i-1}} \times G_{p_{i+1}} \times \cdots \times G_{p_l}$, $|a| = m$ and $|x| = p_i^{r_i}$. Now $|ax| = mp_i^{r_i}$ as $(m, p_i^{r_i}) = 1$. Since $(ax)^{p_i^{r_i}} = a^{p_i^{r_i}}$ has order $m$ we get $[(ax)^{p_i^{r_i}}, y] = [a^{p_i^{r_i}}, y] = 1$. Thus $[ax, y] = [x, y] \in P_1(G)$.

**Corollary 3.** Let $G$ be a finite nilpotent group; then $G / P_1(G)$ is abelian or it has exponent $p$. In both cases $G / P_1(G)$ is a $PC$-group.
Proof. The result is an immediate consequence of the previous propositions and [5, Theorem 4].

Now we prove that the equality $P(G) = P_1(G)$ holds for every nilpotent group $G$.

**Theorem 4.** Let $G$ be a nilpotent group. Then $G/P_1(G)$ is a PC-group.

**Proof.** If $G$ is torsion-free then $G$ is a PC-group (see [3]), so $P_1(G) = \{1\}$ and the result is true. So we may suppose that the torsion subgroup $T$ of $G$ is nontrivial.

First of all, notice that if for elements $x, y \in G \setminus \{1\}$ there exists a positive integer $n$ such that $x^n \neq 1$ and $[x^n, y] = 1$, then $[x, y] \in T$. This is obvious if $x \in T$ or $y \in T$, so we may assume $x, y \notin T$. Then $(x, y)T/T \leq G/T$. So $(xT, yT)$ is torsion-free, and $[(xT)^n, yT] = T$ implies $[x, y] \in T$. This means that $P_1(G) \subseteq T$.

If for any $x, y \in G$ the commutator $[x, y]$ is periodic, then it is easy to see that there exists a positive integer $m$ such that $[x, y^m] = 1$. In fact, $\langle x, y \rangle$ is a FC-group since $\langle x, y \rangle/Z(\langle x, y \rangle)$ is finite, and therefore the set $\{x^t|t \in Z\}$ is finite.

Now notice that if $x \in T$ then $[x, g] \in P_1(G)$ for all $g \in G \setminus T$. In fact, $[x, g] \in T$ implies that there exists a positive integer $m$ such that $[x, g^m] = 1$. So we get $[x, g] \in P_1(G)$ because $g^m \neq 1$.

Finally, let $x, y \in G \setminus P_1(G)$ such that $x^n \notin P_1(G)$ and $[x^n, y] \in P_1(G)$. If $x, y \in T$ then $\langle x, y \rangle$ is a finite nilpotent group and Corollary 3 implies that $\langle x, y \rangle/P_1(\langle x, y \rangle)$ is a finite PC-group. Hence $\langle x, y \rangle/P_1(G) \cap \langle x, y \rangle$ is a PC-group and $[x, y] \in P_1(G)$. If $x \in T$ or $y \in T$ then $[x, y] \in P_1(G)$, as noticed before. So we may suppose $x, y \in G \setminus T$. Since $[x^n, y] \in P_1(G) \subseteq T$, we get $[x^n, y] \in T$ and so there exists a positive integer $m$ such that $[x^n, y^m] = 1$. Therefore $[x, y] \in P_1(G)$, and the proof is complete.

**Theorem 5.** Let $G$ be a locally nilpotent group. Then $P(G) = P_1(G)$.

**Proof.** Let $x, y \in G \setminus P_1(G)$ such that $x^n \notin P_1(G)$ and $[x^n, y] \in P_1(G)$. Then

$$[x^n, y] = \prod_{i=1}^{r} [a_i, b_i],$$

where $a_i, b_i \in G$ for all $i = 1, 2, \ldots, r$, and $[a_i^{a_i}, b_i^{b_i}] = 1$ for some positive integers $\alpha_i$ and $\beta_i$ such that $a_i^{\alpha_i} \neq 1$ and $b_i^{\beta_i} \neq 1$.

Let $H = \langle x, y, a_1, \ldots, a_r, b_1, \ldots, b_r \rangle$. Then $H$ is nilpotent, so $H/P_1(H)$ is a PC-group by Theorem 4. Since $[a_i, b_i] \in P_1(\langle a_i, b_i \rangle) \leq P_1(H)$ for all $i = 1, 2, \ldots, r$, we get $[x^n, y] \in P_1(H)$. Thus $[x, y] \in P_1(H)$, and therefore $[x, y] \in P_1(G)$.

Now it is possible to prove that $P(G) = P_1(G)$ for any finite group $G$ such that $Z(G) \neq \{1\}$.

**Proposition 6.** Let $G$ be a finite group such that $Z(G) \neq \{1\}$. Then $[a, b] \in P_1(G)$ for all $a, b \in G \setminus \{1\}$ such that $|a|, |b| = 1$.

**Proof.** Put $|a| = n$ and $|b| = m$. Then there exists $z \in Z(G) \setminus \{1\}$ such that $|z|$ does not divide $n$ or $m$. Suppose $|z|$ does not divide $n$. Then $[(az)^n, b] = [a^n, b] = [z^n, b] = 1$. Moreover $(az)^n = z^n \neq 1$ and this yields $[az, b] = [a, b] \in P_1(G)$.
Proposition 7. Let $G$ be a finite group such that $Z(G) \neq \{1\}$. Then $G/P_1(G)$ is nilpotent.

Proof. We may assume that the order of $G/P_1(G)$ is not a prime power. Let $p$ be any prime divisor of $|G/P_1(G)|$. Then $p$ divides $|G|$ and $PP_1(G)/P_1(G)$ is a Sylow $p$-subgroup of $G/P_1(G)$ whenever $P$ is a Sylow $p$-subgroup of $G$. We are going to show that $PP_1(G)/P_1(G)$ is normal in $G/P_1(G)$. Let $q \neq p$ be any prime dividing $|G/P_1(G)|$, and let $Q$ be a Sylow $q$-subgroup of $G$. Then $MP_1(G)/P_1(G)$ centralizes $PP_1(G)/P_1(G)$, by Proposition 6. Thus the normalizer in $G/P_1(G)$ of $PP_1(G)/P_1(G)$ contains a Sylow $q$-subgroup of $G/P_1(G)$ for all prime divisors of its order. Therefore this normalizer is actually $G/P_1(G)$, and the result follows.

Theorem 8. Let $G$ be a finite group such that $Z(G) \neq \{1\}$. Then $G/P_1(G)$ is abelian or it has exponent $p$.

Proof. Since $G/P_1(G)$ is nilpotent by Proposition 7, by [5] it suffices to show that $G/P_1(G)$ is a PC-group. Suppose not, and let $G$ be a counterexample of least order. We may assume $G$ is not nilpotent, hence $P_1(G) \not\leq \Phi(G)$. Thus there exists a maximal subgroup $M$ of $G$ such that $P_1(G) \not\leq M$. In particular $G' \not\leq M$. If $Z(G) \not\leq M$, then there exists $z \in Z(G) \setminus M$. Since $M$ is maximal, it follows that $\langle z \rangle M = G$. Hence $M$ is normal in $G$, and $G/M$ is cyclic. This in turn implies that $G' \leq M$, a contradiction. Thus $Z(G) \leq M$, and so $Z(M) \neq \{1\}$. Then $M/P_1(M)$ is a PC-group and therefore $G/P_1(G) \cong (M/P_1(M))/(M \cap P_1(G)/P_1(M))$ is a PC-group, the final contradiction.

References


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