M_r -FACTORS AND Q_r -FACTORS FOR NEAR QUASINORM ON CERTAIN SEQUENCE SPACES

PIYAPONG NIAMSUP AND YONGWIMON LENBURY

Received 15 July 2004 and in revised form 20 June 2005

We study the multiplicativity factor and quadraticity factor for near quasinorm on certain sequence spaces of Maddox, namely, $l(p)$ and $l_\infty(p)$, where $p = (p_k)$ is a bounded sequence of positive real numbers.

1. Introduction

Let X be an algebra over a field F (R or C). A *quasinorm* on X is a function $|\cdot| : X \rightarrow R$ such that

- (i) $|0| = 0$,
- (ii) $|x| \geq 0$, for all $x \in X$,
- (iii) $|-x| = |x|$, for all $x \in X$,
- (iv) $|x + y| \leq |x| + |y|$, for all $x, y \in X$,
- (v) if $t_k \in F$, $|t_k - t| \rightarrow 0$, and $x_k, x \in X$, $|x_k - x| \rightarrow 0$, then $|t_k x_k - tx| \rightarrow 0$.

If $|\cdot|$ satisfies only properties (i) to (iv), then we call $|\cdot|$ a *near quasinorm*. If the quasinorm satisfies $|x| = 0$ if and only if $x = 0$, then it is said to be *total*.

A *quasinormed linear space* (QNLS) is a pair $(X, |\cdot|)$ where $|\cdot|$ is a quasinorm on X . If $(X, |\cdot|)$ is a quasinorm space, then the map $|\cdot| : X \rightarrow R$ is continuous. For $p > 0$, a p -seminorm on X is a function $\|\cdot\| : X \rightarrow R$ satisfying

- (i) $\|x\| \geq 0$, for all $x \in X$,
- (ii) $\|tx\| = |t|^p \|x\|$, for all $t \in F$, for all $x \in X$,
- (iii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

A seminorm is called a norm if it satisfies the following condition:

- (iv) $\|x\| = 0$ if and only if $x = 0$.

A p -seminormed linear space (p -semi-NLS) is a pair $(X, \|\cdot\|)$ where $\|\cdot\|$ is a seminorm on X . p -normed linear spaces (p -normed-LS) are defined similarly.

In [1, 2], *multiplicativity factors* (or M -factors) and *quadrativity factors* (or Q -factors) for seminorms on an algebra X have been introduced and studied in detail. A number $\mu > 0$ is said to be a multiplicativity factor for a seminorm S if and only if $S(xy) \leq \mu S(x)S(y)$, for all $x, y \in X$. Similarly, a number $\lambda > 0$ is said to be a quadrativity factor for S if and

only if $S(x^2) \leq \lambda S(x)^2$, for all $x \in X$. The necessary and sufficient conditions for existence of M -factor and Q -factor for S are answered in the following results.

THEOREM 1.1. *Let X be an algebra and let $S \neq 0$ be a seminorm on X . Then*

(a) *S has M -factors on X if and only if $\text{Ker } S$ is an ideal in X and*

$$\mu_{\inf} \equiv \sup \{S(xy) : x, y \in X, S(x) = S(y) = 1\} < +\infty, \quad (1.1)$$

(b) *if S has M -factors on X and $\mu_{\inf} > 0$, then $\mu_{\inf}$ is the best (least) M -factor for S ,*

(c) *if S has M -factors on X and $\mu_{\inf} = 0$, then μ is an M -factor for S if and only if $\mu > 0$.*

THEOREM 1.2. *Let X be an algebra and let $S \neq 0$ be a seminorm on X . Then*

(a) *S has Q -factors on X if and only if $\text{Ker } S$ is closed under squaring (i.e., $(\text{Ker } S)^2 \subset \text{Ker } S$) and*

$$\lambda_{\inf} \equiv \sup \{S(x^2) : x \in X, S(x) = 1\} < +\infty, \quad (1.2)$$

(b) *if S has Q -factors on X and $\lambda_{\inf} > 0$, then $\lambda_{\inf}$ is the best (least) Q -factor for S ,*

(c) *if S has Q -factors on X and $\lambda_{\inf} = 0$, then λ is a Q -factor for S if and only if $\lambda > 0$.*

If S is a norm, then $\text{Ker } S = \{0\}$. If in addition X is finite-dimensional, then a simple compactness argument shows that $\mu_{\inf}$ is finite. Therefore, by Theorem 1.1, norms on finite-dimensional algebras always have M -factors. If S is a seminorm on a finite-dimensional algebra X , then S has M -factors on X if and only if $\text{Ker } S$ is a (two-sided) ideal in X . In [1, 2] several examples of seminorms having M -factors and Q -factors are given. In [3], scalar multiplicativity factors for near quasinorms on certain sequence spaces of Maddox are studied. Motivated by these results we define M_r -factors and Q_r -factors for a near quasinorm q on an algebra X as follows.

A number $\mu > 0$ is an M_r -factor for q if and only if $q(txy) \leq \mu |t|^r q(x)q(y)$, there exists $r > 0$, for all $t \in F$, for all $x, y \in X$.

A number $\lambda > 0$ is a Q_r -factor for q if and only if $q(tx^2) \leq \lambda |t|^r q(x)^2$, there exists $r > 0$, for all $t \in F$, for all $x \in X$.

Let

$$\begin{aligned} \mu_{\inf} &\equiv \sup \left\{ \frac{q(txy)}{|t|^r q(x)q(y)} : t \in F - \{0\}, x, y \in X - \text{Ker } q \right\}, \\ \lambda_{\inf} &\equiv \sup \left\{ \frac{q(tx^2)}{|t|^r q(x)^2} : t \in F - \{0\}, x \in X - \text{Ker } q \right\}. \end{aligned} \quad (1.3)$$

2. M_r -factors and Q_r -factors for near quasinorms

In this section, we will prove the following theorems.

THEOREM 2.1. *Let X be an algebra over a field F ($F = \mathbb{C}$ or $\mathbb{R}$). Let q be a near quasinorm on X . Then*

(a) *q has M_r -factors on X if and only if $\text{Ker } q$ is a (two-sided) ideal in X and $\mu_{\inf} < +\infty$,*

(b) *if q has M_r -factors on X and $\mu_{\inf} > 0$, then $\mu_{\inf}$ is the best (least) M_r -factor for q ,*

(c) *if q has M_r -factors on X and $\mu_{\inf} = 0$, then μ is an M_r -factor for q if and only if $\mu > 0$.*

THEOREM 2.2. *Let X be an algebra over a field F ($F = C$ or R). Let q be a near quasinorm on X . Then*

- (a) q has Q_r -factors on X if and only if $\text{Ker } q$ is closed under squaring (i.e., $x^2 \in \text{Ker } q$, for all $x \in \text{Ker } q$) and $\lambda_{\inf} < +\infty$,
- (b) if q has Q_r -factors on X and $\lambda_{\inf} > 0$, then $\lambda_{\inf}$ is the best (least) Q_r -factors for q ,
- (c) if q has Q_r -factors on X and $\lambda_{\inf} = 0$, then λ is a Q_r -factors for q if and only if $\lambda > 0$.

Proof of Theorem 2.1. (a) Suppose that q has an M_r -factor μ on X . Clearly, $\text{Ker } q$ is a subspace of X . Now take any $x \in \text{Ker } q$ and $y \in X$. Then $q(xy) \leq \mu q(x)q(y) = 0$ which implies that $xy \in \text{Ker } q$. Similarly, $yx \in \text{Ker } q$, so $\text{Ker } q$ is a (two-sided) ideal in X . Now for $t \in F - \{0\}$ and $x, y \in X - \text{Ker } q$, we have $q(txy) \leq \mu |t|^r q(x)q(y)$ or $q(txy)/|t|^r q(x)q(y) \leq \mu$ which implies that $\mu_{\inf} \leq \mu < +\infty$. Conversely, suppose that $\text{Ker } q$ is a (two-sided) ideal in X and $\mu_{\inf} < +\infty$. If $t = 0$, $x \in \text{Ker } q$, or $y \in \text{Ker } q$, then $txy \in \text{Ker } q$, so $0 = q(txy) = \mu_{\inf} |t|^r q(x)q(y)$. If $t \neq 0$ and $x, y \notin \text{Ker } q$, then $q(txy)/|t|^r q(x)q(y) \leq \mu_{\inf}$ or $q(txy) \leq \mu_{\inf} |t|^r q(x)q(y)$. Therefore, $q(txy) \leq \mu_{\inf} |t|^r q(x)q(y)$, for all $t \in F$ and for all $x, y \in X$ which implies that q has M_r -factors on X .

(b) Let μ be an M_r -factor for q on X and $\mu_{\inf} > 0$. Then $q(txy) \leq \mu |t|^r q(x)q(y)$ for all $t \in F$ and for all $x, y \in X$. Therefore, $q(txy)/|t|^r q(x)q(y) \leq \mu$, for all $t \in F - \{0\}$ and for all $x, y \in \text{Ker } q$, so $\mu_{\inf} \leq \mu$.

(c) This part follows directly from definition of $\mu_{\inf}$ and M_r -factors for q on X . □

Proof of Theorem 2.2. The proof of this theorem is a simple modification of the proof of Theorem 2.1 and will be omitted. □

3. M_r -factors and Q_r -factors for near quasinorm on certain sequence spaces of Maddox

Let $p = (p_k)$ be a bounded sequence of positive real numbers. The sequence spaces of Maddox $l_{\infty}(p)$ and $l(p)$ are defined as follows:

$$\begin{aligned} l_{\infty}(p) &= \left\{ (x_k) : x_k \in C, \sup_k |x_k|^{p_k} < \infty \right\}, \\ l(p) &= \left\{ (x_k) : x_k \in C, \sum_k |x_k|^{p_k} < \infty \right\}. \end{aligned} \tag{3.1}$$

With the usual multiplication (i.e., $(x_k)(y_k) = (x_k y_k)$), both $l_{\infty}(p)$ and $l(p)$ are algebras over C . We define near quasinorms q_1 on $l_{\infty}(p)$ and q_2 on $l(p)$ as follows:

$$\begin{aligned} q_1((x_k)) &= \sup_k |x_k|^{p_k/M}, \quad (x_k) \in l_{\infty}(p), \\ q_2((x_k)) &= \left(\sum_k |x_k|^{p_k} \right)^{1/M}, \quad (x_k) \in l(p), \end{aligned} \tag{3.2}$$

where $M = \max\{1, \sup_k p_k\}$. We observe that q_1 and q_2 may or may not be quasinorms. For example, when $(p_k) = (1/k)$, then q_1 is a near quasinorm but not a quasinorm; if $(p_k) = (1 - 1/(k+1))$, then q_1 is a quasinorm.

In this section, we give necessary and sufficient conditions for sequence spaces $l_\infty(p)$ and $l(p)$ to have M_r -factors and Q_r -factors.

THEOREM 3.1. *Let $p = (p_k)$ and let M be defined as above. Then the following are equivalent.*

- (a) $p_0 = p_k = p_{k+1}$ for all $k \geq 0$ where p_0 is a positive real number.
- (b) q_1 has M_r -factors on $l_\infty(p)$.
- (c) q_1 has Q_r -factors on $l_\infty(p)$.
- (d) q_1 is a p_0/M -seminorm on $l_\infty(p)$.

THEOREM 3.2. *Let $p = (p_k)$ and let M be defined as above. Then the following are equivalent.*

- (a) $p_0 = p_k = p_{k+1}$ for all $k \geq 0$ where p_0 is a positive real number.
- (b) q_2 has M_r -factors on $l(p)$.
- (c) q_2 has Q_r -factors on $l(p)$.
- (d) q_2 is a p_0/M -seminorm on $l(p)$.

Proof of Theorem 3.1. (a) $\Rightarrow$ (b) If $p_0 = p_k = p_{k+1}$ for all $k \geq 1$, then

$$q_1(txy) = \sup_k |txy|^{p_k/M} = \sup_k |txy|^{p_0/M} \leq |t|^{p_0/M} q_1(x)q_1(y) \quad (3.3)$$

for all $x, y \in l_\infty(p)$, so q_1 has an M_r -factor on $l_\infty(p)$.

(b) $\Rightarrow$ (a) Assume that q_1 has M_r -factors on $l_\infty(p)$. This implies that

$$\mu_{\inf} = \sup \left\{ \frac{q_1(txy)}{|t|^r q_1(x)q_1(y)} : t \in F - \{0\}, x, y \in X - \text{Ker } q_1 \right\} < +\infty. \quad (3.4)$$

We shall show that $r = \sup_k p_k/M = \inf_k p_k/M$ which implies that $p_k = p_{k+1}$ for all $k \geq 1$. To this end we observe that

$$\begin{aligned} \mu_{\inf} &= \sup \left\{ \frac{q_1(txy)}{|t|^r q_1(x)q_1(y)} : t \in F - \{0\}, x, y \in X - \text{Ker } q_1 \right\} \\ &\geq \sup \left\{ \frac{q_1(txy)}{|t|^r q_1(x)q_1(y)} : t \in F - \{0\}, x, y = (1, 1, 1, \dots) \right\} \\ &\geq \sup \left\{ \frac{\sup_k |t|^{p_k/M}}{|t|^r} : t \in F, |t| \geq 1 \right\} = \sup \left\{ |t|^{\sup_k p_k/M} : t \in F, |t| \geq 1 \right\} \end{aligned} \quad (3.5)$$

so that

$$\mu_{\inf} \geq \sup \left\{ \frac{|t|^{\sup_k p_k/M}}{|t|^r} : t \in F, |t| \geq 1 \right\}. \quad (3.6)$$

If $r < \sup_k p_k/M$, then $\mu_{\inf} = +\infty$ which is a contradiction. Therefore, $r \geq \sup_k p_k/M$. Similarly, we can show that $r \leq \inf_k p_k/M$ from which it follows that $r = \sup_k p_k/M = \inf_k p_k/M$ and the proof is complete.

- (a) $\Rightarrow$ (c) The same proof as (a) $\Rightarrow$ (b).
- (c) $\Rightarrow$ (a) The same proof as (b) $\Rightarrow$ (a).

(d) $\Rightarrow$ (b) This is obvious.

(b) $\Rightarrow$ (d) Assume that q_1 has M_r -factors. Then, by (a), $p_0 = p_k = p_{k+1}$ for all $k \geq 0$ where p_0 is a positive real number. Moreover, we have

$$q_1(txy) = \sup_k |t \cdot (x_k)(y_k)|^{p_0/M} = |t|^{p_0/M} \sup_k |x_k y_k|^{p_0/M} = |t|^{p_0/M} q_1(xy) \quad (3.7)$$

for all $x = (x_k)$, $y = (y_k) \in l_\infty(p)$ and all $t \in F$. Putting $y = (1, 1, 1, \dots)$ we see that

$$q_1(tx) = |t|^{p_0/M} q_1(x) \quad (3.8)$$

and the proof is complete. $\square$

Proof of Theorem 3.2. The proof is almost the same as in Theorem 3.1 and will be omitted. $\square$

Remark 3.3. If the algebra X has an identity element x_0 for multiplication and $q \neq 0$ is a near-quasinorm on X which has an M_r -factor on X , then we obtain $q(x_0) > 0$, $\mu_{\inf} \geq 1/q(x_0)$ and

$$\frac{1}{q(x_0)\mu_{\inf}} |t|^r q(xy) \leq q(tx y) \leq \mu_{\inf} |t|^r q(x) q(y) \quad (3.9)$$

for all $x, y \in X$ and all $t \in F$.

References

- [1] R. Arens and M. Goldberg, *Multiplicativity factors for seminorms*, J. Math. Anal. Appl. **146** (1990), no. 2, 469–481.
- [2] ———, *A class of seminorms on function algebras*, J. Math. Anal. Appl. **162** (1991), no. 2, 592–609.
- [3] S. Suantai, *Scalar multiplicative factors for near quasi-norms*, Bull. Calcutta Math. Soc. **90** (1998), no. 3, 183–190.

Piyapong Niamsup: Department of Mathematics, Faculty of Science, Chiangmai University, Chiangmai 50200, Thailand

E-mail address: scipnmsp@chiangmai.ac.th

Yongwimon Lenbury: Department of Mathematics, Faculty of Science, Mahidol University, Bangkok 10400, Thailand

E-mail address: scylb@mahidol.ac.th