COMPLETIONS OF NON-\(T_2\) FILTER SPACES

NANDITA RATH

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The well-known completions of \(T_2\) Cauchy spaces and \(T_2\) filter spaces are extended to the completions of non-\(T_2\) filter spaces, and a completion functor on the category of all filter spaces is described.

1. Introduction

The categorical topologists Bentley et al. [1] have shown that the category FIL of filter spaces is isomorphic to the category of filter merotopic spaces which were introduced by Katetov [3]. The category CHY of Cauchy spaces is also known to be a bireflective, finally dense subcategory of FIL [7]. So the category FIL is an important category which deserves special discussion. A completion theory for filter spaces was introduced in [4], where a completion functor was defined on the subcategory \(T_2\) FIL of \(T_2\) filter spaces. This completion theory was later applied to completion of filter semigroups [9]. Several other types of completions and their properties were also studied by Minkler et al. [5] and Csaszár [2]. In this paper, a completion theory is developed for filter spaces without the \(T_2\) restriction on the spaces. Also, a completion functor is defined on a subcategory of FIL, which is constructed by taking all the filter spaces as objects and morphisms as certain special type of continuous maps which we call \(s\)-maps.

2. Preliminaries

For basic definitions and terminologies related to filters, the reader is referred to [11], though a few of the definitions will be mentioned here. Let \(X\) be a set and let \(F(X)\) be the set of all filters on \(X\). If \(\mathcal{F}, \mathcal{G} \in F(X)\) and \(F \cap G \neq \phi\) for all \(F \in \mathcal{F}, G \in \mathcal{G}\), then \(\mathcal{F} \lor \mathcal{G}\) denotes the filter generated by \(\{F \cap G : F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}\). If there exist \(F \in \mathcal{F}\) and \(G \in \mathcal{G}\) such that \(F \cap G = \phi\), we say that \(\mathcal{F} \lor \mathcal{G}\) fails to exist. For each \(x \in X\), we denote by \(\hat{x}\) the ultrafilter generated by \(\{x\}\). If \(\zeta \subset F(X)\) satisfies the conditions

\[(c_1) \ \hat{x} \in \zeta, \ \text{for all } x \in X,\]

\[(c_2) \mathcal{F} \in \zeta, \ \mathcal{G} \supseteq \mathcal{F} \ \text{imply that } \mathcal{G} \in \zeta, \ \text{then the pair } (X, \zeta) \ \text{is called a filter space.}\]

The two filters \(\mathcal{F}, \mathcal{G} \in F(X)\) are said to be \(\zeta\)-linked if there exist a finite number of filters \(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n \in \zeta\) such that \(\mathcal{F} \lor \mathcal{H}_1, \mathcal{H}_1 \lor \mathcal{H}_2, \ldots, \mathcal{H}_{n-1} \lor \mathcal{H}_n, \mathcal{H}_n \lor \mathcal{G}\) all exist. In
particular, if \( \mathcal{F}, \mathcal{G} \in \zeta \), we write \( \mathcal{F} \sim \zeta \mathcal{G} \) if and only if \( \mathcal{F}, \mathcal{G} \) are \( \zeta \)-linked. Note that \( \sim \zeta \) is an equivalence relation on \( \zeta \). For \( \mathcal{F} \in \zeta \), let \([\mathcal{F}]_\zeta\) denote the equivalence class containing \( \mathcal{F} \). There is a \textit{preconvergence structure} \( p_\zeta \) associated with \( \zeta \) in a natural way: \( \mathcal{F} \xrightarrow{p_\zeta} x \) if and only if \( \mathcal{F} \sim \zeta x \). Also, there is a convergence structure \( q_\zeta \) associated with \( \zeta \) defined by \( \mathcal{F} \xrightarrow{q_\zeta} x \) if and only if \( \mathcal{F} \not\sim \zeta x \). Note that for any filter space \( (X, \zeta) \), \( p_\zeta \leq q_\zeta \) (see [4]).

A filter space \( (X, \zeta) \) is a \textit{c-filter space} if, in addition,

\((c_3)\) \( \mathcal{F} \in \zeta, \mathcal{F} \sim \zeta \tilde{x} \) imply that \( \mathcal{F} \cap \tilde{x} \in \zeta \), and it is called a \textit{Cauchy space} if

\((c_4)\) whenever \( \mathcal{F}, \mathcal{G} \in \zeta \) and \( \mathcal{F} \sim \zeta \mathcal{G} \), then \( \mathcal{F} \cap \mathcal{G} \in \zeta \).

**Lemma 2.1.** For a filter space \( (X, \zeta) \), \( p_\zeta = q_\zeta \) if and only if \( (X, \zeta) \) is a c-filter space.

A convergence structure \( q \) on a set \( X \) is said to be \textit{compatible} (resp., \textit{c-filter compatible}, \textit{Cauchy compatible}) if there exists a filter structure (resp., \textit{c-filter structure}, \textit{Cauchy structure}) \( \zeta \) on \( X \) such that \( q = q_\zeta \). As shown above, given a filter space \( (X, \zeta) \), we can always associate a convergence \( q_\zeta \). However, every convergence structure on \( X \) is not \( c \)-filter compatible.

**Example 2.2.** Let \( X = \{a, b, c\} \) and let \( q \) be the convergence structure on \( X \) defined by \( a \rightarrow^q a, b \rightarrow^q b, t \rightarrow^q t, a \cap b \rightarrow^q b, b \cap t \rightarrow^q t \), and all other filters fail to converge. If possible, let \( \zeta \) be the filter structure on \( X \) such that \( q = q_\zeta \). Let \( \mathcal{F} = b \cap t \), so \( \mathcal{F} \not\rightarrow^q t \). However, \( \mathcal{F} \cap b = b \cap t \in \zeta \Rightarrow \mathcal{F} \not\rightarrow^q b \), but \( \mathcal{F} \rightarrow^q b \). So \( q \) is not compatible.

The following lemma states the necessary and sufficient conditions for such compatibilities of a convergence structure on \( X \).

**Lemma 2.3.** A convergence structure \( q \) on \( X \) is

\[(a)\] compatible if and only if \( \mathcal{F} \) is \( q \)-convergent and \( \tilde{x} \geq \mathcal{F} \Rightarrow \mathcal{F} \rightarrow^q x \);

\[(b)\] \( c \)-filter compatible if and only if either \( q(x) = q(y) \) or \( q(x) \cap q(y) = \emptyset \);

\[(c)\] Cauchy compatible if and only if \( \mathcal{F}, \mathcal{G} \in q(x) \Rightarrow \mathcal{F} \cap \mathcal{G} \in q(x) \) and for all \( x, y \in X \),

\[q(x) = q(y) \text{ or } q(x) \cap q(y) = \emptyset.\]

**Proof.** The proof of \((b)\) is similar to [4, Proposition 1.3] and the proof of \((c)\) is well known. So we will prove only \((a)\). Let \( q = q_\zeta \), where \( \zeta \) is a filter structure on \( X \). Let \( \mathcal{F} \) be \( q \)-convergent, \( \mathcal{F} \rightarrow^q y \) (say), and \( \tilde{x} \geq \mathcal{F} \). So \( \mathcal{F} \rightarrow^q y \), that is, \( \mathcal{F} \cap y \in \zeta \). So \( \mathcal{F} = \mathcal{F} \cap \tilde{x} \in \zeta \Rightarrow \mathcal{F} \rightarrow^q \tilde{x} \). Conversely, let \( q \) satisfy the given condition and let \( \zeta_q \) be the set of all \( q \)-convergent filters. We show that \( q = q_\zeta \). If \( \mathcal{F} \rightarrow^q \tilde{x} \), then \( \mathcal{F} \cap \tilde{x} \in \zeta_q \Rightarrow \mathcal{F} \rightarrow^q \tilde{x} \). Also, if \( \mathcal{F} \rightarrow^q \tilde{x} \), then \( \mathcal{F} \cap \tilde{x} \in \zeta_q \Rightarrow \mathcal{F} \cap \tilde{x} \) is \( q \)-convergent and since \( \tilde{x} \geq \mathcal{F} \cap \tilde{x} \), \( \mathcal{F} \cap \tilde{x} \rightarrow^q x \). So \( \mathcal{F} \rightarrow^q x \).

Compatibilities of a preconvergence structure on a filter space is defined in a similar way. The conditions for compatibility, \( c \)-filter compatibility, and Cauchy compatibility of a preconvergence structure \( p \) were established in [4].

Note that if \( q \) is a convergence structure on \( X \) and there is a filter structure \( \zeta \) on \( X \) such that \( q = p_\zeta \), then \( (X, \zeta) \) is a \( c \)-filter space and \( q = q_\zeta \). In particular, the following lemma holds when \( q \) is a pretopology.

**Lemma 2.4.** A pretopology \( \sigma \) on \( X \) is compatible and \( \sigma = p_\zeta \) for some filter structure \( \zeta \) on \( X \) if and only if \( \zeta \) is a \( c \)-filter structure.
However, if the pretopology $\sigma = q_\zeta$ for some filter structure $\zeta$ on $X$, then as illustrated in the following example, the above lemma may not hold in general.

**Example 2.5.** Let $X = \mathbb{R}$, the set of real numbers, $\mathcal{F} = \{(0, 1/n) \mid n \in \mathbb{N}\}$, and $\mathcal{L} = \{\text{all complements of countable sets}\}$. It is clear that $\zeta = \{x \mid x \in X\} \cap \{\Gamma \mid \Gamma \geq \mathcal{F} \cap 0\} \cap \{\Psi \mid \Psi \geq \mathcal{L} \cap 1\}$ is a filter structure on $\mathbb{R}$. For each $x \in X$, let $V_\sigma(x) = \cap \{\mathcal{G} \in \mathcal{F}(X) \mid \mathcal{G} \rightarrow^\sigma \rightarrow^0 x\}$ denote the neighborhood filter of $x$. We define $\sigma$ as follows: $V_\sigma(x) = \hat{x}$, for all $x \neq 0, 1$ and $V_\sigma(0) = \mathcal{F} \cap 0, V_\sigma(1) = \mathcal{L} \cap 1$. It is clear that $\sigma$ is a pretopology and $\sigma = q_\zeta$. However, $\zeta$ is not a $c$-filter structure because $0 \rightarrow^{p_F} 1$ and $1 \rightarrow^{p_F} 0$, but $\hat{0} \cap 1 \rightarrow p_F 1$, which implies $\hat{0} \cap 1 \notin \zeta$. So $p_\zeta \neq q_\zeta$ and hence, by Lemma 2.1, $\zeta$ is not a $c$-filter structure on $X$.

A filter space $(X, \zeta)$ is said to be

(i) $T_2$ (resp., $w$-$T_2$) if and only if $\mathcal{F} \sim \hat{x}, \mathcal{F} \sim \hat{y} \Rightarrow x = y$ (resp., $\mathcal{F} \cap \hat{x}, \mathcal{F} \cap \hat{y} \in \zeta \Rightarrow x = y$),

(ii) complete (resp., $w$-complete) if and only if for each $\mathcal{F} \in \zeta$, $p_\zeta$ converges (resp., for each $\mathcal{F} \in \zeta$, $q_\zeta$ converges),

(iii) regular (resp., $w$-regular) if and only if $cl_{p_\zeta} \mathcal{F} \in \zeta$ whenever $\mathcal{F} \in \zeta$ (resp., $cl_{q_\zeta} \mathcal{F} \in \zeta$ whenever $\mathcal{F} \in \zeta$),

(iv) totally bounded if each ultrafilter on $X$ is in $\zeta$.

Note that if a filter space is $T_2$ (resp., regular, complete), then it is $w$-$T_2$ (resp., regular, $w$-complete). The proof of the following lemma is immediate from the definitions.

**Lemma 2.6.** The following are true for any filter space $(X, \zeta)$:

(I) $(X, \zeta)$ is $T_2$ (resp., $w$-$T_2$) if and only if $(X, p_\zeta)$ (resp., $(X, q_\zeta)$) is $T_2$;

(II) if $(X, \zeta)$ is regular (resp., $w$-regular), then $(X, p_\zeta)$ (resp., $(X, q_\zeta)$) is regular;

(III) if $(X, p_\zeta)$ (resp., $(X, q_\zeta)$) is regular and $(X, \zeta)$ is complete (resp., $w$-complete), then $(X, \zeta)$ is regular (resp., $w$-regular).

**Lemma 2.7.** Any regular filter space is a $c$-filter space.

**Lemma 2.8.** A filter space $(X, \zeta)$ is totally bounded and complete if and only if $(X, p_\zeta)$ is compact. If a filter space $(X, \zeta)$ is totally bounded and $w$-complete, then $(X, q_\zeta)$ is compact. Also $(X, q_\zeta)$ is compact implies $(X, \zeta)$ is totally bounded. However, if $(X, q_\zeta)$ is compact, then $(X, \zeta)$ is not necessarily $w$-complete.

A map $f : (X, \zeta) \rightarrow (Y, \kappa)$ between two filter spaces is called continuous if $f(\mathcal{F}) \in \kappa$ whenever $\mathcal{F} \in \zeta$. We denote by FIL the category of all filter spaces and continuous maps as morphisms. Let CFIL and CHY be the full subcategories of FIL whose objects are $c$-filter spaces and Cauchy spaces, respectively. In [4], a completion of objects in $T_2$ FIL and a completion functor on $T_2$ CFIL and its bireflective subcategory $C_3$ FIL were constructed. The completion functor and the completion subcategory constructed in [4] deal with $T_2$ filter spaces. The underlying reason for this is the existence of unique limits for convergent filters which are also preserved by the continuous map. In this paper, we partially overcome that limitation by using a special type of continuous map called $s$-map which will be introduced later.

The map $f : (X, \zeta) \rightarrow (Y, \kappa)$ between two filter spaces is a homeomorphism if $f$ is bijective and both $f$ and $f^{-1}$ are continuous maps. In this case, $(X, \zeta)$ and $(Y, \kappa)$ are called
homeomorphic filter spaces. Note that the underlying preconvergence spaces \((X, p_\kappa)\) and \((Y, p_\kappa)\) are also homeomorphic. A map \(f : (X, \zeta) \rightarrow (Y, \kappa)\) is an embedding of \((X, \zeta)\) into \((Y, \kappa)\) if \(f : (X, \zeta) \rightarrow (f(X), \kappa_{f(X)})\) is a homeomorphism, where \(\kappa_{f(X)}\) is a subspace structure on \(f(X)\).

In this paper, we construct completions for a filter space without the \(T_2\) restriction on the space. Results obtained in this paper also generalise the completion theory developed for non-\(T_2\) Cauchy spaces obtained in [8] where the author introduced the morphisms on the category CHY as \(s\)-map. The corresponding extension of the notion of \(s\)-maps to filter spaces can be used to form a completion subcategory of FIL.

Definition 2.9. A continuous map between two filter spaces \(f : (X, \zeta) \rightarrow (Y, \kappa)\) is said to be an \(s\)-map if it satisfies the following condition: \(F \in \zeta, p_\kappa\) converges to at most one point in \(X\) implies \(f(F) \in \kappa, p_\kappa\) converges to at most one point in \(Y\).

There are several examples of \(s\)-maps. Any continuous map is an \(s\)-map if the codomain of the map is a \(T_2\) filter space. The identity map on a filter space and the embedding map \(\varphi\) for a stable completion is also an \(s\)-map. Note that it follows from the definition of \(s\)-map that composition of two \(s\)-maps is an \(s\)-map. The class of all filter spaces with the \(s\)-maps as morphisms forms a category which we denote by FIL'. Since every continuous map is not necessarily an \(s\)-map, FIL' is not a full subcategory of FIL.

3. Completion and extension theorems for filter spaces

Throughout this section, \((X, \zeta)\) denotes a filter space (not necessarily \(T_2\)). A completion of a filter space \((X, \zeta)\) is a pair \(((Y, \kappa), \phi)\) consisting of a complete filter space \((Y, \kappa)\) and an embedding \(\phi : (X, \zeta) \rightarrow (Y, \kappa)\) such that \(cl_{p_\kappa}(\phi(X)) = Y\). The completion \(((Y, \kappa), \phi)\) is called a weak completion if \((Y, \kappa)\) is \(w\)-complete and \(cl_{p_\kappa}(\phi(X)) = Y\). A completion \(((Y, \kappa), \varphi)\) of a filter space \((X, \zeta)\) is said to be a \(P\)-completion if \((Y, \kappa)\) has the property \(P\) whenever \((X, \zeta)\) has the property \(P\). For instance, a completion which is \(T_2\) will be called a \(T_2\) completion. The results related to completions in standard form and extension theorems are established here.

When \((X, \zeta)\) is a \(T_2\) filter space, completion and extension theorems were established in [4] and later a few other classes of completions were constructed in [2, 5]. In this section, we will construct non-\(T_2\) completion and non-\(T_2\) weak completion of a filter space and establish some extension theorems.

We construct a completion \(((\tilde{X}, \zeta), j)\) for a filter space \((X, \zeta)\). Let

\[
\eta_\zeta = \{ [F] \mid F \in \zeta, F \not\subset \tilde{x}, \forall x \in X \}.
\]

(3.1)

For an arbitrary filter space \((X, \zeta)\), we define the following:

(i) \(\tilde{X} = \eta_\zeta \cup X\),
(ii) \(j : X \rightarrow \tilde{X}\) is the inclusion map,
(iii) \(\zeta = \{ \mathcal{A} \in \mathbf{F}(\tilde{X}) \mid \exists F \in \zeta, \text{ non-}p_\zeta \text{ convergent such that } \mathcal{A} \supseteq j(F) \cap [\tilde{F}] \cup j(\zeta) \} \).

Definition 3.1. A completion \(((Y, \kappa), \varphi)\) of a filter space \((X, \zeta)\) is said to be in standard form if \(Y = \tilde{X}, \varphi = j\) and for each non-\(p_\zeta\) convergent filter \(F \in \zeta, j(F) \not\supseteq [\tilde{F}]\).
A similar notion of completion was introduced in [10] for $T_2$ Cauchy spaces. We extend that notion to completion of non-$T_2$ filter spaces. In [4], we defined an equivalence relation between $T_2$ completions. For a $T_2$ filter space $(Z, \delta)$, a completion $\mathcal{X}_1 = ((Y_1, \kappa_1), \varphi_1)$ is said to be finer than a completion $\mathcal{X}_2 = ((Y_2, \kappa_2), \varphi_2)$ if there exists a continuous map $h : (Y_1, \kappa_1) \to (Y_2, \kappa_2)$ such that $h \circ \varphi_1 = \varphi_2$. The two completions $\mathcal{X}_1$ and $\mathcal{X}_2$ are equivalent if $\mathcal{X}_1 \succeq \mathcal{X}_2$ and $\mathcal{X}_2 \succeq \mathcal{X}_1$. Observe that in case of equivalence the two completion spaces are isomorphic in the category $T_2$ FIL of $T_2$ filter spaces.

We can define an equivalence relation for non-$T_2$ completions of filter spaces in the same way, but it is not a categorical equivalence in the sense of Preuss [6], since in this case the map $h$ is not necessarily a unique homeomorphism. This motivates the introduction of a weakly stable completion.

**Definition 3.2.** A completion $((Y, \kappa), \varphi)$ of a filter space $(X, \zeta)$ is weakly stable if whenever $z \in Y \setminus \phi(X)$ and $\phi(\mathcal{F}) \not\to z$, for some $\mathcal{F} \in \zeta$, it follows that $z$ is the unique limit of $\phi(\mathcal{F})$ in $Y$.

**Remark 3.3.** (I) Stable completion of a $T_2$ filter space was defined in [4]. Note that any stable completion of a $T_2$ filter space is always weakly stable. Also, if $(X, \zeta)$ is a $T_2$-c-filter space, then a weakly stable completion is stable. If $(X, \zeta)$ is a Cauchy space, then every weakly stable completion is stable.

(II) A weakly stable completion $\mathcal{X}_1 = ((Y_1, \kappa_1), \varphi_1)$ is said to be finer than another weakly stable completion $\mathcal{X}_2 = ((Y_2, \kappa_2), \varphi_2)$ if there is a continuous map $h : (Y_1, \kappa_1) \to (Y_2, \kappa_2)$ such that $h \circ \varphi_1 = \varphi_2$. Note that the map $h$ is a unique homeomorphism when the two weakly stable completions are equivalent.

In the following result, we show that the Wyler completion of a non-$T_2$ filter space has a property similar to the universal property of the $T_2$ completions.

**Theorem 3.4.** $((X, \zeta), j)$ is the finest weakly stable completion of the filter space $(X, \zeta)$ in standard form.

**Proof.** It is clear that $\tilde{\zeta}$ is a filter structure on $\tilde{X}$ and the inclusion map $j$ is an embedding. To show that $(\tilde{X}, \tilde{\zeta})$ is complete, let $\mathcal{A} \in \tilde{\zeta}$. If $\mathcal{A} \succeq j(\mathcal{F})$ for some $\mathcal{F} \in \zeta$, then $\mathcal{F} \not\to x$ implies $\mathcal{A} \not\to x$, and $\mathcal{F}$ is nonconvergent implies $j(\mathcal{F}) \cap [\hat{\mathcal{F}}] \in \tilde{\zeta}$ which shows that $\mathcal{A} \not\to [\hat{\mathcal{F}}]$. If $\mathcal{A} \succeq j(\mathcal{F}) \cap [\hat{\mathcal{F}}]$ for some nonconvergent $\mathcal{F} \in \zeta$, then as we have already shown $\mathcal{A} \not\to [\hat{\mathcal{F}}]$. So $(\tilde{X}, \tilde{\zeta})$ is complete. Also, if $[\hat{\mathcal{F}}] \in \tilde{X} \setminus j(X)$, then $\mathcal{F}$ is nonconvergent and so $j(\mathcal{F}) \not\to [\hat{\mathcal{F}}]$ which implies that $[\hat{\mathcal{F}}] \in \text{cl}_{\tilde{\zeta}} j(X)$. This proves that $((\tilde{X}, \tilde{\zeta}), j)$ is a completion of $(X, \zeta)$. Since for each non-$p_{\zeta}$ convergent filter $\mathcal{F} \in \zeta$, $j(\mathcal{F}) \cap [\hat{\mathcal{F}}] \in \zeta$, we have $j(\mathcal{F}) \not\sim [\hat{\mathcal{F}}]$. So it follows that $j(\mathcal{F}) \not\to [\hat{\mathcal{F}}]$ and this completion is in standard form.

Next we show that it is a weakly stable completion. Let $[\mathcal{F}] \in \tilde{X} \setminus j(X)$, then $\mathcal{F} \in \zeta$ is non-$p_{\zeta}$ convergent and $j(\mathcal{F}) \not\to [\mathcal{G}]$. If there exists $[\mathcal{G}] \in \tilde{X} \setminus j(X)$ such that $j(\mathcal{F}) \not\to [\mathcal{G}]$, then $j(\mathcal{F}) \not\sim [\mathcal{G}]$. So there exist a finite number of filters $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \in \zeta$ such that $j(\mathcal{F}) \setminus \Lambda_1, \Lambda_1 \setminus \Lambda_2, \ldots, \Lambda_n \setminus [\mathcal{G}]$ exist. If $\Lambda_1 \succeq j(\mathcal{H})$ for some $p_{\zeta}$ convergent filter...
\[ \mathcal{H} \in \zeta, \text{ then } j(\mathcal{F}) \lor \Lambda_1 \text{ exists. This implies that } j(\mathcal{F}) \lor j(\mathcal{H}) \text{ exists from which it follows that } \mathcal{F} \lor \mathcal{H} \text{ exists. This leads to a contradiction since } \mathcal{F} \text{ is non-} p_\zeta \text{ convergent. So } \Lambda_1 \geq j(\mathcal{L}_1) \lor [\mathcal{L}_1], \text{ for some non-} p_\zeta \text{ convergent filter } \mathcal{L}_1 \in \zeta. \text{ So } j(\mathcal{F}) \lor \Lambda_1 \text{ implies that } j(\mathcal{F}) \lor (j(\mathcal{L}_1) \lor [\mathcal{L}_1]) \text{ exists from which it follows that either } j(\mathcal{F}) \lor j(\mathcal{L}_1) \text{ or } j(\mathcal{F}) \lor [\mathcal{L}_1] \text{ exists. Since the latter is an impossibility, } j(\mathcal{F}) \lor j(\mathcal{L}_1) \text{ exists. This implies } \mathcal{F} \lor \mathcal{L}_1 \text{ exists, that is, } \mathcal{F} \lor \mathcal{L}_1 \text{ or } [\mathcal{F}] = [\mathcal{L}_1]. \text{ Since } \Lambda_1 \lor \Lambda_2 \text{ exists and } \Lambda_1 \geq j(\mathcal{L}_1) \lor [\mathcal{L}_1], \text{ it follows that } (j(\mathcal{L}_1) \lor [\mathcal{L}_1]) \lor \Lambda_2 \text{ exists, that is, } (j(\mathcal{L}_1) \lor [\mathcal{F}]) \lor \Lambda_2 \text{ exists. Following the same type of argument as above, we can show that there is no } p_\zeta \text{ convergent filter } \mathcal{H} \in \zeta \text{ such that } \Lambda_2 \geq j(\mathcal{H}). \text{ So } \Lambda_2 \geq j(\mathcal{L}_2) \lor [\mathcal{L}_2], \text{ for some non-} p_\zeta \text{ convergent filter } \mathcal{L}_2 \in \zeta. \text{ So } (j(\mathcal{L}_1) \lor [\mathcal{F}]) \lor (j(\mathcal{L}_2) \lor [\mathcal{L}_2]) \text{ exists. This implies at least one of } j(\mathcal{L}_1) \lor j(\mathcal{L}_2), j(\mathcal{L}_1) \lor [\mathcal{L}_2], [\mathcal{F}] \lor j(\mathcal{L}_2), [\mathcal{F}] \lor [\mathcal{L}_2] \text{ exists. Since } [\mathcal{F}], [\mathcal{L}_2] \notin j(X), \text{ neither } j(\mathcal{L}_1) \lor [\mathcal{L}_2] \text{ nor } [\mathcal{F}] \lor j(\mathcal{L}_2) \text{ exists. However, } [\mathcal{F}] \lor [\mathcal{L}_2] \text{ implies that } [\mathcal{F}] = [\mathcal{L}_2] \text{ and } j(\mathcal{L}_1) \lor j(\mathcal{L}_2) \text{ implies that } \mathcal{L}_1 \lor \mathcal{L}_2 \text{ exists, so that } [\mathcal{L}_2] = [\mathcal{L}_1] = [\mathcal{F}]. \text{ Repeating a similar argument, we can show that each } \Lambda_i \geq j(\mathcal{L}_i) \lor [\mathcal{L}_i], \text{ for some non-} p_\zeta \text{ convergent filter } \mathcal{L}_i \in \zeta \text{ and } [\mathcal{L}_i] = [\mathcal{F}], \text{ for } 1 \leq i \leq n. \text{ Also, } \Lambda_n \geq j(\mathcal{L}_n) \lor [\mathcal{L}_n], \text{ for some non-} p_\zeta \text{ convergent filter } \mathcal{L}_n \in \zeta \text{ and } (j(\mathcal{L}_n) \lor [\mathcal{L}_n]) \lor [\mathcal{G}] \text{ implies that } [\mathcal{L}_n] = [\mathcal{G}]. \text{ Hence, } [\mathcal{F}] = [\mathcal{G}]. \text{ This proves that } ((\tilde{x}, \tilde{\zeta}), j) \text{ is a weakly stable completion.}

Let \((\tilde{x}, \beta), j\) be another weakly stable completion of \((X, \zeta)\) in standard form and let \(\Lambda \in \tilde{\zeta} \). If \(\Lambda \geq j(\mathcal{F})\), where \(\mathcal{F}\) is \(p_\zeta\) convergent, then \(\Lambda \in \beta\), since \(j(\mathcal{F}) \in \beta\). Next, let \(\Lambda \geq j(\mathcal{F}) \lor \mathcal{F}\), where \(\mathcal{F}\) is non-\(p_\zeta\) convergent. Then \(j(\mathcal{F}) \rightarrow \mathcal{F}\) since \((\tilde{x}, \beta), j\) is in standard form. So \(j(\mathcal{F}) \lor \mathcal{F} \in \beta\) which implies \(\Lambda \in \beta\). So \((\tilde{x}, \zeta), j\) is the finest weakly stable completion in standard form. \(\Box\)

We will refer to the completion \((\tilde{x}, \tilde{\zeta}), j\) as the Wyler completion of \((X, \zeta)\). Obviously, the mapping \(j\) in \((\tilde{x}, \tilde{\zeta}), j\) is an s-map. Note that if \((X, \zeta)\) is a \(c\)-filter space, then \((\tilde{x}, \tilde{\zeta})\) is a \(c\)-filter space. Also, if \((X, \zeta)\) is \(T_2\), then \((\tilde{x}, \tilde{\zeta}), j\) is a \(T_2\) completion of \((X, \zeta)\). If we identify each \(x \in X\) with the equivalence class \([x]\) of all filters which are \(p_\zeta\) convergent to \(x\), then the Wyler completion coincides with \(((X^*, \zeta^*), j)\) in \([4]\). We will refer to the latter completion as the \(T_2\) Wyler completion of \((X, \zeta)\).

Proposition 3.5. Any weakly stable completion \(((Y, \kappa), \varphi)\) of a filter space \((X, \zeta)\) is equivalent to one in standard form.

Proof. Define \(h : Y \rightarrow \tilde{X}\) by

\[ h(y) = \begin{cases} x, & \text{if } y = \phi(x), \\ [\mathcal{F}], & \text{if } y \in Y \setminus \phi(X), \phi(\mathcal{F}) \rightarrow y. \end{cases} \tag{3.2} \]

Note that \(h\) is well defined and bijective, because \(((Y, \kappa), \varphi)\) is a weakly stable completion. Let \(\zeta_\kappa\) be the quotient filter structure on \(\tilde{X}\). Since \(j = h\varphi\), both \(j\) and \(j^{-1}\) are continuous maps and since \((Y, \kappa)\) is complete and \((\tilde{x}, \zeta_\kappa)\) is the quotient space, the latter is also complete. If \([\mathcal{F}] \in \tilde{X} \setminus j(X)\), then \(\varphi(\mathcal{F}) \rightarrow y\), for some \(y \in Y\). This implies that \(j(\mathcal{F}) = h \circ \varphi(\mathcal{F}) \rightarrow h(y) = [\mathcal{F}]\). Hence, \(\text{cl}_{p_\zeta} j(X) = \tilde{X}\). So \((\tilde{x}, \zeta_\kappa)\) is a completion of \((X, \zeta)\) and by the same argument one can also show that it is in standard form. This proves that \(((\tilde{x}, \zeta_\kappa), j)\)
is a completion of \((X, \zeta)\) in standard form. It remains to show that \(h\) is a homeomorphism. Since the category FIL is a topological category, \(h\) is an injective quotient map implies \(h\) is a monomorphism which is also an extremal epimorphism. Therefore, by [6, Proposition 0.2.7], \(h\) is an isomorphism. This completes the proof of Proposition 3.5

In view of Proposition 3.5, we may therefore assume without loss of generality that all weakly stable completions of a filter space \((X, \zeta)\) are in standard form. If \((X, \zeta)\) is \(T_2\), then any \(T_2\) completion is always stable. Hence, we have the following corollary.

**Corollary 3.6.** Any \(T_2\) completion of a filter space is equivalent to one in standard form.

Given a \(T_2\) filter space \((X, \zeta)\), let \(X^* = \{[\mathcal{F}] | \mathcal{F} \in \zeta\}\), let \(\zeta^* = \{\mathcal{A} | \mathcal{A} \supseteq j(\mathcal{F}), \mathcal{F} \in \zeta \text{ with } \mathcal{F}_c \text{ convergent, or } \mathcal{A} \supseteq j(\mathcal{F}) \cap [\mathcal{F}], \mathcal{F} \in \zeta \text{ with } \mathcal{F} \text{ non-}p_c \text{ convergent}\}\), and let \(j : X \rightarrow X^*\) be defined by \(j(x) = [\hat{x}]\) for all \(x \in X\).

**Proposition 3.7.** A \(w-T_2\) filter space \((X, \zeta)\) has a \(w-T_2\) completion if and only if \((X, \zeta)\) is a \(T_2\) \(c\)-filter space.

**Proof.** \((\Rightarrow)\) Let \(((Y, \kappa), \varphi)\) be a \(w-T_2\) completion of \((X, \zeta)\). Let \(\mathcal{F} \in \zeta\) and \(\mathcal{F} \sim_\zeta \hat{x}\). By Proposition 3.5, it follows that \(\varphi(\mathcal{F}) \overset{q_\zeta}{\rightarrow} \varphi(\hat{x})\), that is, \(\varphi(\mathcal{F}) \cap \varphi(\hat{x}) \in \kappa\). Since \(\varphi\) is an embedding, \(\mathcal{F} \cap \hat{x} \in \zeta\) which shows that \((X, \zeta)\) is a \(c\)-filter space.

\((\Leftarrow)\) Let \((X, \zeta)\) be a \(T_2\) \(c\)-filter space. Let \(X^*, \zeta^*,\) and \(j\) be as above. Note that since \((X, \zeta)\) is \(c\)-filter space, \(p_c = q_\zeta\) and \((X^*, \zeta^*)\) is also a \(c\)-filter space. So \(((X^*, \zeta^*), j)\) is a \(w\)-completion of \((X, \zeta)\). Hence, it remains to show that \((X^*, \zeta^*)\) is \(T_2\). Let \(y_1 \cap y_2 \in \zeta^*\). If \(y_1, y_2 \in X\), then \(y_1 = y_2\) since \((X, \zeta)\) is \(T_2\). If at least one of \(y_1\) or \(y_2\) is in \(X^* \setminus X\), then by the definition of \(\zeta^*\) it follows that \(y_1 \cap y_2 \in \zeta^*\) only when \(y_1 = y_2\). This proves Proposition 3.7.

Following a similar argument as in [8, Proposition 3.13], we can show that the Wyler completion is the finest completion in \(\text{CFIL}'\). But it is not the finest completion in FIL. In fact, in [4] it was shown that there is no such finest completion whenever \(X \setminus j(X)\) is infinite. However, the following proposition states that we can uniquely extend any \(s\)-map on a non-\(T_2\) \(c\)-filter space to its Wyler completion.

**Proposition 3.8.** If \(f : (X, \zeta) \rightarrow (Y, \beta)\) is an \(s\)-map and \((Y, \beta)\) is a \(c\)-filter space, then there is a unique extension \(f^* : (\bar{X}, \bar{\zeta}) \rightarrow (\bar{Y}, \bar{\beta})\) which is also an \(s\)-map and \(f^* \circ j_X = j_Y \circ f\), where \(j_X\) and \(j_Y\) are the corresponding embedding maps.

**Proof.** Define \(f^* : (\bar{X}, \bar{\zeta}) \rightarrow (\bar{Y}, \bar{\beta})\) as follows:

\[
f^*(x) = f(x),
\]

\[
f^*(\{\mathcal{F}\}) = \begin{cases}
[f(\mathcal{F})], & \text{if } f(\mathcal{F}) \text{ is non-convergent}, \\
y, & \text{if } f(\mathcal{F}) \text{ converges to } y.
\end{cases}
\]  

(3.3)

The mapping \(f^*\) so defined is a well-defined map because if \([\mathcal{F}] = [\mathcal{G}]\), then \(f(\mathcal{F}) \sim_\beta f(\mathcal{G})\), so either both \(f(\mathcal{F})\) and \(f(\mathcal{G})\) are non-convergent or convergent. If both are non-convergent, then obviously \(f^*([\mathcal{F}]) = f^*([\mathcal{G}])\). If \(f(\mathcal{F}) \overset{q_\zeta}{\rightarrow} y_1\) and \(f(\mathcal{G}) \overset{q_\zeta}{\rightarrow} y_2\), then
\( f(\mathcal{F}) \sim \beta y_1, y_2 \). This is a contradiction, since \( \mathcal{F} \) is non-\( p_\xi \)-convergent and \( f \) is an s-map. So in either case \( f^*([\mathcal{F}]) = f^*([\mathcal{G}]) \). Also, it can be easily verified that \( f^* \circ j_\mathcal{X} = j_\mathcal{Y} \circ f \).

Next we show that \( f^* \) is an s-map. Let \( \mathcal{A} \in \xi \). If \( \mathcal{A} \geq j_X(\mathcal{F}) \), then \( f^*(\mathcal{A}) \geq f^* \circ j_X(\mathcal{F}) = j_Y \circ f(\mathcal{F}) \in \beta \). If \( \mathcal{A} \not\geq j_X(\mathcal{F}) \cap [\mathcal{F}] \), where \( \mathcal{F} \) is non-\( p_\xi \) convergent, then \( f^*(\mathcal{A}) \geq (j_Y \circ f(\mathcal{F})) \cap f^*([\mathcal{F}]) \). If \( f(\mathcal{F}) \) is nonconvergent in \( Y \), then \( (j_Y \circ f(\mathcal{F})) \cap [f(\mathcal{F})] \in \beta \).

If \( f(\mathcal{F}) \) converges to \( y \in Y \), then since \( (Y, \beta) \) is a c-filter space \( f(\mathcal{F}) \cap y \in \beta \), so it follows that \( (j_Y \circ f(\mathcal{F})) \cap y \in \beta \). Therefore, \( f^* \) is a continuous map. To show that it is an s-map, it suffices to show that if \( \mathcal{A} \in \xi \) converges to only one point, then \( f^*(\mathcal{A}) \) converges to only one point. If \( \mathcal{A} \geq j_X(\mathcal{F}) \), then \( j_Y \circ f(\mathcal{F}) = f^* \circ j_X(\mathcal{F}) \) converges to only one point, because \( f \) is an s-map. If \( \mathcal{A} \geq j_X(\mathcal{F}) \cap [\mathcal{F}] \), then \( \mathcal{F} \) is non-\( p_\xi \) convergent which implies that \( f(\mathcal{F}) \) converges to at most one point. Therefore, \( f^*(j_X(\mathcal{F}) \cap [\mathcal{F}]) = (f^* \circ j_X(\mathcal{F})) \cap f^*([\mathcal{F}]) = (j_Y \circ f(\mathcal{F})) \cap [f(\mathcal{F})] \) or \( (j_Y \circ f(\mathcal{F})) \cap y \) according as \( f(\mathcal{F}) \) is nonconvergent or \( f(\mathcal{F}) \) converges to \( y \). But in either case \( f^*(\mathcal{A}) \) converges to only one point in \( Y \).

Finally, we show that \( f^* \) is a unique extension. Let \( \mathcal{F} : (X, \mathcal{X}) \to (Y, \beta) \) be another s-map such that \( \mathcal{F} \circ j_X(x) = j_Y \circ f(\mathcal{F}) \). It is obvious that \( \mathcal{F} \circ j_X(x) = f^* \circ j_X(x) \), for all \( x \in X \).

Let \( [\mathcal{F}] \in X^* \setminus j_X(X) \). Since \( \mathcal{F} \in \zeta \) is non-\( p_\xi \) convergent, it follows that \( j_X(\mathcal{F}) \to X^* \)

Since \( f^*, \mathcal{F} \) are s-maps, it follows that \( f^* \circ j_X(\mathcal{F}) = \mathcal{F} \circ j_X(\mathcal{F}) = j_Y \circ f(\mathcal{F}) \) converges to \( f^*([\mathcal{F}]) \) and \( \mathcal{F}([\mathcal{F}]) \). But \( \mathcal{F} \) is nonconvergent and \( f, j_Y \) are s-maps imply that \( j_Y \circ f(\mathcal{F}) \) can converge to at most one point in \( Y \). Hence, \( f^* = \mathcal{F} \). This proves Proposition 3.8.

The unique mapping \( f^* \) in Proposition 3.8 is called the s-extension of \( f \).

Remark 3.9. (I) If \( f : (X, \mathcal{X}) \to (Y, \mathcal{Y}) \) is an s-map, where \( (Y, \mathcal{Y}) \) is a complete c-filter space, then there exists a unique s-extension \( f^* : (X, \mathcal{X}) \to (Y, \mathcal{Y}) \) such that \( f = f^* \circ j_X \).

If \( (Y, \mathcal{Y}) \) is a regular filter space, then \( (X^*, \mathcal{X}^*) \) also has the same extension property (by Lemma 2.7). In either case, the s-extension \( f^* \) is defined by \( f^* (x) = f(x) \), for each \( x \in X \) and \( f^*([\mathcal{F}]) = y \), where \( f([\mathcal{F}]) \to y \).

(II) If \( (X, \mathcal{X}) \) is a T₂ filter space, then its T₂ Wyler completion has the extension property. Recall that if the codomain of an s-map is a T₂ space, then the s-map is simply a continuous map. If \( f : (X, \mathcal{X}) \to (Y, \mathcal{Y}) \) is a continuous map, where \( (Y, \mathcal{Y}) \) is a complete T₂ c-filter space [4] or a complete T₃ filter space, then there exists a unique extension \( f^* : (X, \mathcal{X}) \to (Y, \mathcal{Y}) \).

Since the composition of s-maps is an s-map and the identity map is an s-map, the class of all c-filter spaces with s-maps as morphisms forms a subcategory of FIL. We denote this category by CFIL′. Let CFIL′* be the subcategory of CFIL′ consisting of the complete objects of CFIL′. Let \( W : \text{CFIL}' \to \text{CFIL'}* \) be defined for objects by \( W(X, \mathcal{X}) = (X, \mathcal{X}) \) and for morphisms by \( W(f) = f^* \). Then \( W \) is a covariant functor on CFIL′ and is called the Wyler completion functor.

Note that a morphism \( f : (X, \mathcal{X}) \to (Y, \mathcal{Y}) \) in the category CFIL′ is an epimorphism if for each \( z \in Y \setminus f(X) \), there exists a non-\( q_\xi \) convergent filter \( \mathcal{F} \in \zeta \) such that \( f(\mathcal{F}) \to y \). For example, the embedding map \( j \) in the Wyler completion is an epimorphism. Since Wyler completion is the finest completion in the category CFIL′, we have the following corollary.
Corollary 3.10. CFIL′* is an epireflective subcategory of CFIL′.

However, CFIL′ is not a topological category, since it is not closed under initial structures. But it should be noted that $T_2$ CFIL for which we could construct a completion functor [4] also fails to be a topological category. The construction of a completion functor for a subcategory of FIL which is a topological category may need further investigation. Also, constructions of regular stable completions of filter spaces and the corresponding completion categories may lead to generalisation of the existing $T_3$ completions.

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References

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