ON REDUCIBILITY OF SOME OPERATOR SEMIGROUPS AND ALGEBRAS ON LOCALLY CONVEX SPACES

EDVARD KRAMAR

Received 9 March 2005 and in revised form 20 October 2005

A generalization of some results from normed spaces, concerning reducibility and triangularizability of semigroups and algebras of operators, to locally convex spaces is given.

1. Introduction

Let X be a complex Hausdorff locally convex space. A system of seminorms P inducing the topology on X will be called a calibration. We denote by $\mathcal{P}(X)$ the collection of all calibrations on X . For a given $P \in \mathcal{P}(X)$ let $P = \{p_\alpha : \alpha \in \Delta\}$, where Δ is some index set and for each $\alpha \in \Delta$ denote $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Let us denote by $\mathcal{L}(X)$ the set of all linear continuous operators on X , by $\mathcal{K}(X)$ the set of compact operators on X ($T \in \mathcal{K}(X)$ if there exists a neighbourhood U_γ such that $T(U_\gamma)$ is a relatively compact set), by $\mathcal{F}(X)$ the set of all finite-rank operators and by $\mathcal{LB}(X)$ the set of all locally bounded operators (there is some neighbourhood U_γ such that $T(U_\gamma)$ is bounded). The topology of bounded convergence on $\mathcal{L}(X)$ and on X' will be denoted by τ_b . By X'_b we will denote the topological space (X', τ_b) . We will denote by $\mathcal{R}(T)$ the range of T and by $\mathcal{N}(T)$ the null space of T . For a given $T \in \mathcal{L}(X)$ the number $\lambda \in \mathbb{C}$ is in the resolvent set of T if and only if $(\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The spectrum $\sigma(T)$ is the complement of the resolvent set and by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ we denote the spectral radius of T . An operator T is quasinilpotent if $\sigma(T) = \{0\}$. A closed subspace M in X is an *invariant subspace* of an operator T if $T(M) \subseteq M$. A collection of linear operators is *reducible* if it has a common nontrivial invariant subspace and is *irreducible* otherwise. If a family $\mathcal{A} \subset \mathcal{L}(X)$ is an algebra, it is irreducible if and only if it is transitive, that is, the set $\mathcal{A}x := \{Tx : T \in \mathcal{A}\}$ is dense in X for each $x \neq 0$. For $P \in \mathcal{P}(X)$ and $p_\alpha \in P$ let J_α denote the null space of p_α and X_α the quotient space X/J_α . It is a normed space with the norm $\|x_\alpha\|_\alpha := p_\alpha(x)$, where $x_\alpha = x + J_\alpha$. Let $T \in \mathcal{L}(X)$ be such that $T(J_\alpha) \subset J_\alpha$ then the corresponding operator T_α on X_α is well defined by $T_\alpha(x_\alpha) = Tx + J_\alpha$.

2. The results

LEMMA 2.1. *Let X be a locally convex space and $\mathcal{A}$ a transitive τ_b -closed algebra of continuous operators on X which contains a nonzero finite-rank operator. Then there exists a*

τ_b -closed subspace Φ in X'_b such that

- (i) $\mathcal{A}$ contains all rank-one operators of the form $x \otimes f$, $x \in X$, $f \in \Phi$,
- (ii) if $f(z) = 0$ for each $f \in \Phi$, then $z = 0$.

Proof. Let $F \in \mathcal{A}$ be a nonzero finite-rank operator. Denote $E_1 = \mathcal{R}(F)$ and $\mathcal{A}_1 := F\mathcal{A}$. It is clear that each $T \in \mathcal{A}_1$ maps E_1 to E_1 and the restriction $\mathcal{A}_1|_{E_1}$ is a transitive algebra of operators on a finite-dimensional space. By Burnside's theorem [11] it follows that $\mathcal{A}_1|_{E_1} = \mathcal{L}(E_1)$. Especially, there is some $A_0 \in \mathcal{A}$ such that $FA_0|_{E_1}$ has rank one, hence FA_0F is a rank-one operator in $\mathcal{A}$ of the form $x_0 \otimes f_0$, $x_0 \in X$, and $f_0 \in X'$. Choose arbitrary nonzero $x \in X$. Since $\mathcal{A}x_0$ is dense in X , there is some net of operators $\{A_\delta\} \subset \mathcal{A}$ such that $A_\delta x_0 \rightarrow x$. For any chosen seminorm q_α^M defining the topology τ_b we have

$$q_\alpha^M((A_\delta x_0) \otimes f_0 - x \otimes f_0) = \sup_{y \in M} |f_0(y)| p_\alpha(A_\delta x_0 - x) \leq c \cdot p_\alpha(A_\delta x_0 - x). \quad (2.1)$$

Since the right-hand side tends to zero, the same holds for the left-hand side, thus $x \otimes f_0 \in \mathcal{A}$. Let us define $\Phi := \{f \in X' : x \otimes f \in \mathcal{A}, \text{ for all } x \in X\}$. Choose $\{f_\delta\} \subset \Phi$ a net which is τ_b -convergent to some f , and $x \in X$ arbitrary, then for any seminorm q_α^M we have

$$q_\alpha^M(x \otimes f_\delta - x \otimes f) = \sup_{y \in M} |(f_\delta - f)(y)| p_\alpha(x). \quad (2.2)$$

Since $\mathcal{A}$ is τ_b -closed we have $x \otimes f \in \mathcal{A}$, hence $f \in \Phi$. Thus, Φ is a nontrivial τ_b -closed subspace satisfying (i). To verify (ii), let $f(z) = 0$ for each $f \in \Phi$. Choose a nonzero $f_0 \in \Phi$ and for each $A \in \mathcal{A}$ define $f_1 := A'f_0$, where A' is the adjoint operator of A . Since $x \otimes f_1 = x \otimes A'f_0 = (x \otimes f_0)A \in \mathcal{A}$, for any $x \in X$, we have $f_1 \in \Phi$, hence $f_1(z) = 0$, that is $f_0(Az) = 0$ for each $A \in \mathcal{A}$. If there were $z \neq 0$, then f_0 would be zero on a dense set $\mathcal{A}z$ and consequently identically zero, which is a contradiction. $\square$

COROLLARY 2.2. *Let X be a semireflexive locally convex space and $\mathcal{A}$ a transitive τ_b -closed algebra of continuous operators which contains a nonzero finite-rank operator. Then $\mathcal{A}$ contains all finite-rank operators.*

Proof. It is sufficient to show that $\mathcal{A}$ contains all rank-one operators and this will be in case $\Phi = X'$. If there were $\Phi \neq X'$, then by the Hahn-Banach theorem there would be some nonzero $F \in (X'_b)'$, such that $F|_\Phi = 0$. Since X is semireflexive, there is some nonzero $y \in X$ such that $F(f) = f(y)$ for all $f \in X'$ and then $f(y) = F(f) = 0$ for all $f \in \Phi$. By (ii) in the previous lemma then $y = 0$, which is a contradiction. $\square$

A linear operator T is called *nuclear* if it can be written in the form

$$Tx = \sum_{j=1}^{\infty} \lambda_j c_j(x) a_j, \quad x \in X, \quad (2.3)$$

where $\{c_j\}$ is an equicontinuous sequence in X' , $\{\lambda_j\} \in l_1$, and $\{a_j\}$ is a sequence contained in an absolutely convex bounded set B in X , such that $X_B := \bigcup \{nB : n \in \mathbb{N}\}$ is a complete normed subspace in X with respect to the Minkowski's functional of the set B . (see, e.g., [8]). It is easy to see that the family of nuclear operators is an ideal in $\mathcal{L}(X)$ and that each nuclear operator is also compact.

COROLLARY 2.3. *Let X be a semireflexive locally convex space and $\mathcal{A}$ a transitive algebra of continuous operators such that $(\mathcal{A}, \tau_b)$ contains a nonzero finite-rank operator. Then $(\mathcal{A}, \tau_b)$ contains all nuclear operators.*

Proof. By Corollary 2.2, $(\mathcal{A}, \tau_b) \supset \mathcal{F}(X)$, since each nuclear operator can be τ_b -approximated by finite-rank operators by [1], the conclusion follows. $\square$

For a given set $\mathcal{M}$ of compact operators let us denote by $\widetilde{\mathcal{M}}$ the set of all $A \in \mathcal{K}(X)$ which are τ_b -limits of some sequence $\{A_n\} \subset \mathcal{M}$. By [5, Proposition 1] it follows that if X is a barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators on X , then $\widetilde{\mathcal{S}}$ is a semigroup too.

PROPOSITION 2.4. *Let X be a barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators. If $A \in \mathcal{S}$ is such that $r(A) \neq 0$ then there exists a sequence $\{n_i\}$ of integers such that one of the following assertions holds:*

- (a) $A^{n_i} \xrightarrow{\tau_b} E$, where E is idempotent, or
- (b) $\alpha_i A^{n_i} \xrightarrow{\tau_b} E$, for some scalar sequence $\{\alpha_i\}$, where E is nilpotent.

In both cases $E \in \mathbb{R}^+ \widetilde{\mathcal{S}}$ and is of finite rank.

Proof. Following the first part of the proof of [6, Proposition 3] we can find a sequence of operators from $\mathcal{S}$ with the above property. $\square$

THEOREM 2.5. *Let X be a barreled locally convex space and $\mathcal{A}$ a τ_b -closed transitive algebra in $\mathcal{L}(X)$ which contains a nonzero compact operator. Then $\mathcal{A} \cap \mathcal{F}(X)$ is a nontrivial transitive algebra.*

Proof. Define $\mathcal{C} = \mathcal{A} \cap \mathcal{K}(X)$, then it is an ideal in $\mathcal{A}$ and by [6, Lemma 5] it is transitive too. Let K be a nonzero operator in $\mathcal{C}$. Then by Lomonosov's theorem for locally convex spaces [3] there is some $A \in \mathcal{C}$ such that $r(AK) \neq 0$. By Proposition 2.4, $\widetilde{\mathcal{C}} \cap \mathcal{F}(X) \neq \{0\}$. Since $\mathcal{A}$ is τ_b -closed, it is easy to see that $\widetilde{\mathcal{C}} = \mathcal{C}$. Clearly, $\mathcal{A} \cap \mathcal{F}(X) = \mathcal{C} \cap \mathcal{F}(X)$ is an ideal in $\mathcal{A}$ and it is transitive too. $\square$

COROLLARY 2.6. *Let X be a barreled semireflexive locally convex space and $\mathcal{A}$ a τ_b -closed transitive algebra in $\mathcal{L}(X)$ which contains a nonzero compact operator. Then $\mathcal{A}$ contains all finite-rank operators.*

Proof. By Theorem 2.5 and Corollary 2.2, the conclusion follows. $\square$

COROLLARY 2.7. *Let X be a barreled semireflexive locally convex space and $\mathcal{A}$ a transitive τ_b -closed algebra in $\mathcal{L}(X)$ which contains a nonzero compact operator. If X has the property that each compact operator on X is τ_b -limit of a net of finite-rank operators, then $\mathcal{A}$ contains all compact operators.*

PROPOSITION 2.8. *Let X be a semireflexive locally convex space, $\mathcal{S}$ a semigroup in $\mathcal{F}(X)$, and ϕ a nontrivial τ_b -continuous linear functional on $\mathcal{F}(X)$. If ϕ is identically zero on $\mathcal{S}$, then $\mathcal{S}$ is reducible.*

Proof. Let us suppose that $\mathcal{S}$ is irreducible. Then the algebra $\mathcal{A}$ generated by $\mathcal{S}$ is also irreducible and the same holds for $(\mathcal{A}, \tau_b)$. By Corollary 2.2, $(\mathcal{A}, \tau_b) \supset \mathcal{F}(X)$. Since ϕ is equal to zero on $\mathcal{S}$, it is equal to zero also on $\mathcal{F}(X)$, which is a contradiction. $\square$

With respect to the strong topology τ_s on $\mathcal{L}(X)$ the following theorem is proven in [8].

THEOREM 2.9. *Let X be a locally convex space and $\mathcal{A}$ a transitive algebra in $\mathcal{L}(X)$, such that $(\mathcal{A}, \tau_s)$ contains a nontrivial compact operator. Then $\mathcal{A}$ is τ_s -dense in $\mathcal{L}(X)$.*

COROLLARY 2.10. *Let X be a locally convex space and A a nonscalar operator commuting with a nonzero compact operator. Then A has a nontrivial hyperinvariant subspace.*

Proof. Denote by $\mathcal{A} := (A)'$ the commutant of A . Clearly, it is an algebra and let us prove that it is τ_s -closed. Choose any net $\{B_\delta\}$ in $\mathcal{A}$ which is strongly convergent to some B . Then for any seminorm q_x^α for the strong topology one has

$$\begin{aligned} q_x^\alpha(BA - AB) &= p_\alpha((BA - AB)x) \leq p_\alpha((B - B_\delta)Ax) + p_\alpha(A(B - B_\delta)x) \\ &\leq q_y^\alpha(B - B_\delta) + c_\alpha q_x^\beta(B - B_\delta), \end{aligned} \quad (2.4)$$

where $y = Ax$. Since the right-hand side is arbitrary small, the left-hand side is zero. Since q_x^α is arbitrary, we have $BA - AB = 0$. If $\mathcal{A}$ were transitive, then by the above theorem, it would be equal to $\mathcal{L}(X)$ and consequently $A = \lambda I$ for some complex number λ , which is a contradiction. $\square$

COROLLARY 2.11. *Let X be a locally convex space and $A, B \in \mathcal{L}(X)$ two commuting operators, where B is nonscalar and commutes with a nonzero compact operator. Then A has a nontrivial invariant subspace.*

Proof. By Corollary 2.10, B has a hyperinvariant subspace which is invariant for A . $\square$

COROLLARY 2.12. *Let X be a locally convex space and $\mathcal{S}$ a semigroup of operators such that $(\overline{\mathcal{S}}, \tau_s)$ contains a nonzero compact operator and ϕ a nontrivial τ_s -continuous linear functional on $\mathcal{L}(X)$ such that $\phi|_{\mathcal{S}} = 0$. Then $\mathcal{S}$ is reducible.*

Proof. Let $\mathcal{S}$ be irreducible, then the algebra $\mathcal{A}$ generated by $\mathcal{S}$ is also irreducible and then by Theorem 2.9, it is strongly dense in $\mathcal{L}(X)$. Clearly, ϕ is equal to zero also on $\mathcal{A}$ and thus on $\mathcal{L}(X)$, which is a contradiction. $\square$

COROLLARY 2.13. *Let X be a locally convex space and $\mathcal{E}$ a commutative family of compact operators on X . Then $\mathcal{E}$ is reducible.*

Proof. Choose a nonzero $A \in \mathcal{E}$, by Corollary 2.10 it has a nontrivial hyperinvariant subspace which is then an invariant subspace for $\mathcal{E}$. $\square$

Let ϕ be a functional on a semigroup $\mathcal{S} \subset \mathcal{L}(X)$. By [11] ϕ is *permutable* on a family $\mathcal{E} \subset \mathcal{S}$ if for any $A_1, A_2, \dots, A_n \in \mathcal{E}$ and any permutation τ of $\{1, 2, \dots, n\}$ we have $\phi(A_1, A_2, \dots, A_n) = \phi(A_{\tau(1)}, A_{\tau(2)}, \dots, A_{\tau(n)})$.

PROPOSITION 2.14. *Let X be a semireflexive locally convex space and ϕ a nontrivial τ_b -continuous linear functional on $\mathcal{F}(X)$. Let $\mathcal{E}$ be a family of finite-rank operators such that ϕ is permutable on $\mathcal{E}$. Then $\mathcal{E}$ is reducible.*

Proof. Since ϕ is permutable on $\mathcal{E}$, it is also permutable on the algebra $\mathcal{A}$ generated by $\mathcal{E}$. In view of Corollary 2.13 we can assume that $\mathcal{E}$ is noncommutative. Then there are

$A, B \in \mathcal{E}$ such that $C := AB - BA \neq 0$. Denote by $\mathcal{J}$ the ideal in $\mathcal{A}$ generated by C . Clearly, $\phi(SCT) = 0$ for each $S, T \in \mathcal{A}$, consequently, $\phi|_{\mathcal{J}} = 0$. Hence, by Proposition 2.8, $\mathcal{J}$ is reducible, then by [6], $\mathcal{A}$ is reducible and $\mathcal{E}$ is reducible too. $\square$

COROLLARY 2.15. *Let X be a semireflexive locally convex space, $\mathcal{S}$ a semigroup of finite-rank operators, and ϕ a nontrivial τ_b -continuous linear functional on $\mathcal{F}(X)$. Then $\mathcal{S}$ is reducible if one of the following conditions holds:*

- (i) ϕ is multiplicative on $\mathcal{S}$,
- (ii) ϕ is constant on $\mathcal{S}$.

For fixed nonzero $x_0 \in X$, $f \in X'$ and a subspace $\mathcal{M}$ in $\mathcal{L}(X)$ let us define (as in [11]) the so-called *coordinate functional* by the relation $\phi(T) := f(Tx_0)$, $T \in \mathcal{M}$. For this class of functionals we do not need semireflexivity of the space.

LEMMA 2.16. *Let X be a locally convex space, then the coordinate functional ϕ is τ_b -continuous on $\mathcal{L}(X)$ if and only if f is continuous on X .*

Proof. If f is continuous, then $|\phi(T)| = |f(Tx_0)| \leq cp_\beta(Tx_0) \leq cq_\beta^M(T)$, where M is an arbitrary bounded set in X containing x_0 . Let ϕ be τ_b -continuous on $\mathcal{L}(X)$. Choose any $x \in X$. By the Hahn-Banach theorem there is some $g \in X'$ such that $g(x_0) = 1$ and let $S = x \otimes g$, hence $Sx_0 = x$. By continuity of ϕ there exist a seminorm q_α^M and $c > 0$ such that $|f(x)| = |f(Sx_0)| = |\phi(S)| \leq cq_\alpha^M(S) = c \sup_{y \in M} p_\alpha(g(y)x) \leq c_1 p_\alpha(x) \sup_{y \in M} p_y(y) = d_y p_\alpha(x)$ for some constant $d_y > 0$. $\square$

PROPOSITION 2.17. *Let X be a locally convex space, $\mathcal{S}$ a semigroup in $\mathcal{L}(X)$, and ϕ a τ_b -continuous coordinate functional on $\mathcal{L}(X)$. Then $\mathcal{S}$ is reducible if one of the following conditions holds:*

- (i) ϕ is constant on $\mathcal{S}$,
- (ii) ϕ is multiplicative on $\mathcal{S}$.

The proof is the same as for the normed space (see [11, Lemma 8.2.8]).

COROLLARY 2.18. *Let X be a locally convex space, $\mathcal{E}$ a noncommutative family in $\mathcal{L}(X)$, and ϕ a τ_b -continuous coordinate functional which is permutable on $\mathcal{E}$, then $\mathcal{E}$ is reducible.*

Proof. Since permutability is inherited by passing to an algebra (see [11, page 28]), we can assume that $\mathcal{E}$ is an algebra. Let us choose A, B in $\mathcal{E}$ such that $C := AB - BA \neq 0$. Then it is easy to see that ϕ is equal to zero on ideal $\mathcal{J}$ generated by C and by the previous proposition $\mathcal{J}$ is reducible, hence $\mathcal{E}$ is reducible. $\square$

LEMMA 2.19. *Let X be a locally convex space and $A, B \in \mathcal{L}(X)$ such that $\sigma(AB)$ is bounded. Then $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ and $r(AB) = r(BA)$.*

Proof. It is easy to see that if for $\lambda \neq 0$ there exists $C := (\lambda I - AB)^{-1} \in \mathcal{L}(X)$. Then we also have that $(\lambda I - BA)^{-1} = \lambda^{-1}(I + BCA) \in \mathcal{L}(X)$. Hence also $\sigma(BA)$ is bounded and both equalities follow. $\square$

A locally convex space is called *H-locally convex* if its topology can be defined by a calibration P such that each $p_\alpha \in P$ is generated by a semiscalar product: $p_\alpha^2(x) = (x, x)_\alpha$

(see, e.g., [10]). In H -locally convex spaces the trace functional is well-defined on nuclear operators and a generalization to these spaces of Lidskii's theorem holds, which says that the trace of nuclear operator is equal to the sum of its eigenvalues, (see [7]). By this theorem and by the above lemma we also have $\text{tr}(AB) = \text{tr}(BA)$ for each pair of nuclear operators A and B acting on an H -locally convex space.

THEOREM 2.20. *Let $\mathcal{S}$ be a family of nuclear operators on a barreled H -locally convex space. Then $\mathcal{S}$ is triangularizable if and only if the trace functional is permutable on $\mathcal{S}$.*

Proof. We can verify as in [11, Lemma 2.1.14] that the permutability of the trace functional is equivalent to the condition $\text{tr}(ABC) = \text{tr}(ACB)$, for all $A, B, C \in \mathcal{S}$. Let $\mathcal{S}$ be triangularizable. Then for the diagonal elements the following relations hold:

$$d_j(ABC) = d_j(A)d_j(B)d_j(C) = d_j(ACB); \quad A, B, C \in \mathcal{S} \quad (2.5)$$

(see [9]), where it is also shown that the nonzero eigenvalues coincide with nonzero diagonal elements for each operator of $\mathcal{S}$. Thus, for any $A, B, C \in \mathcal{S}$ we have

$$\text{tr}(ABC) = \sum \lambda_j(ABC) = \sum d_j(ABC) = \sum d_j(ACB) = \text{tr}(ACB). \quad (2.6)$$

Let the trace be permutable on $\mathcal{S}$. Then it is also permutable on the algebra $\mathcal{A}$ generated by $\mathcal{S}$. Hence $\text{tr}(A(BC - CB)) = 0$, $A, B, C \in \mathcal{A}$. Taking $A = (BC - CB)^{n-1}$, for $n \in \mathbb{N}$ we obtain $\text{tr}((BC - CB)^n) = 0$, $n \in \mathbb{N}$. Denote $T := BC - CB$, since $T \in \mathcal{H}(X)$, we have $\lambda_j(T^n) = \lambda_j(T)^n$ for each j (see [12]). Hence, by Lidskii's theorem we have

$$\sum_j \lambda_j(T)^n = \sum_j \lambda_j(T^n) = \text{tr}(T^n) = 0, \quad n \in \mathbb{N}. \quad (2.7)$$

As in [4] it follows that $\lambda_j(T) = 0$ for each j , thus, $\sigma(BC - CB) = \{0\}$ for each pair $B, C \in \mathcal{A}$ and by [5, Theorem 2], $\mathcal{A}$ is triangularizable and the same holds for $\mathcal{S}$. $\square$

COROLLARY 2.21. *Let $\mathcal{S}$ be a family of nuclear operators on a barreled H -locally convex space. Then $\mathcal{S}$ is triangularizable if the trace functional satisfies one of the following conditions:*

- (i) *it is multiplicative on $\mathcal{S}$,*
- (ii) *it is constant on $\mathcal{S}$.*

The following result is a generalization of the so-called “downsizing lemma” from [11].

LEMMA 2.22. *Let X be a barreled locally convex space, $\mathcal{S}$ a semigroup of compact operators on X and $\mathcal{P}$ a property on $\mathcal{S}$ such that*

- (i) *each subsemigroup in $\mathcal{S}$ has the property $\mathcal{P}$,*
- (ii) *$\mathcal{S}|_{X_0}$ has the property $\mathcal{P}$, where $X_0 = \overline{\text{span}\{\mathcal{R}(S), S \in \mathcal{S}\}}$,*
- (iii) *$\widehat{\mathbb{R}^+\mathcal{S}}$ has the property $\mathcal{P}$.*

Let $\mathcal{S}$ be irreducible. Then there exist a natural number $k \geq 2$ and an idempotent operator $E \in \mathcal{L}(X)$ of rank k , such that $\mathcal{S}$ contains a subsemigroup $\mathcal{S}_0$ with properties: $\mathcal{S}_0 = E\mathcal{S}_0$, $\mathcal{S}_0|_{\mathcal{R}(E)}$ is irreducible in $\mathcal{L}(\mathbb{C}^k)$ and it has the property $\mathcal{P}$. Moreover, if $\min\{\text{rank}(F) : F \in \widehat{\mathbb{R}^+\mathcal{S}}\} > 1$, the operator E can be chosen from $\widehat{\mathbb{R}^+\mathcal{S}}$.

Proof. By [6, Theorem 3], not all operators in $\mathcal{S}$ are quasinilpotent, hence by Proposition 2.4, $\widehat{\mathbb{R}^+\mathcal{S}}$ contains a nonzero finite-rank operator. By [6, Proposition 2], $\hat{\mathcal{S}} := \widehat{\mathbb{R}^+\mathcal{S}}$ is also a semigroup. Then the proof is the same as for the normed space (see [11, Lemma 8.2.13]), where we take $\hat{\mathcal{S}}$ instead of $\mathcal{S}$. $\square$

PROPOSITION 2.23. *Let X be a barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators on X . If $\mathcal{S}^k$ is triangularizable for some $k \in \mathbb{N}$, then $\mathcal{S}$ is triangularizable.*

Proof. If $\mathcal{S}^k = \{0\}$, then by [6, Theorem 3], $\mathcal{S}$ is reducible. Let $\mathcal{S}^k \neq \{0\}$ and let $\mathcal{S}^k$ be triangularizable, then $\mathcal{S}$ is reducible since $\mathcal{S}^k$ is an ideal in $\mathcal{S}$. The triangularizability of a family of compact operators $\mathcal{S}^k$ is inherited by quotients which follows in the same manner as in the proof of [11, Theorem 7.3.9] for the normed space. Then applying the triangularization lemma [5] the triangularization of $\mathcal{S}$ follows. $\square$

THEOREM 2.24. *Let X be a barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators on X . If $AB - BA$ is quasinilpotent for every $A, B \in \mathcal{S}$, then $\mathcal{S}$ is triangularizable.*

Proof. By [5, Lemma 5], the quasinilpotency is inherited by quotients for compact operators. So, by triangularizing lemma it suffices to prove the reducibility of the semigroup $\mathcal{S}$. Let us verify the conditions of Lemma 2.22. The condition (i) is obvious. Denoting by X_0 the closed span of $\{\mathcal{R}(S) : S \in \mathcal{S}\}$, then for $A, B \in \mathcal{S}$ it is easy to see that $A|_{X_0}B|_{X_0} - B|_{X_0}A|_{X_0} = (AB - BA)|_{X_0}$ and it is clear that $(AB - BA)|_{X_0}$ is also quasinilpotent, hence (ii) holds. By [5, Theorem 1], quasinilpotency is inherited by $\widehat{\mathbb{R}^+\mathcal{S}}$ and so (iii) holds. If $\mathcal{S}$ were irreducible, then by Lemma 2.22, there would exist a subsemigroup $\mathcal{S}_0$ of $\mathcal{S}$ such that $\mathcal{S}_0|_{\mathcal{R}(E)}$ would be irreducible in $\mathcal{L}(\mathbb{C}^k)$, which is impossible [11, Theorem 4.4.12]. $\square$

Let $\mathcal{E}$ be a semigroup in $\mathcal{LB}(X)$. It is known (see, e.g., [12]) that the spectrum for each $T \in \mathcal{LB}(X)$ is bounded. We say that the spectrum is submultiplicative on $\mathcal{E}$ if $\sigma(AB) \subset \sigma(A)\sigma(B) = \{\lambda\mu : \lambda \in \sigma(A), \mu \in \sigma(B)\}$ for all $A, B \in \mathcal{E}$.

THEOREM 2.25. *Let X be an infinite-dimensional barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators on X with the property that the spectrum is submultiplicative on $\mathcal{S}$. Then $\mathcal{S}$ is reducible.*

Proof. Let us denote by $\hat{\mathcal{S}} := \widehat{\mathbb{R}^+\mathcal{S}}$. Suppose that $\mathcal{S}$ is irreducible, then $\hat{\mathcal{S}}$ is irreducible too. By [6, Proposition 3] there exists nonzero finite-rank idempotent operator $E \in \hat{\mathcal{S}}$ which has minimal rank m . Denoting $\mathcal{S}_0 = \hat{\mathcal{S}}E\hat{\mathcal{S}}$, this is an ideal in $\hat{\mathcal{S}}$ and all operators in $\mathcal{S}_0$ have a rank equal to m or 0. Thus, $\mathcal{S}_0$ is irreducible and then the rest of the proof is the same as for the normed space (see [11, Theorem 8.3.5]), where we take $\mathcal{S}_0$ instead of $\mathcal{S}$. $\square$

THEOREM 2.26. *Let X be an infinite-dimensional barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators with the following property: $\sigma(S) \subset \{0, 1\}$ for every $S \in \mathcal{S}$ and if $1 \in \sigma(ST)$, for $S, T \in \mathcal{S}$, let $1 \in \sigma(S) \cap \sigma(T)$. Then $\mathcal{S}$ is reducible.*

Proof. We can assume that X is not normable (see [11, Theorem 8.3.8]). Since each $A \in \mathcal{S}$ is locally bounded, it has 0 in its spectrum [12]. With the above assumption the submultiplicativity of spectrum on $\mathcal{S}$ follows, so $\mathcal{S}$ is reducible by the previous theorem. $\square$

THEOREM 2.27. *Let X be a barreled locally convex space and $\mathcal{S}$ a semigroup of compact idempotent operators. Then $\mathcal{S}$ is triangularizable.*

Proof. In view of [11, Theorem 2.3.5] let X be infinite-dimensional. A quotient of an idempotent is, clearly, idempotent operator and a quotient of compact operator is again compact (see [5]). So, by the triangularization lemma, it suffices to prove the reducibility. Clearly, for an idempotent S and $\lambda \neq 0, \neq 1$, there exists $(\lambda I - S)^{-1} = I/\lambda - S/(\lambda(1 - \lambda)) \in \mathcal{L}(X)$, hence $\sigma(S) \subset \{0, 1\}$. If $1 \in \sigma(ST)$ for $S, T \in \mathcal{S}$, then S and T are nonzero idempotent, hence $1 \in \sigma(S) \cap \sigma(T)$. Thus, the conditions of the preceding theorem are fulfilled and the reducibility of $\mathcal{S}$ follows. $\square$

For a semiball U_γ , $\gamma \in \Delta$, let us denote by $\mathcal{L}_\gamma(X)$ the family of all continuous linear operators T on X , for which $T(U_\gamma)$ is a bounded set. Clearly, this is a subspace and left ideal in $\mathcal{L}(X)$. For each $T \in \mathcal{L}_\gamma(X)$ we have $T(U_\gamma) \subset \lambda_\gamma U_\gamma$ for some $\lambda_\gamma > 0$, hence $T(J_\gamma) \subset J_\gamma$, thus, operator T_γ is well defined on X_γ . For some fixed $T \in \mathcal{L}_\gamma(X)$ the convergence of some sequence in $\mathcal{L}(X)$ is inherited to the operator sequence on the quotient space X_γ in the following sense.

LEMMA 2.28. *Let X be a locally convex space, $T \in \mathcal{L}_\gamma(X)$ for some $\gamma \in \Delta$ and $\{S_n\} \subset \mathcal{L}(X)$ a sequence which is τ_b -convergent to some S in $\mathcal{L}(X)$. Then $\{(S_n T)_\gamma\}$ is convergent to $(ST)_\gamma$ with respect to the norm $\|\cdot\|_\gamma$ in X_γ .*

Proof. Since $M := T(U_\gamma)$ is a bounded set, we have $\|(S_n T)_\gamma - (ST)_\gamma\|_\gamma = \sup_{x \in U_\gamma} p_\gamma((S_n T - ST)x) = \sup_{y \in M} p_\gamma((S_n - S)y) = q_\gamma^M(S_n - S) \rightarrow 0$, as $n \rightarrow \infty$. $\square$

It is well known that in a normed space the spectral radius is continuous on the set of compact operators (see, e.g., [11]), but this is not the case for general locally convex spaces. Let us take as an example $X = s$, the space of all real sequences $\{x_n\}$ with the topology generated by seminorms $P = \{p_m : m \in \mathbb{N}\}$, where $p_m(x) = \sup\{|x_j| : j \leq m\}$, $x \in X$ and a sequence of operators $\{T_n\}$ defined by $T_n(x_1, x_2, \dots) = (0, 0, \dots, x_n, 0, 0, \dots)$. It is easy to see that all T_n are compact and $T_n \xrightarrow{\tau_b} T$, where $T = 0$. Hence, $r(T_n) = 1$ for all $n \in \mathbb{N}$, but $r(T) = 0$. We will prove the continuity of spectral radius in a special case.

LEMMA 2.29. *Let X be a locally convex space, $T \in \mathcal{H}(X)$ and $\{S_n\}$ a sequence of continuous operators which is τ_b -convergent to $S \in \mathcal{L}(X)$. Then*

$$r(S_n T) \longrightarrow r(ST), \quad r(TS_n) \longrightarrow r(TS), \quad \text{as } n \longrightarrow \infty. \quad (2.8)$$

Proof. Since T is locally bounded, there is some $\gamma \in \Delta$ such that $T \in \mathcal{L}_\gamma(X)$ and also $AT \in \mathcal{L}_\gamma(X)$ for each $A \in \mathcal{L}(X)$. The corresponding operator T_γ is also compact on X_γ and for the spectral radius we have $r(T) = r(T_\gamma)$ (see [2]). Then, by Lemma 2.28 and by continuity of the spectral radius for a sequence of compact operators acting on a normed

space, we obtain $r(S_n T) = r((S_n T)_\gamma) \rightarrow r((ST)_\gamma) = r(ST)$, as $n \rightarrow \infty$. By Lemma 2.19 we have also $r(TS_n) \rightarrow r(TS)$, as $n \rightarrow \infty$. $\square$

LEMMA 2.30. *Let X be a locally convex space, $\mathcal{S}$ a semigroup in $\mathcal{L}(X)$ and A, B some nonzero members in $\mathcal{S}$ such that $B\mathcal{S}A = \{0\}$. Then $\mathcal{S}$ is reducible.*

Proof. If $\mathcal{S}\mathcal{R}(A) = \{0\}$, then it is easy to see that $\overline{\mathcal{R}(A)}$ is a nontrivial invariant subspace for $\mathcal{S}$. If $\mathcal{S}\mathcal{R}(A) \neq \{0\}$, then the closed span of this set is a nontrivial invariant subspace for $\mathcal{S}$. $\square$

LEMMA 2.31. *Let X be a locally convex space and $\mathcal{S}$ an irreducible semigroup of compact operators on X . If spectral radius is submultiplicative on $\mathcal{S}$ then no nonzero product $\tilde{S}\tilde{T}$, where $\tilde{S}, \tilde{T} \in \tilde{\mathcal{S}}$, is quasinilpotent.*

Proof. Let us denote $\mathcal{J} = \{\tilde{S}\tilde{T} : \tilde{S}, \tilde{T} \in \tilde{\mathcal{S}}, r(\tilde{S}\tilde{T}) = 0\}$. This is an outer ideal of $\mathcal{S}$. Indeed, choose any product $\tilde{S}\tilde{T} \in \mathcal{J}$ and $C \in \mathcal{S}$, then there exist two sequences $\{S_n\}$ and $\{T_m\}$ in $\mathcal{S}$ such that $S_n \xrightarrow{\tau_b} \tilde{S}$ and $T_m \xrightarrow{\tau_b} \tilde{T}$. For each pair of operators S_n, T_m we have

$$r(CS_n T_m) \leq r(C)r(S_n T_m); \quad n, m \in \mathbb{N}. \quad (2.9)$$

Using Lemma 2.29 twice we obtain $r(C\tilde{S}\tilde{T}) \leq r(C)r(\tilde{S}\tilde{T}) = 0$, hence, $C\tilde{S}\tilde{T} \in \mathcal{J}$. Since $r(\tilde{S}\tilde{T}C) = r(C\tilde{S}\tilde{T})$, also $\tilde{S}\tilde{T}C \in \mathcal{J}$. If $\mathcal{J} \neq \{0\}$, by [6, Theorem 3], it would be reducible and then by [6, Lemma 5], $\mathcal{S}$ would be reducible too. Thus, $\mathcal{J} = \{0\}$. $\square$

THEOREM 2.32. *Let $\mathcal{S}$ be an irreducible semigroup of compact operators on a barreled locally convex space. If spectral radius is submultiplicative on $\mathcal{S}$, then it is multiplicative on $\mathcal{S}$.*

Proof. Since the spectral radius is homogenous for the nonnegative scalars, one can suppose $\mathbb{R}^+\mathcal{S} = \mathcal{S}$. By [6], $\tilde{\mathcal{S}}$ is again a semigroup. Let us prove that it has no quasinilpotent element. Suppose, to the contrary, that there is a nonzero quasinilpotent operator $\tilde{T}$ in $\tilde{\mathcal{S}}$. Then also $r(\tilde{T}^2) = 0$, hence by the previous lemma, $\tilde{T}^2 = 0$. Consequently $r(\tilde{T}\tilde{S}\tilde{T}) = r(\tilde{S}\tilde{T}^2) = 0$ for any $\tilde{S} \in \tilde{\mathcal{S}}$. Thus $\tilde{T}\tilde{\mathcal{S}}\tilde{T} = \{0\}$, hence, by Lemma 2.30, $\tilde{\mathcal{S}}$ is reducible, which is a contradiction. Choose any $A, B \in \mathcal{S}$, where we can assume $r(A) = r(B) = 1$. By Proposition 2.4 there exist two nonzero finite-rank idempotents $E, F \in \tilde{\mathcal{S}}$ such that $A^{n_i} \xrightarrow{\tau_b} E$ and $B^{m_k} \xrightarrow{\tau_b} F$ for some sequences of integers $\{n_i\}$ and $\{m_k\}$. Let us prove that $EF \neq 0$. If $EF = 0$, then $r(F\tilde{S}E) = r(\tilde{S}EF) = 0$ and by the above lemma it would be $F\tilde{S}E = 0$ for each $\tilde{S} \in \tilde{\mathcal{S}}$, hence $F\tilde{\mathcal{S}}E = \{0\}$ and by Lemma 2.30, $\tilde{\mathcal{S}}$ would be reducible. In the sequel, let us prove the following inequality:

$$r(EF^2E) \leq r(EF)^2. \quad (2.10)$$

For each pair of operators from the sequences defining E and F we have by assumption $r(A^{n_i}B^{2m_k}A^{n_i}) \leq r(A^{n_i}B^{m_k})r(B^{m_k}A^{n_i})$ and by Lemma 2.29 we obtain the above inequality. Similarly, we have $r(A^{n_i}B^{m_k}) = r(ABB^{m_k-1}A^{n_i-1}) \leq r(AB)r(A)^{n_i-1}r(B)^{m_k-1} = r(AB)$ and by Lemma 2.29 we obtain

$$r(EF) \leq r(AB). \quad (2.11)$$

From the idempotency of E and F and by the inequality (2.10) it follows that:

$$r(EF) = r(E^2F^2) = r(EF^2E) \leq r(EF)^2. \quad (2.12)$$

Thus, we have $1 \leq r(EF) \leq r(AB) \leq r(A)r(B) = 1$, and the multiplicativity of the spectral radius follows. $\square$

THEOREM 2.33. *Let X be a barreled locally convex space, $\mathcal{S}$ a semigroup of compact operators on X such that each $S \in \mathcal{S}$ is a nonnegative scalar multiple of an idempotent operator and let spectral radius be submultiplicative on $\mathcal{S}$. Then $\mathcal{S}$ is triangularizable.*

Proof. Let us prove that $\mathcal{S}$ is reducible. Denote $\mathcal{S}_0 = \{S/r(S) : S \in \mathcal{S}, S \neq 0\} \cup \{0\}$. Clearly, $\mathcal{S}_0$ is reducible if and only if $\mathcal{S}$ is reducible. Suppose that $\mathcal{S}$ is irreducible. Then, by Theorem 2.32 the spectral radius is multiplicative on $\mathcal{S}$. Consequently, $\mathcal{S}_0$ is a semigroup of compact idempotents. By Theorem 2.27, $\mathcal{S}_0$ is reducible. Thus, $\mathcal{S}$ is reducible and by triangularization lemma it is triangularizable. $\square$

In view of Lemma 2.19 it is easy to see that the spectral radius is permutable on a semigroup $\mathcal{S}$ if and only if $r(ABC) = r(ACB)$ for all $A, B \in \mathcal{S}$.

THEOREM 2.34. *Let X be a locally convex space and $\mathcal{S}$ a semigroup of compact operators on X . Then spectral radius is submultiplicative on $\mathcal{S}$ if and only if it is permutable on $\mathcal{S}$.*

Proof. We will use the property $r(T) = r(T_\gamma)$ for $T \in \mathcal{L}_\gamma(X)$, $\gamma \in \Delta$ (see [2]). With no loss of generality we may assume that the calibration $P \in \mathcal{P}(X)$ is *directed*, that is for each $p_\alpha, p_\beta \in P$ there is some $p_\gamma \in P$ such that $p_\alpha \leq p_\gamma$ and $p_\beta \leq p_\gamma$. Let r be permutable on $\mathcal{S}$. Choose any $A, B \in \mathcal{S}$. Since they are locally bounded and P is directed, there exists $p_\gamma \in P$ such that $A, B \in \mathcal{L}_\gamma(X)$. Denote by $\mathcal{S}_0^\gamma$ the semigroup generated by A_γ, B_γ . By [6, Lemma 1] spectral radius is also permutable on $\mathcal{S}_0^\gamma$ and by [11, Theorem 8.6.3] it is submultiplicative and then $r(AB) = r(A_\gamma B_\gamma) \leq r(A_\gamma)r(B_\gamma) = r(A)r(B)$. Let r be submultiplicative on $\mathcal{S}$. For any $A, B, C \in \mathcal{S}$ there is some $p_\gamma \in P$ such that $A, B, C \in \mathcal{L}_\gamma(X)$, then on the semigroup $\mathcal{S}_1^\gamma$ generated by A_γ, B_γ , and C_γ the submultiplicativity implies the permutability of r and similarly as above we obtain $r(ABC) = r(ACB)$. $\square$

Question. What are the conditions on a family of compact operators on a locally convex space yielding the continuity of the spectral radius on this family?

References

- [1] K. Floret and J. Wloka, *Einführung in die Theorie der lokalkonvexen Räume*, Lecture Notes in Mathematics, no. 56, Springer, Berlin, 1968.
- [2] F. Garibay Bonales and R. Vera Mendoza, *Extending the formula to calculate the spectral radius of an operator*, Proc. Amer. Math. Soc. **126** (1998), no. 1, 97–103.
- [3] D. Kalnins, *Sous-espaces hyperinvariant d'un opérateur compact*, C. R. Acad. Sci. Paris Sér. A **288** (1979), 115–116.
- [4] A. Katavolos and H. Radjavi, *Simultaneous triangularization of operators on a Banach space*, J. London Math. Soc. (2) **41** (1990), no. 3, 547–554.
- [5] E. Kramar, *Triangularization of some families of operators on locally convex spaces*, Comment. Math. Univ. Carolin. **42** (2001), no. 3, 499–506.

- [6] ———, *On reducibility of Volterra semigroups on locally convex spaces*, Acta Sci. Math. (Szeged) **70** (2004), no. 1-2, 289–298.
- [7] G. L. Litvinov, *On the traces of linear operators in locally convex spaces*, Selecta Math. Soviet. **8** (1989), no. 3, 203–212.
- [8] G. L. Litvinov and V. I. Lomonosov, *Density theorems in locally convex spaces and their applications*, Trudy Sem. Vektor. Tenzor. Anal. (1981), no. 20, 210–227 (Russian).
- [9] T.-W. Ma, *On rank one commutators and triangular representations*, Canad. Math. Bull. **29** (1986), no. 3, 268–273.
- [10] T. Precupanu, *Espaces linéaires à semi-normes hilbertiennes*, An. Şti. Univ. “Al. I. Cuza” Iaşi Sect. I a Mat. (N.S.) **15** (1969), 83–93.
- [11] H. Radjavi and P. Rosenthal, *Simultaneous Triangularization*, Universitext, Springer, New York, 2000.
- [12] P. Uss, *Sur les opérateurs bornés dans les espaces localement convexes*, Studia Math. **37** (1970/1971), 139–158.

Edvard Kramar: Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: edvard.kramar@fmf.uni-lj.si