ITERATIVE APPROXIMATION OF FIXED POINT FOR Φ -HEMICONTRACTIVE MAPPING WITHOUT LIPSCHITZ ASSUMPTION

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Let E be an arbitrary real Banach space and let K be a nonempty closed convex subset of E such that $K + K \subset K$. Assume that $T : K \rightarrow K$ is a uniformly continuous and Φ -hemicontractive mapping. It is shown that the Ishikawa iterative sequence with errors converges strongly to the unique fixed point of T .

1. Introduction

Let E be a real Banach space and let E^* be the dual space on E . The normalized duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\| = \|f\|^2\} \quad (1.1)$$

for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E is a uniformly smooth Banach space, then J is single valued and such that $J(-x) = -J(x)$, $J(tx) = tJ(x)$ for all $x \in E$ and $t \geq 0$; and J is uniformly continuous on any bounded subset of E . In the sequel, we shall denote single-valued normalized duality mapping by j by means of the normalized duality mapping J . In the following, we give some concepts.

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ is said to be strongly pseudocontractive if for any $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2 \quad (1.2)$$

for some constant $k \in (0, 1)$. The mapping T is called Φ -strongly pseudocontractive if there exists a strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ such that the inequality

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|)\|x - y\| \quad (1.3)$$

holds for all $x, y \in D(T)$. Let $F(T) = \{x \in D(T) : Tx = x\}$. A mapping T is called Φ -hemicontractive if there exists a strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with

$\Phi(0) = 0$ such that the inequality

$$\langle Tx - Tq, j(x - q) \rangle \leq \|x - q\|^2 - \Phi(\|x - q\|)\|x - q\| \quad (1.4)$$

holds for all $x \in D(T)$ and $q \in F(T)$.

It is shown in [5] that the class of strongly pseudocontractive mapping is a proper subclass of Φ -strongly pseudocontractive mapping. Furthermore, the example in [2] shows that the class of Φ -strongly pseudocontractive mapping with the nonempty fixed point set is a proper subclass of Φ -hemiccontractive mapping. The classes of mappings introduced above have been studied by several authors. In [1], Chidume proved that if $E = L_p$ (or l^p), $p \geq 2$, K is a nonempty closed convex and bounded subset of E , and $T : K \rightarrow K$ is a Lipschitz strongly pseudocontractive mapping, then Mann iteration process converges strongly to the unique fixed point of T . In [4], Deng extended the above result to the Ishikawa iteration process. After Tan and Xu [7] extended the results of both Chidume [1] and Deng [4] to q -uniformly smooth Banach spaces ($1 < q < 2$), Chidume and Osilike [3] extended to real q -uniformly smooth Banach spaces ($1 < q < \infty$). Recently, these results above have been extended from Lipschitz strongly pseudocontractive mapping to Lipschitz Φ -strongly pseudocontractive mapping in real q -uniformly smooth Banach spaces ($1 < q < \infty$). More recently, Osilike [6] proved that if K is a nonempty closed convex subset of arbitrary real Banach space E and $T : K \rightarrow K$ is a Lipschitzian Φ -hemiccontractive mapping, then Ishikawa iteration sequence $\{x_n\}_{n=1}^\infty$ converges strongly to the unique fixed point of T . It is our purpose in this paper to examine the strong convergence theorems of the Ishikawa iterative sequences with errors for Φ -hemiccontractive mapping in arbitrary real Banach spaces.

LEMMA 1.2. *Let E be a real Banach space, then for all $x, y \in E$, there exists $j(x + y) \in J(x + y)$ such that $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle$.*

Proof. By definition of duality mapping, we may obtain directly the results of Lemma 1.2. $\square$

2. Main results

THEOREM 2.1. *Let E be a real Banach space, and let K be a nonempty closed convex subset of E such that $K + K \subset K$. Assume that $T : K \rightarrow K$ is a uniformly continuous Φ -hemiccontractive mapping. Let $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ be two real sequences in $[0, 1]$ satisfying the following conditions: (i) $\alpha_n, \beta_n \rightarrow 0$ as $n \rightarrow \infty$; (ii) $\sum_{n=0}^\infty \alpha_n = \infty$. Suppose that $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ are two sequences in K satisfying that $\sum_{n=0}^\infty \|u_n\| < \infty$ and $\sum_{n=0}^\infty \|v_n\| < \infty$. Define the Ishikawa iterative sequence $\{x_n\}_{n=0}^\infty$ with errors in K by*

$$(IS) \begin{cases} x_0 \in K, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n + v_n, & n \geq 0, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n, & n \geq 0. \end{cases} \quad (2.1)$$

If $\{Ty_n\}_{n=0}^\infty$ and $\{Tx_n\}_{n=0}^\infty$ are bounded, then the sequence $\{x_n\}_{n=0}^\infty$ converges strongly to the unique fixed point of T .

Proof. We first observe that the iterative sequence $\{x_n\}$ defined by (2.1) is well defined, since K is convex and T is a self-mapping from K to itself with $K + K \subset K$. By the definition of T , we know that if $F(T) \neq \emptyset$, then $F(T)$ must be a singleton, let $q \in K$ denote the unique fixed point. And we also obtain that for any $x \in K$, there exists $j(x - q) \in J(x - q)$ such that

$$\langle Tx - Tq, j(x - q) \rangle \leq \|x - q\|^2 - \Phi(\|x - q\|)\|x - q\|. \quad (2.2)$$

Now set

$$\begin{aligned} M &= \sup_{n \geq 0} \|Ty_n - q\| + \|x_0 - q\|, \\ D &= \sum_{n=0}^{\infty} \|u_n\| + M + 1. \end{aligned} \quad (2.3)$$

By using induction, we obtain $\|x_n - q\| \leq M + \sum_{n=0}^{\infty} \|u_n\|, n \geq 0$, which implies that $\|x_n - q\| \leq D, n \geq 0$. Using (2.1) and Lemma 1.2, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq) + u_n\|^2 \\ &\leq \|(1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq)\|^2 + 2D\|u_n\|. \end{aligned} \quad (2.4)$$

Let $A_n = \|Ty_n - T(x_{n+1} - u_n)\|$. Then $A_n \rightarrow 0$ as $n \rightarrow \infty$. Indeed, since T is uniformly continuous, we observe that $\{x_n\}_{n=0}^\infty, \{Tx_n\}_{n=0}^\infty$, and $\{Ty_n\}_{n=0}^\infty$ are all bounded and $\|y_n - (x_{n+1} - u_n)\| \rightarrow 0$ as $n \rightarrow \infty$, so that $A_n \rightarrow 0$ as $n \rightarrow \infty$. Using Lemma 1.2, (2.1), and (2.2), we have

$$\begin{aligned} &\|x_{n+1} - u_n - q\|^2 \\ &= \|(1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq)\|^2 \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle Ty_n - Tq, j(x_{n+1} - u_n - q) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle Ty_n - T(x_{n+1} - u_n), j(x_{n+1} - u_n - q) \rangle \\ &\quad + 2\alpha_n \langle T(x_{n+1} - u_n) - Tq, j(x_{n+1} - u_n - q) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n A_n \|x_{n+1} - u_n - q\| \\ &\quad + 2\alpha_n \|x_{n+1} - u_n - q\|^2 - 2\alpha_n \Phi(\|x_{n+1} - u_n - q\|)\|x_{n+1} - u_n - q\| \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n A_n (1 - \alpha_n) \|x_n - q\| + 2\alpha_n^2 A_n D \end{aligned}$$

$$\begin{aligned}
& + 2\alpha_n \|x_{n+1} - u_n - q\|^2 - 2\alpha_n \Phi(\|x_{n+1} - u_n - q\|) \|x_{n+1} - u_n - q\| \\
& \leq (1 - \alpha_n)^2 \|x_n - q\|^2 + \alpha_n A_n (1 - \alpha_n) (1 + \|x_n - q\|^2) + 2\alpha_n^2 A_n D \\
& \quad + 2\alpha_n \|x_{n+1} - u_n - q\|^2 - 2\alpha_n \Phi(\|x_{n+1} - u_n - q\|) \|x_{n+1} - u_n - q\| \\
& \leq ((1 - \alpha_n)^2 + \alpha_n A_n) \|x_n - q\|^2 + \alpha_n A_n (1 + 2\alpha_n D) \\
& \quad + 2\alpha_n \|x_{n+1} - u_n - q\|^2 - 2\alpha_n \Phi(\|x_{n+1} - u_n - q\|) \|x_{n+1} - u_n - q\|,
\end{aligned} \tag{2.5}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - u_n - q\|^2 & \leq \frac{(1 - \alpha_n)^2 + \alpha_n A_n}{1 - 2\alpha_n} \|x_n - q\|^2 + \frac{\alpha_n A_n (1 + 2\alpha_n D)}{1 - 2\alpha_n} \\
& \quad - \frac{2\alpha_n}{1 - 2\alpha_n} \Phi(\|x_{n+1} - u_n - q\|) \|x_{n+1} - u_n - q\| \\
& \leq \|x_n - q\|^2 + \frac{2\alpha_n}{1 - 2\alpha_n} \left(\frac{D^2 \alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D}{2} \right. \\
& \quad \left. - \Phi(\|x_{n+1} - u_n - q\|) \|x_{n+1} - u_n - q\| \right).
\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.4) yields that

$$\begin{aligned}
\|x_{n+1} - q\|^2 & \leq \|x_n - q\|^2 + \frac{2\alpha_n}{1 - 2\alpha_n} \left(\frac{D^2 \alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D}{2} \right. \\
& \quad \left. - \Phi(\|x_{n+1} - u_n - q\|) \|x_{n+1} - u_n - q\| \right) + 2D \|u_n\| \\
& \leq \|x_n - q\|^2 + \frac{2\alpha_n}{1 - 2\alpha_n} (B_n - \Phi(\|x_{n+1} - u_n - q\|) \|x_{n+1} - u_n - q\|) + 2D \|u_n\|,
\end{aligned} \tag{2.7}$$

where $B_n = D^2 \alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D/2$. Now we consider the following two possible cases.

Case (i). $\lim_{n \rightarrow \infty} \inf \|x_{n+1} - u_n - q\| = r > 0$. Since $B_n \rightarrow 0, \alpha_n \rightarrow 0$ as $n \rightarrow \infty$, then there exists a positive integer N such that $B_n < 1/2 \Phi(r)r, \alpha_n < 1/2$ for all $n \geq N$. It follows from (2.7) that

$$\begin{aligned}
\|x_{n+1} - q\|^2 & \leq \|x_n - q\|^2 + \frac{\alpha_n}{1 - 2\alpha_n} \Phi(r)r - \frac{2\alpha_n}{1 - 2\alpha_n} \Phi(r)r + 2D \|u_n\| \\
& \leq \|x_n - q\|^2 - \frac{\alpha_n}{1 - 2\alpha_n} \Phi(r)r + 2D \|u_n\|
\end{aligned} \tag{2.8}$$

which implies that $\Phi(r)r \sum_{n=N}^{\infty} \alpha_n / (1 - 2\alpha_n) \leq \|x_N - q\|^2 + 2D \sum_{n=N}^{\infty} \|u_n\| < \infty$. This contradicts the assumption that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and so the case (i) is impossible.

Case (ii). $\lim_{n \rightarrow \infty} \inf \|x_{n+1} - u_n - q\| = 0$. In this case, there exists a subsequence $\{x_{n_j+1} - u_{n_j} - q\}$ such that $x_{n_j+1} - u_{n_j} - q \rightarrow 0$ as $j \rightarrow \infty$. Hence, for any $0 < \varepsilon < 1$, there exists a positive integer n_j such that $\|x_{n_j+1} - u_{n_j} - q\| < \varepsilon$ and $B_n < \Phi(\varepsilon)\varepsilon$, $2D \sum_{k=n_j+1}^{\infty} \|u_k\| < \varepsilon$ for all $n \geq n_j$ for all $n \geq n_j$. Now we show that $\|x_{n_j+m}\| < \varepsilon$ for all $m \geq 1$. First, by (2.4), we have $\|x_{n_j+1} - q\|^2 \leq \varepsilon^2 + 2D\|u_{n_j}\|$. Again consider the following two possible cases.

Case(ii-1). $\|x_{n_j+2} - u_{n_j+1} - q\| < \varepsilon$. Using (2.4), we obtain

$$\begin{aligned} \|x_{n_j+2} - q\|^2 &= \|(1 - \alpha_{n_j+1})(x_{n_j+1} - q) + \alpha_{n_j+1}(Ty_{n_j+1} - Tq) + u_{n_j+1}\|^2 \\ &\leq \|x_{n_j+2} - u_{n_j+1} - q\|^2 + 2D\|u_{n_j+1}\| \\ &\leq \varepsilon^2 + 2D\|u_{n_j+1}\|. \end{aligned} \quad (2.9)$$

Case(ii-2). $\|x_{n_j+2} - u_{n_j+1} - q\| \geq \varepsilon$. Then using (2.7) yields that

$$\|x_{n_j+2} - q\|^2 \leq \varepsilon^2 + 2D(\|u_{n_j}\| + \|u_{n_j+1}\|). \quad (2.10)$$

For all $m \geq 1$, using induction, we have $\|x_{n_j+m} - q\|^2 \leq \varepsilon^2 + 2D \sum_{k=n_j}^{n_j+m-1} \|u_k\| < 2\varepsilon$. Thus we prove that $x_n \rightarrow q$ as $n \rightarrow \infty$. This completes the proof. $\square$

Remark 2.2. The assumption $K + K \subset K$ only is used to guarantee that the iterative sequence $\{x_n\}_{n=0}^{\infty}$ is well defined. We can drop this assumption in Theorem 2.1 by using a revised iterative scheme.

COROLLARY 2.3. *Let E be a real Banach space, and let K be a nonempty bounded and convex subset of E . Assume that $T : K \rightarrow K$ is a uniformly continuous Φ -hemicontractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\hat{\alpha}_n\}_{n=0}^{\infty}$, $\{\hat{\beta}_n\}_{n=0}^{\infty}$, and $\{\hat{\gamma}_n\}_{n=0}^{\infty}$ be six real sequences in $[0,1]$ satisfying the following conditions: (i) $\beta_n \rightarrow 0, \hat{\beta}_n \rightarrow 0, \hat{\gamma}_n \rightarrow 0$ as $n \rightarrow \infty$; (ii) $\sum_{n=0}^{\infty} \beta_n = \infty, \sum_{n=0}^{\infty} \gamma_n < \infty$; (iii) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two bounded sequences in K . Define iteratively the Ishikawa sequence $\{x_n\}_{n=0}^{\infty}$ with errors in K as follows:*

$$\begin{aligned} x_0 &\in K, \\ y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n, \quad n \geq 0, \\ x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0. \end{aligned} \quad (2.11)$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (2.11) converges strongly to the unique fixed point of T .

Proof. We observe that (2.11) can be rewritten as follows:

$$\begin{aligned} x_0 &\in K, \\ y_n &= (1 - \hat{\beta}_n)x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n(v_n - x_n), \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n T y_n + \gamma_n(u_n - x_n), \quad n \geq 0. \end{aligned} \quad (2.12)$$

It is easily seen that under the assumptions of Corollary 2.3, the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded. Now the conclusion follows from Theorem 2.1. This completes the proof. $\square$

THEOREM 2.4. *Let E be a real Banach space, and let K be a nonempty closed convex subset of E such that $K + K \subset K$. Assume that $T : K \rightarrow K$ is a uniformly continuous Φ -hemicontractive mapping. Let $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ be two real sequences in $[0,1]$ satisfying the following conditions: (i) $\alpha_n, \beta_n \rightarrow 0$ as $n \rightarrow \infty$; (ii) $\sum_{n=0}^\infty \alpha_n = \infty$. Suppose that $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ are two sequences in K satisfying $\|u_n\|, \|v_n\| \rightarrow 0$ as $n \rightarrow \infty$, where $\|u_n\| = o(\alpha_n)$. Define the Ishikawa iterative sequence $\{x_n\}_{n=0}^\infty$ with errors in K by*

$$(IS) \begin{cases} x_0 \in K, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n + v_n, & n \geq 0, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + u_n, & n \geq 0. \end{cases} \quad (2.13)$$

If $\{T y_n\}_{n=0}^\infty$ and $\{T x_n\}_{n=0}^\infty$ are bounded, then the sequence $\{x_n\}_{n=0}^\infty$ converges strongly to the unique fixed point of T .

Proof. Since $K + K \subset K$ and K is convex, we see that the sequence $\{x_n\}_{n=0}^\infty$ is well defined. By the definition of T , T has a unique fixed point in K . Let q denote the unique fixed point. Now we shall show that $\{x_n\}_{n=0}^\infty$ is bounded. In fact, we may set $\|u_n\| = \varepsilon_n \alpha_n$, where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Set $D = \sup_{n \geq 0} \{\|T y_n - q\| + \varepsilon_n\} + \|x_0 - q\|$, by induction, we can show that $\|x_n - q\| \leq D$ for all $n \geq 0$, so that $\{y_n\}$ is bounded. And we have

$$\langle Tx - Tq, j(x - q) \rangle \leq \|x - q\|^2 - \Phi(\|x - q\|)\|x - q\| \quad (2.14)$$

for each $x \in K$. By using Lemma 1.2 and (2.7), we have

$$\|x_{n+1} - q\|^2 \leq \|(1 - \alpha_n)(x_n - q) + \alpha_n(T y_n - Tq)\|^2 + 2D\|u_n\|. \quad (2.15)$$

After repeating the usage of the proof of Theorem 2.1, we obtain

$$\begin{aligned} & \|(1 - \alpha_n)(x_n - q) + \alpha_n(T y_n - Tq)\|^2 \\ & \leq ((1 - \alpha_n)^2 + \alpha_n A_n) \|x_n - q\|^2 + \alpha_n A_n (1 + 2\alpha_n D) \\ & \quad + 2\alpha_n \|x_{n+1} - u_n - q\|^2 - 2\alpha_n \Phi(\|x_{n+1} - u_n - q\|) \|x_{n+1} - u_n - q\|. \end{aligned} \quad (2.16)$$

Thus, we have

$$\begin{aligned} & \|x_{n+1} - q\|^2 \\ & \leq \|x_n - q\|^2 + \frac{2\alpha_n}{1 - 2\alpha_n} \left(\frac{D^2 \alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D}{2} \right. \\ & \quad \left. - \Phi(\|x_{n+1} - u_n - q\|) \|x_{n+1} - u_n - q\| \right) + 2D\|u_n\| \\ & \leq \|x_n - q\|^2 + \frac{2\alpha_n}{1 - 2\alpha_n} (B_n + C_n - \Phi(\|x_{n+1} - u_n - q\|) \|x_{n+1} - u_n - q\|) + 2D\|u_n\|, \end{aligned} \quad (2.17)$$

where $B_n = D^2 \alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D/2 \rightarrow 0$, $C_n = 1 - 2\alpha_n/\alpha_n D \|u_n\| \rightarrow 0$ as $n \rightarrow \infty$. Then $\lim_{n \rightarrow \infty} \inf \|x_{n+1} - u_n - q\| = 0$. If it is not the case, then there exist $\delta > 0$ and

positive integer N such that $B_n + C_n < 1/2\Phi(r)r, \alpha_n < 1/2$ for all $n \geq N$. It follows that $\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n/1 - 2\alpha_n\Phi(r)r$, which leads to $\Phi(r)r \sum_{n=N}^{\infty} \alpha_n \leq \|x_N - q\|^2 < \infty$, a contradiction. Hence, there exists a subsequence $\{x_{n_j} + 1\}$ such that $x_{n_j} + 1 \rightarrow q$ as $j \rightarrow \infty$. At this point, we can choose a positive integer n_j such that $\|x_{n_j+1} - q\| < \varepsilon$ and $B_n + C_n < \Phi(\varepsilon/2)\varepsilon/4, \|u_n\| < \varepsilon/2$ for all $n \geq n_j$. We show that $\|x_{n_j+2} - q\| < \varepsilon$. If not, we assume that $\|x_{n_j+2} - q\| \geq \varepsilon$, then $\|x_{n_j+2} - u_{n_j+1} - q\| \geq \|x_{n_j+2} - q\| - \|u_{n_j+1}\| \geq \varepsilon/2$ so that $\Phi(x_{n_j+2} - u_{n_j+1} - q) \geq \Phi(\varepsilon/2)$. Thus, using (2.17), we have

$$\|x_{n_j+2} - q\|^2 \leq \|x_{n_j+1} - q\|^2 - \frac{\alpha_{n_j+1}}{1 - 2\alpha_{n_j+1}} \Phi\left(\frac{\varepsilon}{2}\right) \frac{\varepsilon}{2} < \varepsilon^2, \quad (2.18)$$

this is a contradiction and so $\|x_{n_j+2} - q\| < \varepsilon$. By induction, $\|x_{n_j+m} - q\| < \varepsilon$ for all $m \geq 1$. $\square$

COROLLARY 2.5. *Let E be a real Banach space, and let K be a nonempty bounded and convex subset of E . Assume that $T : K \rightarrow K$ is a uniformly continuous Φ -hemiccontractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\hat{\alpha}_n\}_{n=0}^{\infty}, \{\hat{\beta}_n\}_{n=0}^{\infty}$, and $\{\hat{\gamma}_n\}_{n=0}^{\infty}$ be six real sequences in $[0, 1]$ satisfying the following conditions: (i) $\beta_n \rightarrow 0, \hat{\beta}_n \rightarrow 0, \hat{\gamma}_n \rightarrow 0$ as $n \rightarrow \infty$; (ii) $\sum_{n=0}^{\infty} \beta_n = \infty, \gamma_n = o(\beta_n)$; (iii) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1, n \geq 0$. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two bounded sequences in K . Define iteratively the Ishikawa sequence $\{x_n\}_{n=0}^{\infty}$ with errors in K as follows:*

$$\begin{aligned} x_0 &\in K, \\ y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n, \quad n \geq 0, \\ x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0. \end{aligned} \quad (2.19)$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (2.11) converges strongly to the unique fixed point of T .

Proof. We observe that (2.11) can be rewritten as follows:

$$\begin{aligned} x_0 &\in K, \\ y_n &= (1 - \hat{\beta}_n)x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n(v_n - x_n), \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n T y_n + \gamma_n(u_n - x_n), \quad n \geq 0. \end{aligned} \quad (2.20)$$

It is easily to obtain the conclusion from Theorem 2.4. This completes the proof. $\square$

Remark 2.6. Theorems 2.1 and 2.4 extend the results of [5] from real q -uniformly smooth Banach spaces to arbitrary real Banach spaces. It is also easy to see that our results are significant extensions of the results of [1, 2, 3, 4, 7] to arbitrary real Banach spaces and to the more general classes of mapping (Φ -hemiccontractive mapping) considered here. Moreover, our iteration schemes extend from the usual iterative sequences to the iterative sequences with errors.

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