FIXED POINT THEORY FOR MÖNCH-TYPE MAPS DEFINED ON CLOSED SUBSETS OF FRÉCHET SPACES: THE PROJECTIVE LIMIT APPROACH

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New Leray-Schauder alternatives are presented for Mönch-type maps defined between Fréchet spaces. The proof relies on viewing a Fréchet space as the projective limit of a sequence of Banach spaces.

1. Introduction

This paper presents new Leray-Schauder alternatives for Mönch-type maps defined between Fréchet spaces. Two approaches [1, 2, 3, 6, 7] have recently been presented in the literature both of which are based on the fact that a Fréchet space can be viewed as a projective limit of a sequence of Banach spaces $\{E_n\}_{n \in \mathbb{N}}$ (here $\mathbb{N} = \{1, 2, \dots\}$). Both approaches are based on constructing maps F_n defined on subsets of E_n whose fixed points converge to a fixed point of the original operator F . Both approaches have advantages and disadvantages over the other [1] and in this paper, we combine the advantages of both approaches to present very general fixed point results. Our theory in particular extends and improves the theory in [3] (in [3], the single-valued case was discussed).

Finally in this section, we gather together some definitions and a fixed point result which will be needed in Section 2.

Now, let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I, \beta \in I$ for which $\alpha \leq \beta$, let $\pi_{\alpha, \beta} : E_\beta \rightarrow E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : x_\alpha = \pi_{\alpha, \beta}(x_\beta) \quad \forall \alpha, \beta \in I, \alpha \leq \beta \right\} \quad (1.1)$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha, \beta}\}$ or the generalized intersection [4, page 439] $\bigcap_{\alpha \in I} E_\alpha$).

Next, we recall a fixed point result from the literature [9] which we will use in Section 2.

THEOREM 1.1. *Let K be a closed convex subset of a Banach space X , U a relatively open subset of K , $x_0 \in U$, and suppose that $F : \overline{U} \rightarrow CK(K)$ is an upper semicontinuous map (here $CK(K)$ denotes the family of nonempty convex compact subsets of K). Also assume*

that the following conditions hold:

$$M \subseteq \overline{U}, \quad M \subseteq \text{co}(\{x_0\} \cup F(M)) \quad \text{with } \overline{M} = \overline{C},$$

$$C \subseteq M \quad \text{countable, implies } \overline{M} \text{ is compact,} \quad (1.2)$$

$$x \notin (1 - \lambda)\{x_0\} + \lambda Fx \quad \text{for } x \in \overline{U} \setminus U, \lambda \in (0, 1). \quad (1.3)$$

Then there exist a compact set Σ of $\overline{U}$ and an $x \in \Sigma$ with $x \in Fx$.

Remark 1.2. In [9], we see that we could take Σ to be

$$\{y \in \overline{U} : y \in (1 - \lambda)\{x_0\} + \lambda Fy \text{ for some } \lambda \in [0, 1]\}. \quad (1.4)$$

We did not show that Σ is compact in [9] but this is easy to see as we will now show. First, notice that Σ is closed since F is upper semicontinuous. Now let $\{y_n\}_1^\infty$ be a sequence in Σ . Then there exists $\{t_n\}_1^\infty$ in $[0, 1]$ with $y_n \in (1 - t_n)\{x_0\} + t_n Fy_n$ for $n \in \mathbb{N} = \{1, 2, \dots\}$. Without loss of generality, assume that $t_n \rightarrow t \in [0, 1]$. Let $C = \{y_n\}_1^\infty$. Notice that C is countable and $C \subseteq \text{co}(\{x_0\} \cup F(C))$. Now (1.2) with $M = C$ guarantees that $\overline{C}$ is compact (so sequentially compact). Thus there exist a subsequence N_1 of $\mathbb{N}$ and a $y \in \overline{C}$ with $y_n \rightarrow y$ as $n \rightarrow \infty$ in N_1 . This together with $y_n \in (1 - t_n)\{x_0\} + t_n Fy_n$ and the upper semicontinuity of F guarantees that $y \in (1 - t)\{x_0\} + t Fy$, so $y \in \overline{\Sigma} = \Sigma$. Consequently, Σ is sequentially compact and hence compact. In fact, one could also of course take Σ to be

$$\{y \in \overline{U} : y \in Fy\} \quad (1.5)$$

for the compact set in Theorem 1.1.

2. Projective limit approach

Let $E = (E, \{\|\cdot\|_n\}_{n \in \mathbb{N}})$ be a Fréchet space with the topology generated by a family of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$. We assume that the family of seminorms satisfies

$$\|x\|_1 \leq \|x\|_2 \leq \|x\|_3 \leq \dots \quad \text{for every } x \in E. \quad (2.1)$$

To E , we associate a sequence of Banach spaces $\{(E_n, \|\cdot\|_n)\}$ described as follows. For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff } \|x - y\|_n = 0. \quad (2.2)$$

We denote by $E^n = (E/\sim_n, \|\cdot\|_n)$ the quotient space, and by $(E_n, \|\cdot\|_n)$ the completion of E^n with respect to $\|\cdot\|_n$ (the norm on E^n induced by $\|\cdot\|_n$ and its extension to E_n are still denoted by $\|\cdot\|_n$). This construction defines a continuous map $\mu_n : E \rightarrow E_n$. Now since (2.1) is satisfied, the seminorm $\|\cdot\|_n$ induces a seminorm on E_m for every $m \geq n$ (again this seminorm is denoted by $\|\cdot\|_n$). Also (2.2) defines an equivalence relation on E_m from which we obtain a continuous map $\mu_{n,m} : E_m \rightarrow E_n$ since $E_m/\sim_n$ can be regarded as

a subset of E_n . We now assume that the following condition holds:

$$\begin{aligned} &\text{for each } n \in \mathbb{N}, \text{ there exist a Banach space } (E_n, |\cdot|_n) \\ &\text{and an isomorphism (between normed spaces) } j_n : E_n \longrightarrow E_n. \end{aligned} \quad (2.3)$$

Remark 2.1. (i) For convenience, the norm on E_n is denoted by $|\cdot|_n$.

(ii) In our applications, $E_n = \mathbb{E}^n$ for each $n \in \mathbb{N}$.

(iii) Note that if $x \in E_n$ (or $\mathbb{E}^n$), then $x \in E$. However if $x \in E_n$, then x is not necessarily in E and in fact E_n is easier to use in applications as we will see in Theorem 2.3 (even though E_n is isomorphic to $\mathbb{E}^n$).

Finally, we assume that

$$E_1 \supseteq E_2 \supseteq \cdots \quad \text{and for each } n \in \mathbb{N}, \quad |x|_n \leq |x|_{n+1} \quad \forall x \in E_{n+1}. \quad (2.4)$$

Let $\lim_- E_n$ (or $\bigcap_1^\infty E_n$, where $\bigcap_1^\infty$ is the generalized intersection [4]) denote the projective limit of $\{E_n\}_{n \in \mathbb{N}}$ (note that $\pi_{n,m} = j_n \mu_{n,m} j_m^{-1} : E_m \rightarrow E_n$ for $m \geq n$) and note that $\lim_- E_n \cong E$, so for convenience we write $E = \lim_- E_n$.

For each $X \subseteq E$ and each $n \in \mathbb{N}$, we set $X_n = j_n \mu_n(X)$ and we let $\overline{X_n}$ and ∂X_n denote, respectively, the closure and the boundary of X_n with respect to $|\cdot|_n$ in E_n . Also the pseudointerior of X is defined by [2]

$$\text{pseudo-int}(X) = \{x \in X : j_n \mu_n(x) \in \overline{X_n} \setminus \partial X_n \text{ for every } n \in \mathbb{N}\}. \quad (2.5)$$

Our main result in this paper is the extension of Theorem 1.1 to an applicable result in the Fréchet space setting (we refer the reader to [1]; in applications, usually the set U is bounded and as a result has *empty* interior in the nonnormable situation).

THEOREM 2.2. *Let E and E_n be as described above and let $F : X \rightarrow 2^E$, where $X \subseteq E$ (here 2^E denotes the family of nonempty subsets of E). Suppose that the following conditions are satisfied:*

$$x_0 \in \text{pseudo-int}(X), \quad (2.6)$$

$$\text{for each } n \in \mathbb{N}, \quad F : \overline{X_n} \longrightarrow CK(E_n) \text{ is an upper semicontinuous map,} \quad (2.7)$$

$$\text{for each } n \in \mathbb{N}, \quad M \subseteq \overline{X_n} \text{ with } M \subseteq \text{co}(\{j_n \mu_n(x_0)\} \cup F(M)), \quad (2.8)$$

$$\text{with } \overline{M} = \overline{C} \text{ and } C \subseteq M \text{ countable, implies that } \overline{M} \text{ is compact}$$

$$\text{for each } n \in \mathbb{N}, \quad y \notin (1-\lambda)j_n \mu_n(x_0) + \lambda Fy \text{ in } E_n \quad \forall \lambda \in (0,1), y \in \partial X_n, \quad (2.9)$$

$$\text{for each } n \in \{2,3,\dots\} \text{ if } y \in \overline{X_n} \text{ solves } y \in Fy \text{ in } E_n, \text{ then } y \in \overline{X_k}, \quad (2.10)$$

$$\text{for } k \in \{1,\dots,n-1\}.$$

Then F has a fixed point in X .

Proof. Fix $n \in \mathbb{N}$. Let $\sum_n = \{x \in \overline{X_n} : x \in Fx \text{ in } E_n\}$. Now Theorem 1.1 (note that (2.6) implies that $j_n \mu_n(x_0) \in \overline{X_n} \setminus \partial X_n$) guarantees that there exists $y_n \in \sum_n$ with $y_n \in Fy_n$. We look at $\{y_n\}_{n \in \mathbb{N}}$. Now $y_1 \in \sum_1$. Also $y_k \in \sum_1$ for $k \in \mathbb{N} \setminus \{1\}$ since $y_k \in \overline{X_1}$ from (2.10) (see also (2.4)). As a result, $y_n \in \sum_1$ for $n \in \mathbb{N}$ and since $\sum_1$ is compact (see Remark 1.2), there exist a subsequence N_1^* of $\mathbb{N}$ and a $z_1 \in \sum_1$ with $y_n \rightarrow z_1$ in E_1 as $n \rightarrow \infty$ in N_1^* . Let $N_1 = N_1^* \setminus \{1\}$. Now $y_n \in \sum_2$ for $n \in N_1$ so there exist a subsequence N_2^* of N_1 and a $z_2 \in \sum_2$ with $y_n \rightarrow z_2$ in E_2 as $n \rightarrow \infty$ in N_2^* . Note from (2.4) that $z_2 = z_1$ in E_1 since $N_2^* \subseteq N_1$. Let $N_2 = N_2^* \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^* \supseteq N_2^* \supseteq \cdots, \quad N_k^* \subseteq \{k, k+1, \dots\} \quad (2.11)$$

and $z_k \in \sum_k$ with $y_n \rightarrow z_k$ in E_k as $n \rightarrow \infty$ in N_k^* . Note that $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$. Also let $N_k = N_k^* \setminus \{k\}$.

Fix $k \in \mathbb{N}$. Let $y = z_k$ in E_k . Notice that y is well defined and $y \in \lim_{-} E_n = E$. Now $y_n \in Fy_n$ in E_n for $n \in N_k$ and $y_n \rightarrow y$ in E_k as $n \rightarrow \infty$ in N_k (since $y = z_k$ in E_k) together with the fact that $F : \overline{X_k} \rightarrow CK(E_k)$ is upper semicontinuous (note that $y_n \in \sum_k$ for $n \in N_k$) imply that $y \in Fy$ in E_k . We can do this for each $k \in \mathbb{N}$ so as a result, we have $y \in Fy$ in E . $\square$

Next, we present an application of Theorem 2.2. We discuss the differential equation

$$\begin{aligned} y'(t) &= f(t, y(t)) \quad \text{a.e. } t \in [0, T], \\ y(0) &= y_0 \in \mathbb{R}, \end{aligned} \quad (2.12)$$

where $0 < T \leq \infty$ is fixed. First we introduce some notation. If $u \in C[0, T)$, then for every $n \in \mathbb{N}$, we define the seminorms $\rho_n(u)$ by

$$\rho_n(u) = \sup_{t \in [0, t_n]} |u(t)|, \quad (2.13)$$

where $t_n \uparrow T$. Note that $C[0, T)$ is a locally convex linear topological space. The topology on $C[0, T)$, induced by the seminorms $\{\rho_n\}_{n \in \mathbb{N}}$, is the topology of uniform convergence on every compact interval of $[0, T)$.

Recall that a function $g : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function if

- (a) the map $t \mapsto g(t, y)$ is measurable for all $y \in \mathbb{R}$,
- (b) the map $y \mapsto g(t, y)$ is continuous for a.e. $t \in [a, b]$.

Now, $g : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be an L^p -Carathéodory function ($1 \leq p \leq \infty$) if g is a Carathéodory function and

- (c) for any $r > 0$, there exists $\mu_r \in L^p[a, b]$ such that $|y| \leq r$ implies that $|g(t, y)| \leq \mu_r(t)$ for a.e. $t \in [a, b]$.

Finally, a function $g : [0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^p_{loc} -Carathéodory function if (a), (b), and (c) above hold when g is restricted to $[0, t_n] \times \mathbb{R}$ for any $n \in \mathbb{N}$.

THEOREM 2.3. Suppose that the following conditions are satisfied:

$$\text{for each } n \in \mathbb{N}, \quad f : [0, t_n] \times \mathbb{R} \longrightarrow \mathbb{R} \text{ is a Carathéodory function,} \quad (2.14)$$

$$\begin{aligned} &\text{there exists an } L^1_{\text{loc}}[0, T) - \text{Carathéodory function } g : [0, T) \times [0, \infty) \longrightarrow [0, \infty) \\ &\text{such that } |f(t, x)| \leq g(t, |x|) \quad \text{for a.e. } t \in [0, T) \text{ and all } x \in \mathbb{R}, \end{aligned} \quad (2.15)$$

for each $n \in \mathbb{N}$, the problem

$$\begin{aligned} v'(t) &= g(t, v(t)), \quad \text{a.e. } t \in [0, t_n], \\ v(0) &= |y_0| \end{aligned} \quad (2.16)$$

has a maximal solution $r_n(t)$ on $[0, t_n]$ (here $r_n \in C[0, t_n]$).

Then (2.12) has at least one solution $y \in C[0, T)$.

Remark 2.4. One could also obtain a multivalued version of Theorem 2.3 (with (2.12) replaced by a differential inclusion) by using the ideas in the proof below with the ideas in [6].

Proof. Here $E = C[0, T)$, $\mathbf{E}^k$ consists of the class of functions in E which coincide on the interval $[0, t_k]$, $E_k = C[0, t_k]$ with of course $\pi_{n,m} = j_n \mu_{n,m} j_m^{-1} : E_m \rightarrow E_n$ for $m \geq n$ defined by $\pi_{n,m}(x) = x|_{[0, t_n]}$. We will apply Theorem 2.2 with

$$X = \{u \in C[0, T) : |u|_n \leq w_n \text{ for each } n \in \mathbb{N}\}; \quad (2.17)$$

here $|u|_n = \sup_{t \in I_n} |u(t)|$, where $I_n = [0, t_n]$ and $w_n = \sup_{t \in I_n} r_n(t) + 1$. On any interval $I_n = [0, t_n]$ ($n \in \mathbb{N}$), we let F on $C(I_n)$ be defined by

$$Fy(t) = y_0 + \int_0^t f(s, y(s)) ds. \quad (2.18)$$

Fix $n \in \mathbb{N}$. Notice that

$$\overline{X_n} = \{u \in C[0, t_n] : |u|_n \leq w_n\}. \quad (2.19)$$

Clearly, (2.6) holds with $x_0 = 0$ and a standard argument from the literature [8] guarantees that

$$F : \overline{X_n} \longrightarrow E_n \text{ is continuous and compact,} \quad (2.20)$$

so (2.7) and (2.8) hold.

To show that (2.9), fix $n \in \mathbb{N}$ and let $y \in C(I_n)$ be such that $y = \lambda Fy$ for $\lambda \in (0, 1)$. We claim $|y|_n < w_n$ and if this is true, then $y \notin \partial X_n$ and hence (2.9) is true. Let $t \in I_n$ and we now show that $|y(t)| < w_n$. If $|y(t)| \leq |y_0|$, we are finished so it remains to discuss the

case when $|y(t)| > |y_0|$. In this case, there exists $a \in [0, t)$ with

$$|y(s)| > |y_0| \quad \text{for } s \in (a, t], \quad |y(a)| = |y_0|. \quad (2.21)$$

Also

$$|y(s)|' \leq |y'(s)| \leq g(s, |y(s)|) \quad \text{a.e. on } (a, t), \quad (2.22)$$

so

$$\begin{aligned} |y(s)|' &\leq g(s, |y(s)|), \quad \text{a.e. on } (a, t), \\ |y(a)| &= |y_0|. \end{aligned} \quad (2.23)$$

Now a standard comparison theorem for ordinary differential equations in the real case [5, Theorem 1.10.2] guarantees that $|y(s)| \leq r_n(s)$ for $s \in [a, t]$, so in particular $|y(t)| \leq r_n(t) < w_n$, so (2.9) is true.

It remains to show that (2.10). To see this, fix $n \in \{2, 3, \dots\}$ and suppose that $y \in \overline{X_n}$ solves

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad \text{a.e. on } [0, t_n], \\ y(0) &= y_0. \end{aligned} \quad (2.24)$$

Next, fix $k \in \{1, \dots, n-1\}$. We must show that $y \in \overline{X_k}$. Now since $t_n \uparrow T$, notice that $[0, t_k] \subseteq [0, t_n]$ so as a result,

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad \text{a.e. on } [0, t_k], \\ y(0) &= y_0. \end{aligned} \quad (2.25)$$

Let $t \in [0, t_k]$ and essentially the same argument as above guarantees that $|y(t)| < w_k$ so $|y|_k < w_k$. Thus $y \in \overline{X_k}$ and (2.10) holds.

The result now follows immediately from Theorem 2.2. $\square$

Our final result was motivated by Urysohn-type operators.

THEOREM 2.5. *Let E and E_n be as described in the beginning of Section 2 and let $F : X \rightarrow 2^E$, where $X \subseteq E$. Suppose that the following conditions are satisfied:*

$$x_0 \in \text{pseudo-int}(X), \quad (2.26)$$

$$\overline{X_1} \supseteq \overline{X_2} \supseteq \dots, \quad (2.27)$$

$$\text{for each } n \in \mathbb{N}, \quad F_n : \overline{X_n} \rightarrow CK(E_n) \text{ is upper semicontinuous,} \quad (2.28)$$

$$\text{for each } n \in \mathbb{N}, \quad M \subseteq \overline{X_n} \quad \text{with } M \subseteq \text{co}(\{j_n \mu_n(x_0)\} \cup F_n(M)) \quad (2.29)$$

$$\text{with } \overline{M} = \overline{C} \text{ and } C \subseteq M \text{ countable, implies that } \overline{M} \text{ is compact,}$$

$$\text{for each } n \in \mathbb{N}, \quad y \notin (1-\lambda)j_n \mu_n(x_0) + \lambda F_n y \text{ in } E_n \quad \forall \lambda \in (0, 1), \quad y \in \partial X_n, \quad (2.30)$$

for each $n \in \mathbb{N}$, the map $\mathcal{H}_n : \overline{X_n} \rightarrow 2^{E_n}$, given by

$$\mathcal{H}_n(y) = \bigcup_{m=n}^{\infty} F_m(y) \text{ (see Remark 2.6), satisfies that} \quad (2.31)$$

if $C \subseteq \overline{X_n}$ is countable with $C \subseteq \mathcal{H}_n(C)$, then $\overline{C}$ is compact,

if there exist a $w \in X$ and a sequence $\{y_n\}_{n \in \mathbb{N}}$ with $y_n \in \overline{X_n}$ and $y_n \in F_n y_n$ in E_n

such that for every $k \in \mathbb{N}$ there exists a subsequence $S \subseteq \{k+1, k+2, \dots\}$ of $\mathbb{N}$

with $y_n \rightarrow w$ in E_k as $n \rightarrow \infty$ in S , then $w \in Fw$ in E .

(2.32)

Then F has a fixed point in X .

Remark 2.6. The definition of $\mathcal{H}_n$ is as follows. If $y \in \overline{X_n}$ and $y \notin \overline{X_{n+1}}$, then $\mathcal{H}_n(y) = F_n(y)$, whereas if $y \in \overline{X_{n+1}}$ and $y \notin \overline{X_{n+2}}$, then $\mathcal{H}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in \mathbb{N}$. Let $\sum_n = \{x \in \overline{X_n} : x \in F_n x \text{ in } E_n\}$. Now, Theorem 1.1 guarantees that there exists $y_n \in \sum_n$ with $y_n \in F_n y_n$ in E_n . We look at $\{y_n\}_{n \in \mathbb{N}}$. Note that $y_n \in \overline{X_1}$ for $n \in \mathbb{N}$ from (2.27). In addition with $C = \{y_n\}_{n=1}^{\infty}$, we have from assumption (2.31) that $\overline{C} (\subseteq E_1)$ is compact; note that $y_n \in \mathcal{H}_1(y_n)$ in E_1 for each $n \in \mathbb{N}$. Thus there exist a subsequence N_1^* of $\mathbb{N}$ and a $z_1 \in \overline{X_1}$ with $y_n \rightarrow z_1$ in E_1 as $n \rightarrow \infty$ in N_1^* . Let $N_1 = N_1^* \setminus \{1\}$. Proceed inductively to obtain subsequences of integers

$$N_1^* \supseteq N_2^* \supseteq \dots, \quad N_k^* \subseteq \{k, k+1, \dots\} \quad (2.33)$$

and $z_k \in \overline{X_k}$ with $y_n \rightarrow z_k$ in E_k as $n \rightarrow \infty$ in N_k^* . Note that $z_{k+1} = z_k$ in E_k for $k \in \mathbb{N}$. Also let $N_k = N_k^* \setminus \{k\}$.

Fix $k \in \mathbb{N}$. Let $y = z_k$ in E_k . Notice that y is well defined and $y \in \lim_{n \rightarrow \infty} E_n = E$. Now $y_n \in F_n y_n$ in E_n for $n \in N_k$ and $y_n \rightarrow y$ in E_k as $n \rightarrow \infty$ in N_k (since $y = z_k$ in E_k) together with (2.32) imply that $y \in Fy$ in E . $\square$

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