GENERALIZED $(\mathcal{F}, b, \phi, \rho, \theta)$-UNIVEX n-SET FUNCTIONS AND GLOBAL PARAMETRIC SUFFICIENT OPTIMALITY CONDITIONS IN MINMAX FRACTIONAL SUBSET PROGRAMMING

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We first define a new class of generalized convex n-set functions, called $(\mathcal{F}, b, \phi, \rho, \theta)$-*univex functions*, and then establish a fairly large number of global parametric sufficient optimality conditions under a variety of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions for a discrete minmax fractional subset programming problem.

1. Introduction

In this paper, we will present a number of global parametric sufficient optimality conditions under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity hypotheses for the following discrete minmax fractional subset programming problem:

$$\text{Minimize} \max_{1 \le i \le p} \frac{F_i(S)}{G_i(S)} \quad \text{subject to } H_j(S) \le 0, \quad j \in \underline{q},\ S \in \mathbb{A}^n, \tag{1.1}$$

where $\mathbb{A}^n$ is the n-fold product of the σ-algebra $\mathbb{A}$ of subsets of a given set X, F_i, G_i, $i \in \underline{p} \equiv \{1, 2, \ldots, p\}$, and H_j, $j \in \underline{q}$, are real-valued functions defined on $\mathbb{A}^n$, and for each $i \in \underline{p}$, $G_i(S) > 0$ for all $S \in \mathbb{A}^n$ such that $H_j(S) \le 0$, $j \in \underline{q}$.

Optimization problems of this type in which the functions F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are defined on a subset of $\mathbb{R}^n$ (n-dimensional Euclidean space) are called *generalized fractional programming problems*. These problems have arisen in multiobjective programming [1], approximation theory [2, 3, 20, 34], goal programming [8, 19], and economics [33].

The notion of duality for a generalized linear fractional programming problem with point functions was originally considered by von Neumann [33] in the context of an economic equilibrium problem. More recently, various optimality conditions, duality results, and computational algorithms for several classes of generalized fractional programs with point functions have appeared in the related literature. A fairly extensive list of references pertaining to different aspects of these problems is given in [40].

In the area of subset programming problems, minmax fractional programs like (1.1) were first discussed in [37, 38]. In [37], necessary and sufficient optimality conditions and several duality results were established under generalized ρ-convexity assumptions.

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This was accomplished by combining the necessary optimality conditions of [9] for a nonlinear program involving differentiable n-set functions, which are the n-set versions of the seminal results of Morris [28], with a Dinkelbach-type parametric approach [11]. Subsequently, a Lagrangian-type dual problem was constructed for (1.1) in [38] via a Gordan-type theorem of the alternative, and appropriate duality theorems were proved without imposing any differentiability requirements. Later, some results of [37] were generalized in [30] by replacing the notion of ρ-convexity with $(\mathcal{F},\rho)$-convexity, and in [7] by placing generalized ρ-convexity hypotheses on different combinations of the problem functions; different derivations of the dual problem of [38] were given in [4, 18]. In addition, in [18] the n-set counterpart of a Lagrangian-type dual problem originally formulated by Xu [35] was presented. Recently, parameter-free versions of the results of [37] were established in [21], some optimality and duality results for (1.1) were obtained in [6] under generalized b-vexity assumptions, several optimality results and duality relations for (1.1) with nonsmooth generalized $(\mathcal{F},\rho,\theta)$-convex functions were discussed in [22], and a number of generalized sufficient optimality criteria and duality theorems for (1.1) were proved in [42] under various $(\mathcal{F},\alpha,\rho,\theta)$-V-convexity hypotheses. The optimality results developed here and the complementary duality results obtained in the companion paper [36] under various generalized $(\mathcal{F},b,\phi,\rho,\theta)$-univexity hypotheses subsume a great variety of optimality and duality results obtained previously for several classes of subset programming problems, including those of [6, 7, 9, 25, 28, 30, 37, 39].

For brief surveys and lists of references pertaining to various aspects of subset programming problems, including areas of applications, optimality conditions, and duality models, the reader is referred to [21, 30, 32, 39].

The rest of this paper is organized as follows. In Section 2, we recall the definitions of differentiability, convexity, and certain types of generalized convexity for n-set functions, which will be used frequently throughout the sequel. We begin our discussion of sufficiency criteria for (1.1) in Section 3 where we state and prove a number of sufficiency results. More general sets of sufficiency conditions are formulated and discussed in Section 4 with the help of two partitioning schemes. The first of these schemes was originally used in [27] for constructing generalized dual problems for nonlinear programs with point functions, whereas the second appears to be new and leads to a number of different sufficiency criteria for generalized fractional programming problems.

Evidently, all these optimality results are also applicable, when appropriately specialized, to the following three classes of problems with discrete max, fractional, and conventional objective functions, which are particular cases of (1.1):

$$\begin{gathered}\underset{S\in\mathbb{F}}{\text{Minimize}}\ \max_{1\le i\le p} F_i(S),\\ \underset{S\in\mathbb{F}}{\text{Minimize}}\ \frac{F_1(S)}{G_1(S)},\\ \underset{S\in\mathbb{F}}{\text{Minimize}}\ F_1(S),\end{gathered} \tag{1.2}$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (1.1), that is,

$$\mathbb{F} = \{S \in \mathbb{A}^n : H_j(S) \le 0,\ j \in \underline{q}\}. \tag{1.3}$$

Since in most cases, the optimality results established for (1.1) can easily be modified and restated for each one of the above problems, we will not explicitly state these results.

2. Preliminaries

In this section, we gather, for convenience of reference, a number of basic definitions along with a few auxiliary results, which will be used often throughout the sequel.

Let $(X,\mathbb{A},\mu)$ be a finite atomless measure space with $L_1(X,\mathbb{A},\mu)$ separable, and let d be the pseudometric on $\mathbb{A}^n$ defined by

$$d(R,S) = \left[\sum_{i=1}^{n} \mu^2(R_i \triangle S_i)\right]^{1/2}, \quad R = (R_1,\dots,R_n),\ S = (S_1,\dots,S_n) \in \mathbb{A}^n, \tag{2.1}$$

where $\triangle$ denotes symmetric difference; thus $(\mathbb{A}^n,d)$ is a pseudometric space. For $h \in L_1(X,\mathbb{A},\mu)$ and $T \in \mathbb{A}$ with characteristic function $\chi_T \in L_\infty(X,\mathbb{A},\mu)$, the integral $\int_T h\,d\mu$ will be denoted by $\langle h,\chi_T\rangle$.

We next define the notions of differentiability and convexity for n-set functions. They were originally introduced by Morris [28] for set functions, and subsequently extended by Corley [9] for n-set functions.

Definition 2.1. A function $F:\mathbb{A}\to\mathbb{R}$ is said to be *differentiable* at S^* if there exists $DF(S^*) \in L_1(X,\mathbb{A},\mu)$, called the *derivative* of F at S^*, such that for each $S \in \mathbb{A}$,

$$F(S) = F(S^*) + \langle DF(S^*),\chi_S - \chi_{S^*}\rangle + V_F(S,S^*), \tag{2.2}$$

where $V_F(S,S^*)$ is $o(d(S,S^*))$, that is, $\lim_{d(S,S^*)\to 0} V_F(S,S^*)/d(S,S^*) = 0$.

Definition 2.2. A function $G:\mathbb{A}^n\to\mathbb{R}$ is said to have a *partial derivative* at $S^* = (S_1^*,\dots,S_n^*) \in \mathbb{A}^n$ with respect to its ith argument if the function $F(S_i) = G(S_1^*,\dots,S_{i-1}^*,S_i,S_{i+1}^*,\dots,S_n^*)$ has derivative $DF(S_i^*)$, $i \in \underline{n}$; in that case, the ith partial derivative of G at S^* is defined to be $D_iG(S^*) = DF(S_i^*)$, $i \in \underline{n}$.

Definition 2.3. A function $G:\mathbb{A}^n\to\mathbb{R}$ is said to be *differentiable* at S^* if all the partial derivatives $D_iG(S^*)$, $i \in \underline{n}$, exist and

$$G(S) = G(S^*) + \sum_{i=1}^{n} \langle D_iG(S^*),\chi_{S_i} - \chi_{S_i^*}\rangle + W_G(S,S^*), \tag{2.3}$$

where $W_G(S,S^*)$ is $o(d(S,S^*))$ for all $S \in \mathbb{A}^n$.

It was shown by Morris [28] that for any triple $(S,T,\lambda) \in \mathbb{A}\times\mathbb{A}\times[0,1]$, there exist sequences $\{S_k\}$ and $\{T_k\}$ in $\mathbb{A}$ such that

$$\chi_{S_k} \xrightarrow{w^*} \lambda\chi_{S\setminus T}, \qquad \chi_{T_k} \xrightarrow{w^*} (1-\lambda)\chi_{T\setminus S} \tag{2.4}$$

imply

$$\chi_{S_k\cup T_k\cup(S\cap T)} \xrightarrow{w^*} \lambda\chi_S + (1-\lambda)\chi_T, \tag{2.5}$$

where $\overset{w^*}{\to}$ denotes weak* convergence of elements in $L_\infty(X,\mathbb{A},\mu)$, and $S\backslash T$ is the complement of T relative to S. The sequence $\{V_k(\lambda)\} = \{S_k \cup T_k \cup (S \cap T)\}$ satisfying (2.4) and (2.5) is called the *Morris sequence* associated with (S,T,λ).

Definition 2.4. A function $F : \mathbb{A}^n \to \mathbb{R}$ is said to be (strictly) convex if for every $(S,T,\lambda) \in \mathbb{A}^n \times \mathbb{A}^n \times [0,1]$, there exists a Morris sequence $\{V_k(\lambda)\}$ in $\mathbb{A}^n$ such that

$$\limsup_{k\to\infty} F\big(V_k(\lambda)\big)(<) \le \lambda F(S) + (1-\lambda)F(T). \tag{2.6}$$

It was shown in [9, 28] that if a differentiable function $F : \mathbb{S} \to \mathbb{R}$ is (strictly) convex, then

$$F(S)(>) \ge F(T) + \sum_{i=1}^{n} \langle D_i F(T), \chi_{S_i} - \chi_{T_i} \rangle \tag{2.7}$$

for all $S,T \in \mathbb{A}^n$.

Following the introduction of the notion of convexity for set functions by Morris [28] and its extension for n-set functions by Corley [9], various generalizations of convexity for set and n-set functions were proposed in [6, 22, 23, 24, 30, 31, 36, 37, 41, 42]. More specifically, quasiconvexity and pseudoconvexity for set functions were defined in [23], and for n-set functions in [24]; generalized ρ-convexity for n-set functions was defined in [37], $(\mathcal{F},\rho)$-convexity in [30], b-vexity in [6], (ρ,b)-vexity in [31], $(\mathcal{F},\rho,\theta)$-convexity for nondifferentiable set functions in [22], and $(\mathcal{F},\alpha,\rho,\theta)$-V-convexity in [41, 42]. For predecessors and point function counterparts of these convexity concepts, the reader is referred to the original papers where the extensions to set and n-set functions are discussed. A survey of recent advances in the area of generalized convex functions and their role in developing optimality conditions and duality relations for optimization problems is given in [29].

For the purpose of formulating and proving various collections of sufficiency criteria for (1.1), in this study, we will use a new class of generalized convex n-set functions, called $(\mathcal{F},b,\phi,\rho,\theta)$-*univex functions*, which will be defined later in this section. This class of functions may be viewed as a combination of several previously defined types of generalized convex functions. Its main ingredients are $\mathcal{F}$-convex functions and univex functions, which were introduced in [15] and [5], respectively. These functions were proposed as generalizations of the class of invex functions.

Prior to giving the definitions of the new classes of n-set functions, it will be useful for purposes of reference and comparison to recall the definitions of the point function analogues of the principal components of these functions mentioned above. We will keep this review to a bare minimum because our primary objective is only to put a number of interrelated generalized convexity concepts into proper perspective. For this reason, we will only reproduce the essential forms of the definitions without elaborating on their refinements, variants, special cases, and other manifestations. For full discussions of the consequences and applications of the underlying ideas, the reader may consult the original sources. We begin by defining an invex function, which occupies a pivotal position in a vast array of generalized convex functions, some of which are specified in the following definitions.

Definition 2.5 (see [14]). Let f be a real-valued differentiable function defined on an open subset $\mathbb{S}$ of $\mathbb{R}^n$. Then f is said to be *η-invex* (*invex with respect to η*) at x^* if there exists a function $\eta : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^n$ such that for each $x \in \mathbb{S}$,

$$f(x) - f(x^*) \geq \nabla f(x^*)^T \eta(x, x^*), \tag{2.8}$$

where $\nabla f(x^*)$ is the gradient of f at x^*, and T denotes transposition; f is said to be *η-invex* (*invex with respect to η*) on $\mathbb{S}$ if there exists a function $\eta : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^n$ such that for all $x, y \in \mathbb{S}$,

$$f(x) - f(y) \geq \nabla f(y)^T \eta(x, y). \tag{2.9}$$

From the above definitions, it is clear that every real-valued differentiable function is invex with respect to $\eta(x, y) = x - y$. This generalization of the concept of convexity was originally proposed by Hanson [14] who showed that for a nonlinear programming problem of the form

$$\text{Minimize } f(x) \quad \text{subject to } g_i(x) \leq 0, \quad i \in \underline{m},\ x \in \mathbb{R}^n, \tag{2.10}$$

where the differentiable functions $f, g_i : \mathbb{R}^n \to \mathbb{R}$ are invex with respect to the same function η, the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term *invex* (for *in*variant con*vex*) was coined by Craven [10] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define η-pseudoinvex and η-quasi-invex functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been extended in several directions. Some recent surveys and syntheses of results pertaining to various generalizations of invex functions and their applications along with extensive lists of relevant references are available in [12, 13, 16, 17, 26, 29]. Two of the earliest generalizations of invex functions are $\mathcal{F}$-convex and (ρ, η)-invex functions. An $\mathcal{F}$-convex function is defined in terms of a sublinear function, that is, a function that is subadditive and positively homogeneous.

Definition 2.6. A function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ is said to be *sublinear* if $\mathcal{F}(x + y) \leq \mathcal{F}(x) + \mathcal{F}(y)$ for all $x, y \in \mathbb{R}^n$, and $\mathcal{F}(ax) = a\mathcal{F}(x)$ for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}_+ \equiv [0, \infty)$.

The function $\mathcal{F}$ is said to be *superlinear* if the conditions specified in the above definition hold with the inequality reversed, that is, with $\leq$ replaced by $\geq$.

Now combining the definitions of $\mathcal{F}$-convex and (ρ, η)-invex functions given in [15, 16], respectively, we can define $(\mathcal{F}, \rho)$-convex, $(\mathcal{F}, \rho)$-pseudoconvex, and $(\mathcal{F}, \rho)$-quasiconvex functions.

Let g be a real-valued differentiable function defined on the open subset $\mathbb{S}$ of $\mathbb{R}^n$, and assume that for each $x, y \in \mathbb{S}$, the function $\mathcal{F}(x, y; \cdot) : \mathbb{R}^n \to \mathbb{R}$ is sublinear.

Definition 2.7. The function g is said to be *$(\mathcal{F}, \rho)$-convex* at y if there exists a real number ρ such that for each $x \in \mathbb{S}$,

$$g(x) - g(y) \geq \mathcal{F}(x, y; \nabla g(y)) + \rho \|x - y\|^2. \tag{2.11}$$

Definition 2.8. The function g is said to be $(\mathcal{F},\rho)$-*pseudoconvex* at y if there exists a real number ρ such that for each $x \in \mathbb{S}$,

$$\mathcal{F}(x,y;\nabla g(y)) \geq -\rho\|x-y\|^2 \Longrightarrow g(x) \geq g(y). \tag{2.12}$$

Definition 2.9. The function g is said to be $(\mathcal{F},\rho)$-*quasiconvex* at y if there exists a real number ρ such that for each $x \in \mathbb{S}$,

$$g(x) \leq g(y) \Longrightarrow \mathcal{F}(x,y;\nabla g(y)) \leq -\rho\|x-y\|^2. \tag{2.13}$$

Evidently, if in Definitions 2.7, 2.8, and 2.9 we choose $\mathcal{F}(x,y;\nabla g(y)) = \nabla g(y)^T\eta(x,y)$, where $\eta : \mathbb{S}\times\mathbb{S} \to \mathbb{R}^n$ is a given function, and set $\rho = 0$, then we see that they reduce to the definitions of η-invexity, η-pseudoinvexity, and η-quasi-invexity for the function g.

The foregoing classes of generalized convex functions have been utilized for establishing numerous sets of sufficient optimality conditions and a variety of duality results for several categories of static and dynamic optimization problems. For a wealth of information as well as long lists of references concerning these results, the reader is referred to [17, 29].

Another significant generalization of the notion of invexity, called *univexity*, which subsumes a number of previously proposed classes of generalized convex functions, was recently given in [5]. We recall the definitions of univex, pseudounivex, and quasiunivex functions.

Let h be a real-valued differentiable function defined on an open subset $\mathbb{S}$ of $\mathbb{R}^n$, let η be a function from $\mathbb{S}\times\mathbb{S}$ to $\mathbb{R}^n$, let Φ be a real-valued function defined on $\mathbb{R}$, and let b be a function from $\mathbb{S}\times\mathbb{S}$ to $\mathbb{R}_+ \setminus \{0\} \equiv (0,\infty)$.

Definition 2.10 (see [5]). The function h is said to be *univex* at y with respect to η, Φ, and b if for each $x \in \mathbb{S}$,

$$b(x,y)\Phi(h(x)-h(y)) \geq \nabla h(y)^T\eta(x,y). \tag{2.14}$$

Definition 2.11 (see [5]). The function h is said to be *pseudounivex* at y with respect to η, Φ, and b if for each $x \in \mathbb{S}$,

$$\nabla h(y)^T\eta(x,y) \geq 0 \Longrightarrow b(x,y)\Phi(h(x)-h(y)) \geq 0. \tag{2.15}$$

Definition 2.12 (see [5]). The function h is said to be *quasiunivex* at y with respect to η, Φ, and b if for each $x \in \mathbb{S}$,

$$\Phi(h(x)-h(y)) \leq 0 \Longrightarrow b(x,y)\nabla h(y)^T\eta(x,y) \leq 0. \tag{2.16}$$

Finally, we are in a position to give our definitions of generalized $(\mathcal{F},b,\phi,\rho,\theta)$-univex n-set functions. They are formulated by combining the n-set versions of Definitions 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, and 2.12.

Let $S,S^* \in \mathbb{A}^n$, and let the function $F : \mathbb{A}^n \to \mathbb{R}$ be differentiable at S^*.

Definition 2.13. The function F is said to be *(strictly)* $(\mathcal{F},b,\phi,\rho,\theta)$-*univex* at S^* if there exist a sublinear function $\mathcal{F}(S,S^*;\cdot) : L_1^n(X,\mathbb{A},\mu) \to \mathbb{R}$, a function $b : \mathbb{A}^n\times\mathbb{A}^n \to \mathbb{R}$ with

positive values, a function $\theta : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$ such that $S \neq S^* \Rightarrow \theta(S,S^*) \neq (0,0)$, a function $\phi : \mathbb{R} \to \mathbb{R}$, and a real number ρ such that for each $S \in \mathbb{A}^n$,

$$\phi(F(S) - F(S^*))(>) \geq \mathcal{F}(S,S^*;b(S,S^*)DF(S^*)) + \rho d^2(\theta(S,S^*)). \tag{2.17}$$

Definition 2.14. The function F is said to be *(strictly)* $(\mathcal{F},b,\phi,\rho,\theta)$-*pseudounivex* at S^* if there exist a sublinear function $\mathcal{F}(S,S^*;\cdot) : L_1^n(X,\mathbb{A},\mu) \to \mathbb{R}$, a function $b : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}$ with positive values, a function $\theta : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$ such that $S \neq S^* \Rightarrow \theta(S,S^*) \neq (0,0)$, a function $\phi : \mathbb{R} \to \mathbb{R}$, and a real number ρ such that for each $S \in \mathbb{A}^n (S \neq S^*)$,

$$\mathcal{F}(S,S^*;b(S,S^*)DF(S^*)) \geq -\rho d^2(\theta(S,S^*)) \Longrightarrow \phi(F(S) - F(S^*))(>) \geq 0. \tag{2.18}$$

Definition 2.15. The function F is said to be $(\mathcal{F},b,\phi,\rho,\theta)$-*quasiunivex* at S^* if there exist a sublinear function $\mathcal{F}(S,S^*;\cdot) : L_1^n(X,\mathbb{A},\mu) \to \mathbb{R}$, a function $b : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}$ with positive values, a function $\theta : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$ such that $S \neq S^* \Rightarrow \theta(S,S^*) \neq (0,0)$, a function $\phi : \mathbb{R} \to \mathbb{R}$, and a real number ρ such that for each $S \in \mathbb{A}^n$,

$$\phi(F(S) - F(S^*)) \leq 0 \Longrightarrow \mathcal{F}(S,S^*;b(S,S^*)DF(S^*)) \leq -\rho d^2(\theta(S,S^*)). \tag{2.19}$$

Definition 2.16. The function F is said to be *prestrictly* $(\mathcal{F},b,\phi,\rho,\theta)$-*quasiunivex* at S^* if there exist a sublinear function $\mathcal{F}(S,S^*;\cdot) : L_1^n(X,\mathbb{A},\mu) \to \mathbb{R}$, a function $b : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}$ with positive values, a function $\theta : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$ such that $S \neq S^* \Rightarrow \theta(S,S^*) \neq (0,0)$, a function $\phi : \mathbb{R} \to \mathbb{R}$, and a real number ρ such that for each $S \in \mathbb{A}^n$, $S \neq S^*$,

$$\phi(F(S) - F(S^*)) < 0 \Longrightarrow \mathcal{F}(S,S^*;b(S,S^*)DF(S^*)) \leq -\rho d^2(\theta(S,S^*)). \tag{2.20}$$

From the above definitions, it is clear that if F is $(\mathcal{F},b,\phi,\rho,\theta)$-univex at S^*, then it is both $(\mathcal{F},b,\phi,\rho,\theta)$-pseudounivex and $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivex at S^*; if F is $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivex at S^*, then it is prestrictly $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivex at S^*; and if F is strictly $(\mathcal{F},b,\phi,\rho,\theta)$-pseudounivex at S^*, then it is $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivex at S^*.

In the proofs of the sufficiency theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivexity can be defined in the following equivalent way: F is said to be $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivex at S^* if for each $S \in \mathbb{A}^n$,

$$\mathcal{F}(S,S^*;b(S,S^*)DF(S^*)) > -\rho d^2(\theta(S,S^*)) \Longrightarrow \phi(F(S) - F(S^*)) > 0. \tag{2.21}$$

Needless to say, the new classes of generalized convex n-set functions specified in Definitions 2.13, 2.14, 2.15, and 2.16 contain a variety of special cases; in particular, they subsume all the previously defined types of generalized n-set functions. This can easily be seen by appropriate choices of $\mathcal{F}$, b, ϕ, ρ, and θ.

We next recall a set of parametric necessary optimality conditions whose form and features will be used as guidelines for formulating our sufficiency criteria for (1.1).

Theorem 2.17 (see [37]). *Assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at $S^* \in \mathbb{A}^n$, and there exists $\hat{S} \in \mathbb{A}^n$ such that*

$$H_j(S^*) + \sum_{k=1}^{n} \langle D_k H_j(S^*), \chi_{\hat{S}_k} - \chi_{S_k^*} \rangle < 0, \quad j \in \underline{q}. \tag{2.22}$$

If S^* is an optimal solution of (1.1), then there exist $u^* \in U$, $v^* \in \mathbb{R}_+^q$, and $\lambda^* \in \mathbb{R}$ such that

$$\begin{gathered}\left\langle \sum_{i=1}^{p} u_i^* [D_k F_i(S^*) - \lambda^* D_k G_i(S^*)] + \sum_{j=1}^{q} v_j^* D_k H_j(S^*), \chi_{S_k} - \chi_{S_k^*} \right\rangle \geq 0 \quad \forall S_k \in \mathbb{A},\ k \in \underline{n}, \\ u_i^* [F_i(S^*) - \lambda^* G_i(S^*)] = 0, \quad i \in \underline{p}, \\ v_j^* H_j(S^*) = 0, \quad j \in \underline{q},\end{gathered} \tag{2.23}$$

where $U = \{u \in \mathbb{R}_+^p : \sum_{i=1}^{p} u_i = 1\}$ and $\mathbb{R}_+^p$ denotes the nonnegative orthant of $\mathbb{R}^p$.

We will also need the following result which provides an alternative expression for the objective function of (1.1).

Lemma 2.18 (see [37]). *For each $S \in \mathbb{A}^n$,*

$$\varphi(S) \equiv \max_{1 \leq i \leq p} \frac{F_i(S)}{G_i(S)} = \max_{u \in U} \frac{\sum_{i=1}^{p} u_i F_i(S)}{\sum_{i=1}^{p} u_i G_i(S)}. \tag{2.24}$$

3. Sufficient optimality conditions

In this section, we formulate several sets of sufficient optimality conditions for (1.1) with a variety of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions. We begin by introducing some notation.

Let the functions $\mathcal{A}_i(\cdot, \lambda^*)$, $\mathcal{A}(\cdot, u^*, \lambda^*)$, and $\mathcal{B}(\cdot, v^*) : \mathbb{A}^n \to \mathbb{R}$ be defined, for fixed λ^*, u^*, and v^*, by

$$\begin{gathered}\mathcal{A}_i(S, \lambda^*) = F_i(S) - \lambda^* G_i(S), \quad i \in \underline{p}, \\ \mathcal{A}(S, u^*, \lambda^*) = \sum_{i=1}^{p} u_i^* [F_i(S) - \lambda^* G_i(S)], \\ \mathcal{B}(S, v^*) = \sum_{j=1}^{q} v_j^* H_j(S).\end{gathered} \tag{3.1}$$

For given $u^* \in U$ and $v^* \in \mathbb{R}_+^q$, let $I_+(u^*) = \{i \in \underline{p} : u_i^* > 0\}$ and $J_+(v^*) = \{j \in \underline{q} : v_j^* > 0\}$.

Theorem 3.1. *Let $S^* \in \mathbb{F}$ with $F_i(S^*) \geq 0$, $i \in \underline{p}$, let $\lambda^* = \varphi(S^*)$, and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$*

such that

$$\mathcal{F}\left(S,S^*;\sum_{i=1}^{p}u_i^*\left[DF_i(S^*)-\lambda^*DG_i(S^*)\right]+\sum_{j=1}^{q}v_j^*DH_j(S^*)\right)\geq 0 \quad \forall S\in\mathbb{F}, \tag{3.2}$$

$$u_i^*\left[F_i(S^*)-\lambda^*G_i(S^*)\right]=0, \quad i\in\underline{p}, \tag{3.3}$$

$$v_j^*H_j(S^*)=0, \quad j\in\underline{q}, \tag{3.4}$$

where $\mathcal{F}(S,S^*;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$ is a sublinear function. Assume, furthermore, that any of the following three sets of hypotheses is satisfied:

(a) (i) *for each $i\in\underline{p}$, F_i is $(\mathcal{F},b,\bar{\phi},\bar{\rho}_i,\theta)$-univex at S^*, and $-G_i$ is $(\mathcal{F},b,\bar{\phi},\hat{\rho}_i,\theta)$-univex at S^*, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;*

(ii) *for each $j\in J_+\equiv J_+(v^*)$, H_j is $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0)=0$;*

(iii) *$\rho^*+\sum_{j\in J_+}v_j^*\tilde{\rho}_j\geq 0$, where $\rho^*=\sum_{i=1}^{p}u_i^*(\bar{\rho}_i+\lambda^*\hat{\rho}_i)$;*

(b) (i) *for each $i\in\underline{p}$, F_i is $(\mathcal{F},b,\bar{\phi},\bar{\rho}_i,\theta)$-univex at S^*, and $-G_i$ is $(\mathcal{F},b,\bar{\phi},\hat{\rho}_i,\theta)$-univex at S^*, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$;*

(ii) *$\mathcal{B}(\cdot,v^*)$ is $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0)=0$;*

(iii) *$\rho^*+\tilde{\rho}\geq 0$;*

(c) (i) *the Lagrangian-type function $L(\cdot,u^*,v^*,\lambda^*):\mathbb{A}^n\to\mathbb{R}$ defined, for fixed u^*, v^*, and λ^*, by*

$$L(S,u^*,v^*,\lambda^*)=\sum_{i=1}^{p}u_i^*\left[F_i(S)-\lambda^*G_i(S)\right]+\sum_{j=1}^{q}v_j^*H_j(S) \tag{3.5}$$

is $(\mathcal{F},b,\bar{\phi},0,\theta)$-pseudounivex at S^*, and $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$.

Then S^* is an optimal solution of (1.1).

Proof. (a) Let S be an arbitrary feasible solution of (1.1). Using the hypotheses specified in (i), we have for each $i\in\underline{p}$,

$$\begin{aligned}\bar{\phi}(F_i(S)-F_i(S^*))&\geq\mathcal{F}(S,S^*;b(S,S^*)DF_i(S^*))+\bar{\rho}_i d^2(\theta(S,S^*)),\\ \bar{\phi}(-G_i(S)+G_i(S^*))&\geq\mathcal{F}(S,S^*;-b(S,S^*)DG_i(S^*))+\hat{\rho}_i d^2(\theta(S,S^*)).\end{aligned} \tag{3.6}$$

Inasmuch as $\lambda^*\geq 0$, $u^*\geq 0$, $\sum_{i=1}^{p}u_i^*=1$, $\bar{\phi}$ is superlinear, and $\mathcal{F}(S,S^*;\cdot)$ is sublinear, we deduce from the above inequalities that

$$\begin{aligned}&\bar{\phi}\left(\sum_{i=1}^{p}u_i^*\left[F_i(S)-\lambda^*G_i(S)\right]-\sum_{i=1}^{p}u_i^*\left[F_i(S^*)-\lambda^*G_i(S^*)\right]\right)\\ &\quad\geq\mathcal{F}\left(S,S^*;b(S,S^*)\sum_{i=1}^{p}u_i^*\left[DF_i(S^*)-\lambda^*DG_i(S^*)\right]\right)+\sum_{i=1}^{p}u_i^*(\bar{\rho}_i+\lambda^*\hat{\rho}_i)d^2(\theta(S,S^*)).\end{aligned} \tag{3.7}$$

Since $S \in \mathbb{F}$, it follows from (3.4) that for each $j \in J_+$, $H_j(S) \le 0 = H_j(S^*)$, and so using the properties of $\widetilde{\phi}_j$, we get for each $j \in J_+$,

$$\widetilde{\phi}_j(H_j(S) - H_j(S^*)) \le 0, \tag{3.8}$$

which in view of (ii) implies that

$$\mathcal{F}(S,S^*;b(S,S^*)DH_j(S^*)) \le -\widetilde{\rho}_j d^2(\theta(S,S^*)). \tag{3.9}$$

Because $v^* \ge 0$, $v_j^* = 0$ for each $j \in \underline{q} \setminus J_+$, and $\mathcal{F}(S,S^*;\cdot)$ is sublinear, the above inequalities yield

$$\mathcal{F}\left(S,S^*;b(S,S^*)\sum_{j=1}^{q} v_j^* DH_j(S^*)\right) \le -\sum_{j\in J_+} v_j^* \widetilde{\rho}_j d^2(\theta(S,S^*)). \tag{3.10}$$

From the positivity of $b(S,S^*)$, sublinearity of $\mathcal{F}(S,S^*;\cdot)$, and (3.2) it is clear that

$$\mathcal{F}\left(S,S^*;b(S,S^*)\sum_{i=1}^{p} u_i^*[DF_i(S^*)-\lambda^* DG_i(S^*)]\right)+\mathcal{F}\left(S,S^*;b(S,S^*)\sum_{j=1}^{q} v_j^* DH_j(S^*)\right)\ge 0. \tag{3.11}$$

Combining (3.7), (3.10), and (3.11) and using (3.3) and (iii), we obtain

$$\bar{\phi}\left(\sum_{i=1}^{p} u_i^*[F_i(S) - \lambda^* G_i(S)]\right) \ge \left(\rho^* + \sum_{j\in J_+} v_j^* \widetilde{\rho}_j\right) d^2(\theta(S,S^*)) \ge 0. \tag{3.12}$$

Since $\bar{\phi}(a) \ge 0 \Rightarrow a \ge 0$, the above inequality reduces to

$$\sum_{i=1}^{p} u_i^*[F_i(S) - \lambda^* G_i(S)] \ge 0. \tag{3.13}$$

Now using Lemma 2.18 and (3.13), we see that

$$\begin{aligned} \varphi(S) &= \max_{1\le i\le p} \frac{F_i(S)}{G_i(S)} = \max_{u\in U} \frac{\sum_{i=1}^{p} u_i F_i(S)}{\sum_{i=1}^{p} u_i G_i(S)} \quad \text{(by Lemma 2.18)} \\ &\ge \frac{\sum_{i=1}^{p} u_i^* F_i(S)}{\sum_{i=1}^{p} u_i^* G_i(S)} \ge \lambda^* \quad \text{(by (3.13))}. \end{aligned} \tag{3.14}$$

Since $\lambda^* = \varphi(S^*)$ and $S \in \mathbb{F}$ was arbitrary, we conclude from the above inequality that S^* is an optimal solution of (1.1).

(b) The proof is similar to that of part (a).

(c) Since $b(S,S^*) > 0$, $\mathcal{F}(S,S^*;\cdot)$ is sublinear, and $L(\cdot,u^*,v^*,\lambda^*)$ is $(\mathcal{F},b,\bar{\phi},0,\theta)$-pseudounivex at S^*, it follows from (3.2) that $\bar{\phi}(L(S,u^*,v^*,\lambda^*) - L(S^*,u^*,v^*,\lambda^*)) \ge 0$. But $\bar{\phi}(a) \ge 0 \Rightarrow a \ge 0$, and so $L(S,u^*,v^*,\lambda^*) \ge L(S^*,u^*,v^*,\lambda^*) = 0$, where the equality follows from (3.3) and (3.4). Because $v^* \ge 0$ and $S \in \mathbb{F}$, this inequality reduces to (3.13) which leads to the desired conclusion that S^* is an optimal solution of (1.1). □

In Theorem 3.1, separate $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity conditions were imposed on the functions F_i and $-G_i$, $i \in \underline{p}$. In the remainder of this section, we will present a number of sufficiency results in which various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity requirements will be placed on certain combinations of these functions.

Theorem 3.2. *Let $S^* \in \mathbb{F}$, let $\lambda^* = \varphi(S^*)$, and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$ and $v^* \in \mathbb{R}^q_+$ such that (3.2), (3.3), and (3.4) hold. Assume, furthermore, that any of the following six sets of hypotheses is satisfied:*

(a) (i) *$\mathcal{A}(\cdot, u^*, \lambda^*)$ is $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-pseudounivex at S^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*
(ii) *for each $j \in J_+ \equiv J_+(v^*)$, H_j is $(\mathcal{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$;*
(iii) $\bar{\rho} + \sum_{j \in J_+} v_j^* \tilde{\rho}_j \geq 0$;

(b) (i) *$\mathcal{A}(\cdot, u^*, \lambda^*)$ is $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-pseudounivex at S^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*
(ii) *$\mathcal{B}(\cdot, v^*)$ is $(\mathcal{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$;*
(iii) $\bar{\rho} + \tilde{\rho} \geq 0$;

(c) (i) *$\mathcal{A}(\cdot, u^*, \lambda^*)$ is prestrictly $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*
(ii) *for each $j \in J_+$, H_j is $(\mathcal{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$;*
(iii) $\bar{\rho} + \sum_{j \in J_+} v_j^* \tilde{\rho}_j > 0$;

(d) (i) *$\mathcal{A}(\cdot, u^*, \lambda^*)$ is prestrictly $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*
(ii) *$\mathcal{B}(\cdot, v^*)$ is $(\mathcal{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$;*
(iii) $\bar{\rho} + \tilde{\rho} > 0$;

(e) (i) *$\mathcal{A}(\cdot, u^*, \lambda^*)$ is prestrictly $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*
(ii) *for each $j \in J_+$, H_j is strictly $(\mathcal{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$-pseudounivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$;*
(iii) $\bar{\rho} + \sum_{j \in J_+} v_j^* \tilde{\rho}_j \geq 0$;

(f) (i) *$\mathcal{A}(\cdot, u^*, \lambda^*)$ is prestrictly $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*
(ii) *$\mathcal{B}(\cdot, v^*)$ is strictly $(\mathcal{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$-pseudounivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$;*
(iii) $\bar{\rho} + \tilde{\rho} \geq 0$.

Then S^* is an optimal solution of (1.1).

Proof. (a) Let S be an arbitrary feasible solution of (1.1). Then, as seen in the proof of Theorem 3.1, our hypotheses in (ii) lead to (3.10), which when combined with (3.11) yields

$$\begin{aligned} &\mathcal{F}\left(S, S^*; b(S, S^*) \sum_{i=1}^{p} u_i^* \left[DF_i(S^*) - \lambda^* DG_i(S^*)\right]\right) \\ &\quad \geq \sum_{j \in J_+} v_j^* \tilde{\rho}_j d^2(\theta(S, S^*)) \geq -\bar{\rho} d^2(\theta(S, S^*)), \end{aligned} \tag{3.15}$$

where the second inequality follows from (iii). By virtue of (i), (3.15) implies that

$$\bar{\phi}(\mathcal{A}(S,u^*,\lambda^*) - \mathcal{A}(S^*,u^*,\lambda^*)) \geq 0, \tag{3.16}$$

which because of the property of the function $\bar{\phi}$, reduces to $\mathcal{A}(S,u^*,\lambda^*) \geq \mathcal{A}(S^*,u^*,\lambda^*)$. But by (3.3), $\mathcal{A}(S^*,u^*,\lambda^*) = 0$, and hence we have that $\mathcal{A}(S,u^*,\lambda^*) \geq 0$, which is precisely (3.13). Therefore, we conclude, as in the proof of Theorem 3.1, that S^* is an optimal solution of (1.1).

(b) The proof is similar to that of part (a).

(c) Proceeding as in the proof of part (a), we obtain the first inequality of (3.15). Thus in view of (iii) we have

$$\mathcal{F}\left(S,S^*;b(S,S^*)\sum_{i=1}^{p} u_i^*\left[DF_i(S^*) - \lambda^* DG_i(S^*)\right]\right) > -\bar{\rho}d^2(\theta(S,S^*)), \tag{3.17}$$

which by virtue of (i) implies that

$$\bar{\phi}(\mathcal{A}(S,u^*,\lambda^*) - \mathcal{A}(S^*,u^*,\lambda^*)) \geq 0. \tag{3.18}$$

But $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$, and hence we get $\mathcal{A}(S,u^*,\lambda^*) \geq \mathcal{A}(S^*,u^*,\lambda^*)$. From (3.3), it is clear that $\mathcal{A}(S^*,u^*,\lambda^*) = 0$ and so we have

$$\mathcal{A}(S,u^*,\lambda^*) = \sum_{i=1}^{p} u_i^*\left[F_i(S^*) - \lambda^* G_i(S^*)\right] \geq 0. \tag{3.19}$$

Now proceeding as in the proof of Theorem 3.1 and using Lemma 2.18 along with this inequality, we find that $\varphi(S) \geq \lambda^* = \varphi(S^*)$, showing that S^* is an optimal solution of (1.1).

(d)–(f) The proofs are similar to that of part (c). □

Theorem 3.3. *Let $S^* \in \mathbb{F}$, let $\lambda^* = \varphi(S^*)$, and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that (3.2), (3.3), and (3.4) hold. Assume, furthermore, that any of the following six sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u^*)$, $\mathcal{A}_i(\cdot,\lambda^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $j \in J_+ \equiv J_+(v^*)$, H_j is $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$;*

(iii) *$\rho^\circ + \sum_{j\in J_+} v_j^* \tilde{\rho}_j \geq 0$, where $\rho^\circ = \sum_{i\in I_+} u_i^* \bar{\rho}_i$;*

(b) (i) *for each $i \in I_+$, $\mathcal{A}_i(\cdot,\lambda^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *$\mathcal{B}(\cdot,v^*)$ is $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$;*

(iii) *$\rho^\circ + \tilde{\rho} \geq 0$;*

(c) (i) *for each $i \in I_+$, $\mathcal{A}_i(\cdot,\lambda^*)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}(0) = 0$;*

(ii) *for each $j \in J_+$, H_j is $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$;*

(iii) *$\rho^\circ + \sum_{j\in J_+} v_j^* \tilde{\rho}_j > 0$;*

(d) (i) *for each* $i \in I_+$, $\mathcal{A}_i(\cdot,\lambda^*)$ *is prestrictly* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$*-quasiunivex at* S^*, $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}(0) = 0$*;*

(ii) $\mathcal{B}(\cdot,\nu^*)$ *is* $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$*-quasiunivex at* S^*, $\tilde{\phi}$ *is increasing, and* $\tilde{\phi}(0) = 0$*;*

(iii) $\rho^\circ + \tilde{\rho} > 0$*;*

(e) (i) *for each* $i \in I_+$, $\mathcal{A}_i(\cdot,\lambda^*)$ *is prestrictly* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$*-quasiunivex at* S^*, $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}(0) = 0$*;*

(ii) *for each* $j \in J_+$, H_j *is strictly* $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$*-pseudounivex at* S^*, $\tilde{\phi}_j$ *is increasing, and* $\tilde{\phi}_j(0) = 0$*;*

(iii) $\rho^\circ + \sum_{j\in J_+} \nu_j^* \tilde{\rho}_j \geq 0$*;*

(f) (i) *for each* $i \in I_+$, $\mathcal{A}_i(\cdot,\lambda^*)$ *is prestrictly* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$*-quasiunivex at* S^*, $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}(0) = 0$*;*

(ii) $\mathcal{B}(\cdot,\nu^*)$ *is strictly* $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$*-pseudounivex at* S^*, $\tilde{\phi}$ *is increasing, and* $\tilde{\phi}(0) = 0$*;*

(iii) $\rho^\circ + \tilde{\rho} \geq 0$*;*

Then S^* *is an optimal solution of (1.1).*

Proof. (a) Suppose to the contrary that S^* is not an optimal solution of (1.1). Then there is a feasible solution $\bar{S}$ of (1.1) such that $\varphi(\bar{S}) < \varphi(S^*) = \lambda^*$, and so for each $i \in \underline{p}$, $F_i(\bar{S}) < \lambda^* G_i(\bar{S})$. From these inequalities and (3.3) it is clear that for each $i \in I_+$,

$$F_i(\bar{S}) - \lambda^* G_i(\bar{S}) < 0 = F_i(S^*) - \lambda^* G_i(S^*), \tag{3.20}$$

which in view of the properties of $\bar{\phi}_i$, can be expressed as

$$\bar{\phi}_i(F_i(\bar{S}) - \lambda^* G_i(\bar{S}) - [F_i(S^*) - \lambda^* G_i(S^*)]) < 0. \tag{3.21}$$

By (i), this implies that for each $i \in I_+$,

$$\mathcal{F}(\bar{S},S^*;b(\bar{S},S^*)[DF_i(S^*) - \lambda^* DG_i(S^*)]) < -\bar{\rho}_i d^2(\theta(\bar{S},S^*)). \tag{3.22}$$

Since $u^* \geq 0$, $u_i^* = 0$ for each $i \in \underline{p} \setminus I_+$, $\sum_{i\in I_+} u_i^* = 1$, and $\mathcal{F}(\bar{S},S^*;\cdot)$ is sublinear, the above inequalities yield

$$\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\sum_{i=1}^{p} u_i^*[DF_i(S^*) - \lambda^* DG_i(S^*)]\right) < -\sum_{i\in I_+} u_i^* \bar{\rho}_i d^2(\theta(\bar{S},S^*)). \tag{3.23}$$

From (3.11), (3.23), and (iii), it is clear that

$$\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\sum_{j=1}^{q} \nu_j^* DH_j(S^*)\right) > \sum_{i\in I_+} u_i^* \bar{\rho}_i d^2(\theta(\bar{S},S^*)) \geq -\sum_{j\in J_+} \nu_j^* \tilde{\rho}_j d^2(\theta(\bar{S},S^*)). \tag{3.24}$$

But this contradicts (3.10) (with S replaced by $\bar{S}$), which is valid for the present case because of the assumptions specified in (ii). Hence we conclude that S^* is an optimal solution of (1.1).

(b) The proof is similar to that of part (a).

(c) Suppose to the contrary that S^* is not an optimal solution of (1.1). Now proceeding as in the proof of part (a), we obtain, for some $\bar{S} \in \mathbb{F}$, the inequality

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*) \sum_{i=1}^{p} u_i^* \left[DF_i(S^*) - \lambda^* DG_i(S^*)\right]\right) \leq -\rho^\circ d^2(\theta(\bar{S}, S^*)), \tag{3.25}$$

which when combined with (3.11) (with S replaced by $\bar{S}$) gives

$$\mathcal{F}\left(\bar{S}, S^*; b(S, S^*) \sum_{j=1}^{q} v_j^* DH_j(S^*)\right) \geq \rho^\circ d^2(\theta(\bar{S}, S^*)) > -\sum_{j \in J_+} v_j^* \tilde{\rho}_j d^2(\theta(\bar{S}, S^*)), \tag{3.26}$$

where the strict inequality follows from (iii). This obviously contradicts (3.10) (with S replaced by $\bar{S}$), which is valid for the present case because of our assumptions specified in (ii), and so we conclude that S^* is an optimal solution of (1.1).

(d)–(f) The proofs are similar to that of part (c). □

In Theorem 3.3, it was required that each $\mathcal{A}_i(\cdot, \lambda^*)$, $i \in I_+$, possess the same generalized $(\mathcal{F}, b, \phi_i, \rho_i, \theta)$-univexity property, namely, $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivexity in parts (a) and (b), and prestrict $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasiunivexity in parts (c)–(f). It is also possible to partition I_+ into disjoint subsets and then assume that different collections of the $\mathcal{A}_i(\cdot, \lambda^*)$'s corresponding to these subsets have different generalized $(\mathcal{F}, b, \phi_i, \rho_i, \theta)$-univexity properties. Of course, similar partitioning can be applied to J_+. This process can generate some additional sets of sufficiency results for (1.1). We next formulate a variant of Theorem 3.3 whose proof is similar to that of Theorem 3.3, and hence omitted.

Theorem 3.4. *Let $S^* \in \mathbb{F}$, let $\lambda^* = \varphi(S^*)$, and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that (3.2), (3.3), and (3.4) hold. Furthermore, assume that any of the following four sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_{1+} \neq \emptyset$, $\mathcal{A}_i(\cdot, \lambda^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, for each $i \in I_{2+}$, $\mathcal{A}_i(\cdot, \lambda^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasiunivex at S^*, and for each $i \in I_+ \equiv I_+(u^*)$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0) = 0$, where $\{I_{1+}, I_{2+}\}$ is a partition of I_+;*

(ii) *for each $j \in J_+ \equiv J_+(v^*)$, H_j is $(\mathcal{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$;*

(iii) *$\rho^\circ + \sum_{j \in J_+} v_j^* \tilde{\rho}_j \geq 0$, where $\rho^\circ = \sum_{i \in I_+} u_i^* \bar{\rho}_i$;*

(b) (i) *for each $i \in I_{1+} \neq \emptyset$, $\mathcal{A}_i(\cdot, \lambda^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, for each $i \in I_{2+}$, $\mathcal{A}_i(\cdot, \lambda^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasiunivex at S^*, and for each $i \in I_+$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0) = 0$, where $\{I_{1+}, I_{2+}\}$ is a partition of I_+;*

(ii) *$\mathcal{B}(\cdot, v^*)$ is $(\mathcal{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$;*

(iii) *$\rho^\circ + \tilde{\rho} \geq 0$;*

(c) (i) *for each $i \in I_+$, $\mathcal{A}_i(\cdot, \lambda^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $j \in J_{1+}$, H_j is $(\mathcal{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$-quasiunivex at S^*, for each $j \in J_{2+}$, H_j is strictly $(\mathcal{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$-pseudounivex at S^*, and for each $j \in J_+$, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$, where $\{J_{1+}, J_{2+}\}$ is a partition of J_+;*

(iii) *$\rho^\circ + \sum_{j \in J_+} v_j^* \tilde{\rho}_j \geq 0$;*

(d) (i) *for each* $i \in I_{1+}$, $\mathcal{A}_i(\cdot,\lambda^*)$ *is* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$*-pseudounivex at* S^*, *for each* $i \in I_{2+}$, $\mathcal{A}_i(\cdot,\lambda^*)$ *is* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$*-quasiunivex at* S^*, *and for each* $i \in I_+$, $\bar{\phi}_i$ *is increasing and* $\bar{\phi}_i(0) = 0$, *where* $\{I_{1+},I_{2+}\}$ *is a partition of* I_+;

(ii) *for each* $j \in J_{1+}$, H_j *is* $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$*-quasiunivex at* S^*, *for each* $j \in J_{2+}$, H_j *is strictly* $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$*-pseudounivex at* S^*, *and for each* $j \in J_+$, $\tilde{\phi}_j$ *is increasing and* $\tilde{\phi}_j(0) = 0$, *where* $\{J_{1+},J_{2+}\}$ *is a partition of* J_+;

(iii) $\rho^\circ + \sum_{j\in J_+} v_j^* \tilde{\rho}_j \geq 0$;

(iv) $I_{1+} \neq \emptyset$, $J_{2+} \neq \emptyset$, *or* $\rho^\circ + \sum_{j\in J_+} v_j^* \tilde{\rho}_j > 0$.

Then S^* *is an optimal solution of (1.1).*

4. Generalized sufficiency criteria

In this section, we formulate and discuss several families of generalized sufficiency results for (1.1) with the help of a partitioning scheme that was originally proposed in [27] for constructing generalized dual problems for nonlinear programs with point functions.

Let $\{J_0,J_1,\ldots,J_m\}$ be a partition of the index set $\underline{q}$; thus $J_r \subset \underline{q}$ for each $r \in \{0,1,\ldots,m\}$, $J_r \cap J_s = \emptyset$ for each $r,s \in \{0,1,\ldots,m\}$ with $r \neq s$, and $\cup_{r=0}^m J_r = \underline{q}$. In addition, we will make use of the functions $\mathbf{A}_i(\cdot,v^*,\lambda^*)$, $\mathbf{A}(\cdot,u^*,v^*,\lambda^*)$, and $\mathbf{B}_t(\cdot,v^*) : \mathbb{A}^n \to \mathbb{R}$ defined, for fixed λ^*, u^*, and v^*, by

$$\begin{aligned}
\mathbf{A}_i(T,v^*,\lambda^*) &= F_i(T) - \lambda^* G_i(T) + \sum_{j\in J_0} v_j^* H_j(T), \quad i \in \underline{p},\\
\mathbf{A}(T,u^*,v^*,\lambda^*) &= \sum_{i=1}^{p} u_i^* \left[F_i(T) - \lambda^* G_i(T)\right] + \sum_{j\in J_0} v_j^* H_j(T),\\
\mathbf{B}_t(T,v^*) &= \sum_{j\in J_t} v_j^* H_j(T), \quad t \in \underline{m}.
\end{aligned} \tag{4.1}$$

Using these sets and functions, we next state and prove a number of generalized sufficiency results for (1.1).

Theorem 4.1. *Let* $S^* \in \mathbb{F}$, *let* $\lambda^* = \varphi(S^*)$, *and assume that* F_i, G_i, $i \in \underline{p}$, *and* H_j, $j \in \underline{q}$, *are differentiable at* S^*, *and that there exist* $u^* \in U$ *and* $v^* \in \mathbb{R}_+^q$ *such that (3.2), (3.3), and (3.4) hold. Assume, furthermore, that any of the following four sets of hypotheses is satisfied:*

(a) (i) $\mathbf{A}(\cdot,u^*,v^*,\lambda^*)$ *is* $(\mathcal{F},b,\bar{\phi},\bar{\rho},\theta)$*-pseudounivex at* S^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *for each* $t \in \underline{m}$, $\mathbf{B}_t(\cdot,v^*)$ *is* $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$*-quasiunivex at* S^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$;

(iii) $\bar{\rho} + \sum_{t=1}^m \tilde{\rho}_t \geq 0$;

(b) (i) $\mathbf{A}(\cdot,u^*,v^*,\lambda^*)$ *is prestrictly* $(\mathcal{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at* S^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *for each* $t \in \underline{m}$, $\mathbf{B}_t(\cdot,v^*)$ *is strictly* $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$*-pseudounivex at* S^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$;

(iii) $\bar{\rho} + \sum_{t=1}^m \tilde{\rho}_t \geq 0$;

(c) (i) $\mathbf{A}(\cdot,u^*,v^*,\lambda^*)$ *is prestrictly* $(\mathcal{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at* S^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *for each* $t \in \underline{m}$, $\mathbf{B}_t(\cdot, v^*)$ *is* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-quasiunivex at* S^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$;

(iii) $\bar{\rho} + \sum_{t=1}^{m} \tilde{\rho}_t > 0$;

(d) (i) $\mathbf{A}(\cdot, u^*, v^*, \lambda^*)$ *is prestrictly* $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$*-quasiunivex at* S^*, *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) *for each* $t \in \underline{m_1}$, $\mathbf{B}_t(\cdot, v^*)$ *is* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-quasiunivex at* S^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$, *for each* $t \in \underline{m_2} \neq \emptyset$, $\mathbf{B}_t(\cdot, v^*)$ *is strictly* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-pseudounivex at* S^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$, *where* $\{\underline{m_1}, \underline{m_2}\}$ *is a partition of* $\underline{m}$;

(iii) $\bar{\rho} + \sum_{t=1}^{m} \tilde{\rho}_t \geq 0$.

Then S^* *is an optimal solution of (1.1).*

Proof. (a) Let S be an arbitrary feasible solution of (1.1). As $v^* \geq 0$, it is clear from (3.4) that for each $t \in \underline{m}$,

$$\mathbf{B}_t(S, v^*) = \sum_{j \in J_t} v_j^* H_j(S) \leq 0 = \sum_{j \in J_t} v_j^* H_j(S^*) = \mathbf{B}_t(S^*, v^*), \tag{4.2}$$

and hence using the properties of $\tilde{\phi}_t$, we get

$$\tilde{\phi}_t(\mathbf{B}_t(S, v^*) - \mathbf{B}_t(S^*, v^*)) \leq 0, \tag{4.3}$$

which by (ii) implies that for each $t \in \underline{m}$,

$$\mathcal{F}\left(S, S^*; b(S, S^*) \sum_{j \in J_t} v_j^* DH_j(S^*)\right) \leq -\tilde{\rho}_t d^2(\theta(S, S^*)). \tag{4.4}$$

Adding these inequalities and using the sublinearity of $\mathcal{F}(S, S^*; \cdot)$, we obtain

$$\mathcal{F}\left(S, S^*; b(S, S^*) \sum_{t=1}^{m} \sum_{j \in J_t} v_j^* DH_j(S^*)\right) \leq -\sum_{t=1}^{m} \tilde{\rho}_t d^2(\theta(S, S^*)). \tag{4.5}$$

From the sublinearity of $\mathcal{F}(S, S^*; \cdot)$ and (3.2) it follows that

$$\begin{aligned} &\mathcal{F}\left(S, S^*; b(S, S^*)\left[\sum_{i=1}^{p} u_i^* [DF_i(S^*) - \lambda^* DG_i(S^*)] + \sum_{j \in J_0} v_j^* DH_j(S^*)\right]\right) \\ &\quad + \mathcal{F}\left(S, S^*; b(S, S^*) \sum_{t=1}^{m} \sum_{j \in J_t} v_j^* DH_j(S^*)\right) \geq 0. \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6) and using (iii), we obtain the inequality

$$\begin{aligned} &\mathcal{F}\left(S, S^*; b(S, S^*)\left[\sum_{i=1}^{p} u_i^* [DF_i(S^*) - \lambda^* DG_i(S^*)] + \sum_{j \in J_0} v_j^* DH_j(S^*)\right]\right) \\ &\quad \geq \sum_{t=1}^{m} \tilde{\rho}_t d^2(\theta(S, S^*)) \geq -\bar{\rho} d^2(\theta(S, S^*)), \end{aligned} \tag{4.7}$$

which in view of (i) implies that

$$\bar{\phi}(\mathbf{A}(S,u^*,v^*,\lambda^*) - \mathbf{A}(S^*,u^*,v^*,\lambda^*)) \geq 0. \tag{4.8}$$

Because of the property of the function $\bar{\phi}$, the above inequality yields

$$\mathbf{A}(S,u^*,v^*,\lambda^*) - \mathbf{A}(S^*,u^*,v^*,\lambda^*) \geq 0. \tag{4.9}$$

But in view of (3.3) and (3.4), $\mathbf{A}(S^*,u^*,v^*,\lambda^*) = 0$, and so we have that $\mathbf{A}(S,u^*,v^*,\lambda^*) \geq 0$. Since $v_j^* H_j(S) \leq 0$ for each $j \in \underline{q}$, this inequality reduces to $\sum_{i=1}^{p} u_i^* [F_i(S^*) - \lambda^* G_i(S^*)] \geq 0$. Now using Lemma 2.18 and this inequality, as in the proof of Theorem 3.1, we obtain the desired conclusion that S^* is an optimal solution of (1.1).

(b)–(d) The proofs are similar to that of part (a). □

Theorem 4.2. *Let $S^* \in \mathbb{F}$, let $\lambda^* = \varphi(S^*)$, and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that (3.2), (3.3), and (3.4) hold. Assume, furthermore, that any of the following six sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u^*)$, $\mathbf{A}_i(\cdot,v^*,\lambda^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\mathbf{B}_t(\cdot,v^*)$ is $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geq 0$;

(b) (i) *for each $i \in I_+$, $\mathbf{A}_i(\cdot,v^*,\lambda^*)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\mathbf{B}_t(\cdot,v^*)$ is strictly $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-pseudounivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geq 0$;

(c) (i) *for each $i \in I_+$, $\mathbf{A}_i(\cdot,v^*,\lambda^*)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\mathbf{B}_t(\cdot,v^*)$ is $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t > 0$;

(d) (i) *for each $i \in I_{1+} \neq \emptyset$, $\mathbf{A}_i(\cdot,v^*,\lambda^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, for each $i \in I_{2+}$, $\mathbf{A}_i(\cdot,v^*,\lambda^*)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, for each $i \in I_+$, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$, where $\{I_{1+},I_{2+}\}$ is a partition of I_+;*

(ii) *for each $t \in \underline{m}$, $\mathbf{B}_t(\cdot,v^*)$ is $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geq 0$;

(e) (i) *for each $i \in I_+$, $\mathbf{A}_i(\cdot,v^*,\lambda^*)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m_1} \neq \emptyset$, $\mathbf{B}_t(\cdot,v^*)$ is strictly $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-pseudounivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$, and for each $t \in \underline{m_2}$, $\mathbf{B}_t(\cdot,v^*)$ is $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$, where $\{\underline{m_1},\underline{m_2}\}$ is a partition of $\underline{m}$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geq 0$;

(f) (i) *for each* $i \in I_{1+}$, $\mathbf{A}_i(\cdot, v^*, \lambda^*)$ *is* $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$*-pseudounivex at* S^*, $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}_i(0) = 0$, *and for each* $i \in I_{2+}$, $\mathbf{A}_i(\cdot, v^*, \lambda^*)$ *is prestrictly* $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$*-quasiunivex at* S^*, $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}_i(0) = 0$, *where* $\{I_{1+}, I_{2+}\}$ *is a partition of* I_+*;*

(ii) *for each* $t \in \underline{m_1}$, $\mathbf{B}_t(\cdot, v^*)$ *is strictly* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-pseudounivex at* S^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$, *and for each* $t \in \underline{m_2}$, $\mathbf{B}_t(\cdot, v^*)$ *is* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-quasiunivex at* S^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$, *where* $\{\underline{m_1}, \underline{m_2}\}$ *is a partition of* $\underline{m}$*;*

(iii) $\sum_{i \in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geq 0$*;*

(iv) $I_{1+} \neq \emptyset$, $m_1 \neq \emptyset$, *or* $\sum_{i \in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t > 0$.

Then S^* *is an optimal solution of (1.1).*

Proof. (a) Suppose to the contrary that S^* is not an optimal solution of (1.1). As seen in the proof of Theorem 3.3, this supposition leads to the inequalities $F_i(\bar{S}) - \lambda^* G_i(\bar{S}) < 0$, $i \in \underline{p}$, for some $\bar{S} \in \mathbb{F}$. Since for each $i \in I_+$,

$$\begin{aligned}
\mathbf{A}_i(\bar{S}, v^*, \lambda^*) &= F_i(\bar{S}) - \lambda^* G_i(\bar{S}) + \sum_{j \in J_0} v_j^* H_j(\bar{S}) \\
&\leq F_i(\bar{S}) - \lambda^* G_i(\bar{S}) \quad (\text{since } v_j^* H_j(\bar{S}) \leq 0 \text{ for each } j \in \underline{q}) \\
&< 0 \\
&= F_i(S^*) - \lambda^* G_i(S^*) \quad (\text{by } (3.3)) \\
&= F_i(S^*) - \lambda^* G_i(S^*) + \sum_{j \in J_0} v_j^* H_j(S^*) \quad (\text{by } (3.4)) \\
&= \mathbf{A}_i(S^*, v^*, \lambda^*),
\end{aligned} \tag{4.10}$$

it follows from the properties of $\bar{\phi}_i$ that for each $i \in I_+$,

$$\bar{\phi}(\mathbf{A}_i(\bar{S}, v^*, \lambda^*) - \mathbf{A}_i(S^*, v^*, \lambda^*)) < 0, \tag{4.11}$$

which in view of (i) implies that

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)\left[DF_i(S^*) - \lambda^* DG_i(S^*) + \sum_{j \in J_0} v_j^* DH_j(S^*)\right]\right) < -\bar{\rho}_i d^2(\theta(\bar{S}, S^*)). \tag{4.12}$$

Inasmuch as $u^* \geq 0$, $u_i^* = 0$ for each $i \in \underline{p} \setminus I_+$, $\sum_{i \in I_+} u_i^* = 1$, and $\mathcal{F}(\bar{S}, S^*; \cdot)$ is sublinear, the above inequalities yield

$$\begin{aligned}
&\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)\left[\sum_{i=1}^{p} u_i^* [DF_i(S^*) - \lambda^* DG_i(S^*)] + \sum_{j \in J_0} v_j^* DH_j(S^*)\right]\right) \\
&\quad < -\sum_{i \in I_+} u_i^* \bar{\rho}_i d^2(\theta(\bar{S}, S^*)).
\end{aligned} \tag{4.13}$$

Now combining this inequality with (4.6) and using (iii), we obtain

$$\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\sum_{t=1}^{m}\sum_{j\in J_t}v_j^*DH_j(S^*)\right)>\sum_{i\in I_+}u_i^*\bar{\rho}_i d^2(\theta(\bar{S},S^*))\geq-\sum_{t=1}^{m}\tilde{\rho}_t d^2(\theta(\bar{S},S^*)), \tag{4.14}$$

contradicting (4.5) (with S replaced by $\bar{S}$), which is valid for the present case because of our hypotheses in (ii). Hence, S^* is an optimal solution of (1.1).

(b)–(f) The proofs are similar to that of part (a). □

Each of the ten sets of conditions specified in Theorems 4.1 and 4.2 can be viewed as a collection of sufficiency results for (1.1). Their special cases can easily be identified by appropriate choices of the partitioning sets J_r, $r=0,1,\dots,m$. We illustrate this possibility by stating explicitly some important special cases of Theorem 4.2(a). They are collected in the following corollary.

Corollary 4.3. *Let $S^*\in\mathbb{F}$, let $\lambda^*=\varphi(S^*)$, and assume that F_i, G_i, $i\in\underline{p}$, and H_j, $j\in\underline{q}$, are differentiable at S^*, and that there exist $u^*\in U$ and $v^*\in\mathbb{R}_+^q$ such that (3.2), (3.3), and (3.4) hold. Assume, furthermore, that any of the following five sets of hypotheses is satisfied:*

(a) *for each $i\in I_+\equiv I_+(u^*)$, the function $T\to F_i(T)-\lambda^*G_i(T)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$; the function $T\to\sum_{j=1}^q v_j^*H_j(T)$ is $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0)=0$; and $\sum_{i\in I_+}u_i^*\bar{\rho}_i+\tilde{\rho}\geq 0$;*

(b) *for each $i\in I_+$, the function $T\to F_i(T)-\lambda^*G_i(T)+\sum_{j=1}^q v_j^*H_j(T)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$; and $\sum_{i\in I_+}u_i^*\bar{\rho}_i\geq 0$;*

(c) *for each $i\in I_+$, $T\to F_i(T)-\lambda^*G_i(T)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$; for each $j\in\underline{q}$, $T\to v_j^*H_j(T)$ is $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$−quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0)=0$; and $\sum_{i\in I_+}u_i^*\bar{\rho}_i+\sum_{j=1}^q\tilde{\rho}_j\geq 0$;*

(d) *for each $i\in I_+$, $T\to F_i(T)-\lambda^*G_i(T)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$; for each $t\in\underline{m}$, $T\to\sum_{j\in J_t}v_j^*H_j(T)$ is $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0)=0$; and $\sum_{i\in I_+}u_i^*\bar{\rho}_i+\sum_{t=1}^m\tilde{\rho}_t\geq 0$;*

(e) *for each $i\in I_+$, $T\to F_i(T)-\lambda^*G_i(T)+\sum_{j\in J_0}v_j^*H_j(T)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$; $T\to\sum_{j\in J_1}v_j^*H_j(T)$ is $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0)=0$; and $\sum_{i\in I_+}u_i^*\bar{\rho}_i+\tilde{\rho}\geq 0$.*

Then S^* is an optimal solution of (1.1).

Proof. In Theorem 4.2(a), let (a) $J_1=\underline{q}$, (b) $J_0=\underline{q}$, (c) $m=q$ and $J_t=\{t\}$, $t\in\underline{q}$, (d) $J_0=\varnothing$, and (e) $J_t=\varnothing$ for $t=2,3,\dots,m$. □

Comparing parts (a) and (c) of the above corollary, we see that they represent two extreme cases with regard to the $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivexity assumptions in the sense that in (a) all the functions $T\to v_j^*H_j(T)$ are lumped together, whereas in (c) separate $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivexity conditions are imposed on the individual functions. It is also possible to devise sufficiency conditions that lie between these two extremes. For example,

one may consider the following variant of (a):

(a) for each $i \in I_+$, $T \to F_i(T) - \lambda^* G_i(T)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$, $T \to \sum_{j\in J_1}^{q} v_j^* H_j(T)$ is $(\mathcal{F}, b, \widetilde{\phi}, \widetilde{\rho}, \theta)$-quasiunivex at S^*, $\widetilde{\phi}$ is increasing, and $\widetilde{\phi}(0) = 0$; for each $j \in J_2$, $T \to v_j^* H_j(T)$ is $(\mathcal{F}, b, \widetilde{\phi}_j, \widetilde{\rho}_j, \theta)$-quasiunivex at S^*, $\widetilde{\phi}_j$ is increasing, and $\widetilde{\phi}_j(0) = 0$; and $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \widetilde{\rho} + \sum_{j\in J_2} \widetilde{\rho}_j \geq 0$, where $\{J_1, J_2\}$ is a partition of $\underline{q}$.

In a similar manner, one can determine numerous special cases and variants of the other nine sets of sufficient optimality conditions given in Theorems 4.1 and 4.2.

In the remainder of this section, we present several sets of sufficiency results for (1.1) that are different from those stated in Theorems 4.1 and 4.2. These results involve generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions placed on different combinations of the functions $T \to v_j^* H_j(T)$ and $T \to F_i(T) - \lambda^* G_i(T)$ arising from a partition of the index set $\underline{p}$. Results of this type have not appeared before for any kind of optimization problems with generalized fractional objective functions.

Let $\{I_0, I_1, \ldots, I_k\}$ be a partition of $\underline{p}$ such that $K \equiv \{0, 1, \ldots, k\} \subset M \equiv \{0, 1, \ldots, m\}$, and let the function $\mathbf{C}_t(\cdot, u^*, v^*, \lambda^*) : \mathbb{A}^n \to \mathbb{R}$ be defined, for fixed u^*, v^*, and λ^*, by

$$\mathbf{C}_t(T, u^*, v^*, \lambda^*) = \sum_{i\in I_t} u_i^* [F_i(T) - \lambda^* G_i(T)] + \sum_{j\in J_t} v_j^* H_j(T), \quad t \in K. \tag{4.15}$$

Theorem 4.4. *Let $S^* \in \mathbb{F}$, let $\lambda^* = \varphi(S^*)$, and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$, $u^* > 0$, and $v^* \in \mathbb{R}_+^{\underline{q}}$ such that (3.2), (3.3), and (3.4) hold. Assume, furthermore, that any of the following six sets of hypotheses is satisfied:*

(a) (i) *for each $t \in K$, $\mathbf{C}_t(\cdot, u^*, v^*, \lambda^*)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-pseudounivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*

(ii) *for each $t \in M\backslash K$, $\mathbf{B}_t(\cdot, v^*)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*

(iii) $\sum_{t\in M} \rho_t \geq 0$;

(b) (i) *for each $t \in K$, $\mathbf{C}_t(\cdot, u^*, v^*, \lambda^*)$ is prestrictly $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*

(ii) *for each $t \in M\backslash K$, $\mathbf{B}_t(\cdot, v^*)$ is strictly $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-pseudounivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*

(iii) $\sum_{t\in M} \rho_t \geq 0$;

(c) (i) *for each $t \in K$, $\mathbf{C}_t(\cdot, u^*, v^*, \lambda^*)$ is prestrictly $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*

(ii) *for each $t \in M\backslash K$, $\mathbf{B}_t(\cdot, v^*)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*

(iii) $\sum_{t\in M} \rho_t > 0$;

(d) (i) *for each $t \in K_1 \neq \emptyset$, $\mathbf{C}_t(\cdot, u^*, v^*, \lambda^*)$ is $(\mathcal{F}, b, \bar{\phi}_t, \rho_t, \theta)$-pseudounivex at S^*, for each $t \in K_2$, $\mathbf{C}_t(\cdot, u^*, v^*, \lambda^*)$ is prestrictly $(\mathcal{F}, b, \bar{\phi}_t, \rho_t, \theta)$-quasiunivex at S^*, for each $t \in K$, $\bar{\phi}_t$ is increasing, and $\bar{\phi}_t(0) = 0$, where $\{K_1, K_2\}$ is a partition of K;*

(ii) *for each $t \in M\backslash K$, $\mathbf{B}_t(\cdot, v^*)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*

(iii) $\sum_{t\in M} \rho_t \geq 0$;

(e) (i) *for each* $t \in K$, $\mathbf{C}_t(\cdot, u^*, v^*, \lambda^*)$ *is prestrictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* S^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$*;*

(ii) *for each* $t \in (M \setminus K)_1 \neq \emptyset$, $\mathbf{B}_t(\cdot, v^*)$ *is strictly* $(\mathcal{F}, b, \widetilde{\phi}_t, \rho_t, \theta)$*-pseudounivex at* S^*, $\widetilde{\phi}_t$ *is increasing, and* $\widetilde{\phi}_t(0) = 0$*, and for each* $t \in (M \setminus K)_2$, $\mathbf{B}_t(\cdot, v^*)$ *is* $(\mathcal{F}, b, \widetilde{\phi}_t, \rho_t, \theta)$*-quasiunivex at* S^*, $\widetilde{\phi}_t$ *is increasing, and* $\widetilde{\phi}_t(0) = 0$*, where* $\{(M \setminus K)_1, (M \setminus K)_2\}$ *is a partition of* $M \setminus K$*;*

(iii) $\sum_{t \in M} \rho_t \geq 0$*;*

(f) (i) *for each* $t \in K_1$, $\mathbf{C}_t(\cdot, u^*, v^*, \lambda^*)$ *is* $(\mathcal{F}, b, \bar{\phi}_t, \bar{\rho}_t, \theta)$*-pseudounivex at* S^*, $\bar{\phi}_t$ *is increasing, and* $\bar{\phi}_t(0) = 0$*, and for each* $t \in K_2$, $\mathbf{C}_t(\cdot, u^*, v^*, \lambda^*)$ *is prestrictly* $(\mathcal{F}, b, \bar{\phi}_t, \bar{\rho}_t, \theta)$*-quasiunivex at* S^*, $\bar{\phi}_t$ *is increasing, and* $\bar{\phi}_t(0) = 0$*, where* $\{K_1, K_2\}$ *is a partition of* K*;*

(ii) *for each* $t \in (M \setminus K)_1$, $\mathbf{B}_t(\cdot, v^*)$ *is strictly* $(\mathcal{F}, b, \widetilde{\phi}_t, \widetilde{\rho}_t, \theta)$*-pseudounivex at* S^*, $\widetilde{\phi}_t$ *is increasing, and* $\widetilde{\phi}_t(0) = 0$*, and for each* $t \in (M \setminus K)_2$, $\mathbf{B}_t(\cdot, v^*)$ *is* $(\mathcal{F}, b, \widetilde{\phi}_t, \widetilde{\rho}_t, \theta)$*-quasiunivex at* S^*, $\widetilde{\phi}_t$ *is increasing, and* $\widetilde{\phi}_t(0) = 0$*, where* $\{(M \setminus K)_1, (M \setminus K)_2\}$ *is a partition of* $M \setminus K$*;*

(iii) $\sum_{t \in M} \rho_t \geq 0$*;*

(iv) $K_1 \neq \emptyset$, $(M \setminus K)_1 \neq \emptyset$, *or* $\sum_{t \in M} \rho_t > 0$.

Then S^* *is an optimal solution of (1.1).*

Proof. (a) Suppose to the contrary that S^* is not an optimal solution of (1.1). As seen in the proof of Theorem 3.3, this supposition leads to the inequalities $F_i(\bar{S}) - \lambda^* G_i(\bar{S}) < 0$, $i \in \underline{p}$, for some $\bar{S} \in \mathbb{F}$. Since $u^* > 0$, these inequalities yield

$$\sum_{i \in I_t} u_i^* \left[F_i(\bar{S}) - \lambda^* G_i(\bar{S}) \right] < 0, \quad t \in K. \tag{4.16}$$

Inasmuch as $v_j^* H_j(\bar{S}) \leq 0$ for each $j \in \underline{q}$, and $\bar{S}, S^* \in \mathbb{F}$, it follows from (3.3), (3.4), and (4.16) that for each $t \in K$,

$$\begin{aligned} \mathbf{C}_t(\bar{S}, u^*, v^*, \lambda^*) &= \sum_{i \in I_t} u_i^* \left[F_i(\bar{S}) - \lambda^* G_i(\bar{S}) \right] + \sum_{j \in J_t} v_j^* H_j(\bar{S}) \\ &< \sum_{i \in I_t} u_i^* \left[F_i(S^*) - \lambda^* G_i(S^*) \right] + \sum_{j \in J_t} v_j^* H_j(S^*) \\ &= \mathbf{C}_t(S^*, u^*, v^*, \lambda^*), \end{aligned} \tag{4.17}$$

and so using the properties of ϕ_t, $t \in K$, we have that for each $t \in K$,

$$\phi_t\left(\mathbf{C}_t(\bar{S}, u^*, v^*, \lambda^*) - \mathbf{C}_t(S^*, u^*, v^*, \lambda^*)\right) < 0, \tag{4.18}$$

which in view of (i) implies that for each $t \in K$,

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*) \left[DF_i(S^*) - \lambda^* DG_i(S^*) \right] + \sum_{j \in J_t} v_j^* DH_j(S^*)\right) < -\rho_t d^2(\theta(\bar{S}, S^*)). \tag{4.19}$$

Adding these inequalities and using the sublinearity of $\mathcal{F}(\bar{S},S^*;\cdot)$, we obtain

$$\begin{aligned}&\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\sum_{i=1}^{p}u_i^*\left[DF_i(S^*)-\lambda^*DG_i(S^*)\right]+\sum_{t\in K}\sum_{j\in J_t}v_j^*DH_j(S^*)\right)\\&\quad<-\sum_{t\in K}\rho_t d^2(\theta(\bar{S},S^*)).\end{aligned}\tag{4.20}$$

Since for each $t\in M\backslash K$,

$$\sum_{j\in J_t}v_j^*H_j(\bar{S})\le 0=\sum_{j\in J_t}v_j^*H_j(S^*),\tag{4.21}$$

it follows from the properties of ϕ_t that

$$\phi_t\left(\sum_{j\in J_t}v_j^*H_j(\bar{S})-\sum_{j\in J_t}v_j^*H_j(S^*)\right)\le 0,\tag{4.22}$$

which by (ii) implies that

$$\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\sum_{j\in J_t}v_j^*DH_j(S^*)\right)\le-\rho_t d^2(\theta(\bar{S},S^*)).\tag{4.23}$$

Adding these inequalities and using the sublinearity of $\mathcal{F}(\bar{S},S^*;\cdot)$, we obtain

$$\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\sum_{t\in M\backslash K}\sum_{j\in J_t}v_j^*DH_j(S^*)\right)\le-\sum_{t\in M\backslash K}\rho_t d^2(\theta(\bar{S},S^*)).\tag{4.24}$$

Now combining (4.20) and (4.24) and using the sublinearity of $\mathcal{F}(\bar{S},S^*;\cdot)$ and (iii), we get

$$\begin{aligned}&\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\sum_{i=1}^{p}u_i^*\left[DF_i(S^*)-\lambda^*DG_i(S^*)\right]+\sum_{j=1}^{q}v_j^*DH_j(S^*)\right)\\&\quad<-\sum_{t\in M}\rho_t d^2(\theta(\bar{S},S^*))\le 0,\end{aligned}\tag{4.25}$$

which contradicts (3.2). Therefore, S^* is an optimal solution of (1.1).

(b)–(f) The proofs are similar to that of part (a). □

Following the approach employed in generating Corollary 4.3, we can easily identify numerous special cases of the six sets of sufficient optimality conditions given in Theorem 4.4.

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