We obtain necessary and sufficient conditions for the asymptotic stability of the linear delay difference equation $x_{n+1} + p \sum_{j=1}^{N} x_{n-k+(j-1)}l = 0$, where $n = 0, 1, 2, \ldots$, $p$ is a real number, and $k$, $l$, and $N$ are positive integers such that $k > (N - 1)l$.

1. Introduction

In [4], the asymptotic stability condition of the linear delay difference equation

$$x_{n+1} - x_n + p \sum_{j=1}^{N} x_{n-k+(j-1)}l = 0,$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $p$ is a real number, and $k$, $l$, and $N$ are positive integers with $k > (N - 1)l$ is given as follows.

**Theorem 1.1.** Let $k$, $l$, and $N$ be positive integers with $k > (N - 1)l$. Then the zero solution of (1.1) is asymptotically stable if and only if

$$0 < p < \frac{2 \sin(\pi/2M) \sin(l\pi/2M)}{\sin(Nl\pi/2M)},$$

where $M = 2k + 1 - (N - 1)l$.

Theorem 1.1 generalizes asymptotic stability conditions given in [1, page 87], [2, 3, 5], and [6, page 65]. In this paper, we are interested in the situation when (1.1) does not depend on $x_n$, namely we are interested in the asymptotic stability of the linear delay difference equation of the form

$$x_{n+1} + p \sum_{j=1}^{N} x_{n-k+(j-1)}l = 0,$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $p$ is a real number, and $k$, $l$, and $N$ are positive integers with $k \geq (N - 1)l$. Our main theorem is the following.
Theorem 1.2. Let $k$, $l$, and $N$ be positive integers with $k \geq (N - 1)l$. Then the zero solution of (1.3) is asymptotically stable if and only if

$$- \frac{1}{N} < p < p_{\min},$$

where $p_{\min}$ is the smallest positive real value of $p$ for which the characteristic equation of (1.3) has a root on the unit circle.

2. Proof of theorem

The characteristic equation of (1.3) is given by

$$F(z) = z^{k+1} + p(z^{(N-1)l} + \cdots + z^l + 1) = 0.$$  (2.1)

For $p = 0$, $F(z)$ has exactly one root at 0 of multiplicity $k + 1$. We first consider the location of the roots of (2.1) as $p$ varies. Throughout the paper, we denote the unit circle by $C$ and let $M = 2k + 2 - (N - 1)l$.

Proposition 2.1. Let $z$ be a root of (2.1) which lies on $C$. Then the roots $z$ and $p$ are of the form

$$z = e^{w_m i},$$

$$p = (-1)^{m+1} \frac{\sin(lw_m/2)}{\sin(Nlw_m/2)} \equiv p_m$$

for some $m = 0, 1, \ldots, M - 1$, where $w_m = (2m/M)\pi$. Conversely, if $p$ is given by (2.3), then $z = e^{w_m i}$ is a root of (2.1).

Proof. Note that $z = 1$ is a root of (2.1) if and only if $p = -1/N$, which agrees with (2.2) and (2.3) for $w_m = 0$. We now consider the roots of (2.1) which lie on $C$ except the root $z = 1$. Suppose that the value $z$ satisfies $z^Nl = 1$ and $z^l \neq 1$. Then $z^{Nl} - 1 = (z^l - 1)(z^{(N-1)l} + \cdots + z^l + 1) = 0$ which gives $z^{(N-1)l} + \cdots + z^l + 1 = 0$, and hence $z$ is not a root of (2.1).

As a result, to determine the roots of (2.1) which lie on $C$, it suffices to consider only the value $z$ such that $z^Nl \neq 1$ or $z^l = 1$. For these values of $z$, we may write (2.1) as

$$p = -\frac{z^{k+1}}{z^{(N-1)l} + \cdots + z^l + 1}.$$  (2.4)

Since $p$ is real, we have

$$p = -\frac{\overline{z}^{k+1}}{\overline{z}^{(N-1)l} + \cdots + \overline{z}^l + 1} = -\frac{z^{-k-1+(N-1)l}}{z^{(N-1)l} + \cdots + z^l + 1},$$

where $\overline{z}$ denotes the conjugate of $z$. It follows from (2.4) and (2.5) that

$$z^{2k+2-(N-1)l} = 1$$

which implies that (2.2) is valid for $m = 0, 1, \ldots, M - 1$ except for those integers $m$ such that $e^{Nlw_m i} = 1$ and $e^{lw_m i} \neq 1$. We now show that $p$ is of the form stated in (2.3). There are two cases to be considered as follows.
Case 1. \( z \) is of the form \( e^{\omega_m i} \) for some \( m = 1, 2, \ldots, M - 1 \) and \( z^{N_l} \neq 1 \).

From (2.4), we have

\[
p = -\frac{z^{k+1}(z^l - 1)}{z^{N_l} - 1} = -\frac{e^{(k+1)\omega_m i}(e^{\omega_m i} - 1)}{e^{N_l\omega_m i} - 1} = -\frac{e^{(k+1-(N-1)(l/2))\omega_m i}(e^{\omega_m i/2} - e^{-\omega_m i/2})}{e^{N_l\omega_m i/2} - e^{-N_l\omega_m i/2}}
\]

\[
= -e^{(k+1-(N-1)(l/2))\omega_m i} \frac{\sin(l\omega_m/2)}{\sin(N_l\omega_m/2)}
\]

\[
= -e^{m\pi i} \frac{\sin(l\omega_m/2)}{\sin(N_l\omega_m/2)} = (-1)^{m+1} \frac{\sin(l\omega_m/2)}{\sin(N_l\omega_m/2)} \equiv p_m.
\]

Case 2. \( z \) is of the form \( e^{\omega_m i} \) for some \( m = 1, 2, \ldots, M - 1 \) and \( z^l = 1 \).

In this case, we have \( l\omega_m = 2q\pi \) for some positive integer \( q \). Then taking the limit of \( p_m \) as \( l\omega_m \to 2q\pi \), we obtain

\[
p = -\frac{(-1)^{m+q(N-1)}}{N}.
\]

From these two cases, we conclude that \( p \) is of the form in (2.3) for \( m = 1, 2, \ldots, M - 1 \) except for those \( m \) such that \( e^{N_l\omega_m i} = 1 \) and \( e^{\omega_m i} \neq 1 \).

Conversely, if \( p \) is given by (2.3), then it is obvious that \( z = e^{\omega_m i} \) is a root of (2.1). This completes the proof of the proposition. \( \square \)

From Proposition 2.1, we may consider \( p \) as a holomorphic function of \( z \) in a neighborhood of each \( z_m \). In other words, in a neighborhood of each \( z_m \), we may consider \( p \) as a holomorphic function of \( z \) given by

\[
p(z) = -\frac{z^{k+1}}{z^{(N-1)l} + \cdots + z^l + 1}.
\]

Then we have

\[
\frac{dp(z)}{dz} = -\frac{(k+1)z^k}{z^{(N-1)l} + \cdots + z^l + 1} + \frac{z^k \{(N-1)lz^{(N-1)l} + \cdots + lz^l\}}{(z^{(N-1)l} + \cdots + z^l + 1)^2}.
\]

From this, we have the following lemma.

**Lemma 2.2.** \( dp/dz \big|_{z = e^{\omega_m i}} \neq 0 \). In particular, the roots of (2.1) which lie on \( C \) are simple.

**Proof.** Suppose on the contrary that \( dp/dz \big|_{z = e^{\omega_m i}} = 0 \). We divide (2.10) by \( p(z)/z \) to obtain

\[
k + 1 = \frac{l\{(N-1)z^{(N-1)l} + \cdots + z^l\}}{z^{(N-1)l} + \cdots + z^l + 1} = 0.
\]

Substituting \( z \) by \( 1/z \) in (2.10), we obtain

\[
k + 1 = \frac{l\{(N-1) + (N-2)z^l + \cdots + z^{(N-2)l}\}}{z^{(N-1)l} + \cdots + z^l + 1} = 0.
\]
By adding (2.11) and (2.12), we obtain
\[ 2k + 2 - (N - 1)l = 0 \] (2.13)
which contradicts \( k \geq (N - 1)l \). This completes the proof. \( \square \)

From Lemma 2.2, there exists a neighborhood of \( z = e^{wm_i} \) such that the mapping \( p(z) \) is one to one and the inverse of \( p(z) \) exists locally. Now, let \( z \) be expressed as \( z = re^{i\theta} \). Then we have
\[
\frac{dz}{dp} = \frac{z}{r} \left\{ \frac{dr}{dp} + ir \frac{d\theta}{dp} \right\} \] (2.14)
which implies that
\[
\frac{dr}{dp} = \text{Re} \left\{ \frac{r}{z} \frac{dz}{dp} \right\} \] (2.15)
as \( p \) varies and remains real. The following result describes the behavior of the roots of (2.1) as \( p \) varies.

**Proposition 2.3.** The moduli of the roots of (2.1) at \( z = e^{wm_i} \) increase as \( |p| \) increases.

**Proof.** Let \( r \) be the modulus of \( z \). Let \( z = e^{wm_i} \) be a root of (2.1) on \( C \). To prove this proposition, it suffices to show that
\[
\frac{dr}{dp} \cdot p \bigg|_{z=e^{wm_i}} > 0. \] (2.16)
There are two cases to be considered.

**Case 1** \( (z'^{Nl} \neq 1) \). In this case, we have
\[
p(z) = -\frac{z^{k+1}(z^l - 1)}{z^{Nl} - 1} = -\frac{z^k f(z)}{z^{Nl} - 1}, \] (2.17)
where \( f(z) = z(z^l - 1) \). Then
\[
\frac{dp}{dz} = -\frac{z^{k-1}g(z)}{(z^{Nl} - 1)^2}, \] (2.18)
where \( g(z) = (kf(z) + zf'(z))(z^{Nl} - 1) - Nlz^{Nl} f(z) \). Letting \( w(z) = -(z^{Nl} - 1)^2/(z^k g(z)) \), we obtain
\[
\frac{dr}{dp} = \text{Re} \left\{ \frac{r}{z} \frac{dz}{dp} \right\} = r \text{Re}(w). \] (2.19)
We now compute \( \text{Re}(w) \). We note that
\[
f(z) = \frac{f(z)}{z^{i\pi/2}}, \quad f'(z) = \frac{h(z)}{z^l}, \] (2.20)
where $h(z) = l + 1 - z^l$. From the above equalities and as $z^M = 1$, we have

$$
\frac{z^k g(z)}{2} = \frac{1}{2} \left\{ (z^{Nl} - 1)^2 + \frac{(z^{Nl} - 1)^2}{z^k g(z)} \right\}
$$

It follows that

$$
\text{Re}(w) = \frac{w + \bar{w}}{2}
$$

Since

$$
2k f(z) + z(f'(z) - h(z)) - Nlf(z) = Mf(z),
$$

we obtain

$$
\text{Re}(w) = \frac{(z^{Nl} - 1)^4 M}{2z^{2Nl}|g(z)|^2} \frac{-z^k f(z)}{z^{Nl} - 1} = \frac{(z^{Nl} - 1)^4 M_p}{2z^{2Nl}|g(z)|^2}.
$$

The value of $\text{Re}(w)$ at $z = e^{\nu_{mi}}$ is

$$
\text{Re}(w) = \frac{(z^{Nl} - 1)^4 M_p}{2z^{2Nl}|g(z)|^2} = (2 \cos Nl \nu_m - 2)^2 \cdot \frac{M_p}{2 |g(z)|^2} > 0.
$$
Therefore,
\[
\frac{dr}{dp} = \frac{2r(\cos Nlw_m - 1)^2 M_p}{|g(z)|^2}
\] (2.26)
and it follows that (2.16) holds at \( z = e^{\text{wmi}} \).

**Case 2** (\( z^l = 1 \)). With an argument similar to Case 1, we obtain
\[
\frac{dr}{dp} = \frac{2rN^2 M_p}{|(M + 1)z - M + 1|^2}
\] (2.27)
which implies that (2.16) is valid for \( z = e^{\text{wmi}} \).

This completes the proof.

We now determine the minimum of the absolute values of \( p_m \) given by (2.3). We have the following result.

**Proposition 2.4.** \( |p_0| = \min\{|p_m| : m = 0, 1, \ldots, M - 1\} \).

To prove Proposition 2.4, we need the following lemma, which was proved in [4].

**Lemma 2.5.** Let \( N \) be a positive integer, then
\[
\left| \frac{\sin Nt}{\sin t} \right| \leq N
\] (2.28)
holds for all \( t \in \mathbb{R} \).

**Proof of Proposition 2.4.** From (2.3), \( p_m = (-1)^{m+1} \frac{(\sin(lw_m/2)/\sin(Nlw_m/2))}{M} \). For \( m = 0 \), it follows from L'Hospital's rule that \( p_0 = -1/N \). For \( m = 1, 2, \ldots, M - 1 \), we have
\[
|p_m| = \left| (-1)^{m+1} \frac{\sin(lw_m/2)}{\sin(Nlw_m/2)} \right| \geq \frac{1}{N}
\] (2.29)
by Lemma 2.5. This completes the proof.

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Note that \( F(1) = 1 + Np \leq 0 \) if and only if \( p \leq -1/N \). Since \( \lim_{z \to \infty} F(z) = +\infty \), it follows that (2.1) has a positive root \( \alpha \) such that \( \alpha > 1 \) when \( p \leq -1/N \). We claim that if \( |p| \) is sufficiently small, then all the roots of (2.1) are inside the unit disk. To this end, we note that when \( p = 0 \), (2.1) has exactly one root at 0 of multiplicity \( k + 1 \). By the continuity of the roots with respect to \( p \), this implies that our claim is true. By Proposition 2.4, \( p_0 = -1/N \) and \( |p_m| \geq 1/N \) which implies that \( |p_0| = 1/N \) is the smallest positive value of \( p \) such that a root of (2.1) intersects the unit circle as \( |p| \) increases. Moreover, Proposition 2.3 implies that if \( p > p_{\text{min}} \), then there exists a root \( \alpha \) of (2.1) such that \( |\alpha| \geq 1 \), where \( p_{\text{min}} \) is the smallest positive real value of \( p \) for which (2.1) has a root on \( C \). We conclude that all the roots of (2.1) are inside the unit disk if and only if \( -1/N < p < p_{\text{min}} \). In other words, the zero solution of (1.3) is asymptotically stable if and only if condition (1.4) holds. This completes the proof.
3. Examples

Example 3.1. In (1.3), let \( l \) and \( k \) be even positive integers, then we have

\[
F(-1) = -1 + pN.
\]  

(3.1)

Thus if \( p = 1/N \), then \( F(-1) = 0 \) and we conclude that (1.3) is asymptotically stable if and only if \(-1/N < p < 1/N\).

Example 3.2. In (1.3), let \( N = 3, l = 3, \) and \( k = 6 \). Then \( M = 8 \) and we obtain \( p_0 = -1/3, p_1 = \sin(3/8)\pi/\sin(9/8)\pi, p_2 = -\sin(3/4)\pi/\sin(9/4)\pi, p_3 = \sin(9/8)\pi/\sin(27/8)\pi, p_4 = -\sin(3/2)\pi/\sin(9/2)\pi, p_5 = \sin(15/8)\pi/\sin(45/8)\pi, p_6 = -\sin(9/4)\pi/\sin(27/4)\pi, \) and \( p_7 = \sin(21/8)\pi/\sin(63/8)\pi \). Thus, \( p_3 = p_5 = \sin(\pi/8)/\sin(3\pi/8) \) is the smallest positive real value of \( p \) such that (2.1) has a root on \( C \). We conclude that (1.3) is asymptotically stable if and only if \(-1/3 < p < \sin(\pi/8)/\sin(3\pi/8)\).

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