Some conditions equivalent to a strong quasi-divisor property (SQDP) for a partly ordered group $G$ are derived. It is proved that if $G$ is defined by a family of $t$-valuations of finite character, then $G$ admits an SQDP if and only if it admits a quasi-divisor property and any finitely generated $t$-ideal is generated by two elements. A topological density condition in topological group of finitely generated $t$-ideals and/or compatible elements are proved to be equivalent to SQDP.

1. Introduction

Let $G$ be a partly ordered commutative group ($\text{po}$-group). Then $G$ is said to have a quasi-divisor property if there exist commutative lattice-ordered group ($\text{l}$-group) $(\Gamma, \cdot, \wedge)$ and an order isomorphism $h$ (the so-called quasi-divisor morphism) from $G$ into $\Gamma$ such that for any $\alpha \in \Gamma$, there exist $g_1, \ldots, g_n \in G$ such that $\alpha = h(g_1) \wedge \cdots \wedge h(g_n)$. Moreover, if this embedding $h$ satisfies the condition

$$\forall \alpha, \beta \in \Gamma^+ \, \exists \gamma \in \Gamma^+ \, \alpha \cdot \gamma \in h(G), \quad \beta \wedge \gamma = 1,$$

then $G$ is said to have a strong quasi-divisor property. Many papers have dealt with $\text{po}$-groups with (strong) quasi-divisor property (e.g., see [1, 3, 4, 5, 6, 7, 8]). It is well known that there are some generic examples of such $\text{l}$-group $\Gamma$. Namely, if $h : G \to \Gamma$ is a quasi-divisor morphism, then $\Gamma$ is o-isomorphic to the group $(\mathcal{F}_t(G), \times_t)$ of finitely generated $t$-ideals of $G$. Recall that a $t$-ideal $X_t$ of $G$ generated by a lower bounded subset $X \subseteq G$ is a set $X_t = \{ g \in G : (\forall s \in G) \ s \leq X \Rightarrow g \geq s \}$. Then the set $\mathcal{F}_t(G)$ of all finitely generated $t$-ideals of $G$ is a semigroup with operation $\times_t$ defined such that $X_t \times_t Y_t = (X \cdot Y)_t$ (see [2]). It is clear that a map $d : G \to \mathcal{F}_t(G)$ defined by $d(g) = \{ g \}_t$ is an embedding. Another example of a group $\Gamma$ is a group $\mathcal{K}(W)$ of compatible elements of a defining family of $t$-valuations $W$ (see the definitions below). In this note, we want to show that properties of a group $\mathcal{K}(W)$ can be used for deriving new conditions under which quasi-divisor property is also a strong quasi-divisor property.
Let $w : G \to G_1$ be an $o$-homomorphism. Then, $w$ is called $t$-homomorphism if $w(X) \subseteq (w(X))_t$ for any lower bounded subset $X \subseteq G$. Moreover, if $G_1$ is a totally ordered group (i.e., $o$-group), then $w$ is called $t$-valuation. Recall that a family $W$ of $t$-valuations $w : G \to G_w$ is called a defining family for $G$ if

$$(\forall g \in G) \quad g \geq 1 \iff (\forall w \in W) \quad w(g) \geq 1.$$  

(1.2)

We say that $W$ is of finite character if

$$(\forall g \in G) \quad (\forall' w \in W) \quad w(g) = 1,$$  

(1.3)

where $\forall'$ means “for all but a finite number.” Hence any defining family $W$ of finite character creates an embedding of $G$ into a sum $\sum_{w \in W} G_w$ of $o$-groups $G_w, w \in W$. Then a quasi-divisors property of $G$ is said to be of finite character, if there exists a defining family of $t$-valuations of finite character for $G$. If $w_1, w_2$ are two $t$-valuations of a po-group $G$, then $w_1$ is said to be coarser than $w_2$ ($w_1 \geq w_2$) if there exists an $o$-epimorphism $d_{w_1,w_2} : G_{w_1} \to G_{w_2}$ such that $w_2 = d_{w_1,w_2}w_1$. It may be then proved that for any two $t$-valuations $w_1, w_2$, there exists a $t$-valuation $w_1 \wedge w_2$ which is the infimum of $w_1, w_2$ with respect to this preorder relation. Then, $d_{w_1,w_1 \wedge w_2}$ (resp., $d_{w_2,w_1 \wedge w_2}$) is an $o$-epimorphism such that $w_1 \wedge w_2 = d_{w_1,w_1 \wedge w_2}, w_1 = d_{w_2,w_1 \wedge w_2}$. For simplicity, we set $d_{w_1,w_2} = d_{w_1,w_1 \wedge w_2}, d_{w_2,w_1} = d_{w_2,w_1 \wedge w_2}$ (see the difference between $d_{w_1,w_2}$ and $d_{w_2,w_2}$). If $W$ is a system of $t$-valuations $w : G \to G_w$ of a po-group $G$ and $W' \subseteq W$, then a system $(g_w)_w \in \prod_{w \in W} G_w$ of elements is called compatible provided that $d_{w,v}(g_w) = d_{v,w}(g_v)$ for all $w, v \in W'$. Finally, $(g_w)_{w \in W'}$ is called $W'$-complete if $\bigcup_{w \in W'} W(g_w) = W'$, where $W(g_w) = \{v \in W : d_{w,v}(g_w) \neq 1\}$ for $g_w \neq 1_w$ and $W(1_w) = \{w\}$ for any $w \in W$.

Let $W$ be a defining family of $t$-valuations of $G$. Then, we set

$$\mathcal{K}(W) = \left\{ (a_w)_w \in \prod_{w \in W} G_w : (a_w)_w \text{ is compatible} \right\}.$$  

(1.4)

It can be proved that $\mathcal{K}(W)$ is an $l$-subgroup in $\prod_{w \in W} G_w$ (see [8]). Now we say that $G$ with a defining family of $t$-valuations satisfies the positive weak approximation theorem (PWAT) if for any finite subset $F \subseteq W$ and any compatible system $(\alpha_w)_{w \in F} \in \prod_{w \in F} G_w^*$, there exists $g \in G^*$ such that $w(g) = \alpha_w, w \in F$. Finally, we say that $G$ with $W$ satisfies the approximation theorem (AT) if for any finite subset $F \subseteq W$ and any compatible and $F$-complete system $(\alpha_w)_{w \in F} \in \prod_{w \in F} G_w$, there exists $g \in G$ such that

$$w(g) = \alpha_w, \quad w \in F,$$

$$w(g) \geq 1, \quad w \in W \setminus F.$$  

(1.5)

2. Results

In the theory of quasi-divisors of a po-group, a $t$-ideal theory has an important position. In the next propositions, we want to show that all $t$-ideals in a po-group $G$ with a quasi-divisor property of finite character can be derived from the set of compatible elements $\mathcal{K}(W)$ of $G$, where $W$ is some defining family of $t$-valuations of $G$. 
Lemma 2.1. Let \((\alpha_w)_w \in \mathcal{K}(W)\) and let \(W_0 = \{ w \in W : \alpha_w \neq 1 \} \). Then \((\alpha_w)_{w \in W'}\) is \(W'\)-complete for any \(W_0 \subseteq W' \subseteq W\).

Proof. Let \(v \in \bigcup_{w \in W} W(\alpha_w)\). Then there exists \(w \in W_0\) such that \(v \in W(\alpha_w)\). Because \((\alpha_u, \alpha_v)\) is compatible, we have \(1 \neq d_{uv}(\alpha_w) = d_{uv}(\alpha_v)\) and it follows that \(\alpha_v \neq 1\). Hence, \(v \in W_0 \subseteq W'\).

Proposition 2.2. Let \(G\) be a po-group with a quasi-divisor property of finite character and let \(W\) be a defining family of \(t\)-valuations of \(G\). Let \((\alpha_w)_w \in \mathcal{K}(W)\). Then \(X = \{ g \in G : (\forall w \in W) w(g) \geq \alpha_w \}\) is a finitely generated \(t\)-ideal of \(G\).

Proof. Because the \(t\)-system is defined by a family \(W\) of \(t\)-valuations, according to [8, Theorem 2.6], the group \(\mathcal{K}(W)\) is \(o\)-isomorphic to a Lorenzen \(l\)-group \(\Lambda_t(G)\). It follows that a map \(d : G \to \mathcal{K}(W)\) such that \(d(g) = (w(g))_w\) is a quasi-divisors morphism. Then for any \((\alpha_w)_w \in \mathcal{K}(W)\), there exist \(g_1, \ldots, g_n \in G\) such that \(d(g_1) \wedge \cdots \wedge d(g_n) = (\alpha_w)_w\). Then \(X = (g_1, \ldots, g_n)_t\). In fact, for \(g \in X\), we have \(w(g) \geq \alpha_w\) and it follows that \(w(g) \in (w(g_1), \ldots, w(g_n))_t\). Because the \(t\)-system is defined by \(W\), we have \(g \in (g_1, \ldots, g_n)_t\), analogously for the other inclusion.

Corollary 2.3. Let \(G\) be a po-group with a quasi-divisor property of finite character and let \(W\) be a defining family of \(t\)-valuations of \(G\). Then there exists an \(o\)-isomorphism

\[
\sigma : \mathcal{K}(W) \to \mathcal{I}^t(G)
\]

such that for \((\alpha_w)_w \in \mathcal{K}(W)\) and \(J \in \mathcal{I}^t(G)\),

\[
\sigma((\alpha_w)_w) = \{ g \in G : (\forall w \in W) w(g) \geq \alpha_w \},
\]

\[
\sigma^{-1}(J) = (\{ w \in W : w(g) \geq \alpha \})_w.
\]

It is well known that the existence of quasi-divisor property is equivalent to the existence of a defining family of essential \(t\)-valuations (see [3, Theorem 2.1]). Recall that a \(t\)-valuation \(w\) of \(G\) is essential if \(\ker w\) is a directed subgroup of \(G\) and \(w\) is an \(o\)-epimorphism.

Lemma 2.4. Let \(w, v\) be essential \(t\)-valuations of \(G\) and let \(\alpha \in G_v\) be such that \(d_{vw}(\alpha) = 1\). Then there exists \(g \in G\) such that \(w(g) = 1\), \(v(g) \geq \alpha\).

Proof. We may assume that \(\alpha > 1\). Let \(J = \{ x \in G : v(x) \geq \alpha \}\). Let us suppose on contrary that the statement of the lemma is not true. Then for any \(x \in J\), we have \(w(x) > 1\). Let \(H\) be the largest convex subgroup in \(G\) such that \(\alpha \notin H\) and let \(w' : G \twoheadrightarrow G_v\to G_v/H\) be the composition of \(v\) and canonical morphism. Then \(w' \leq w\). In fact, let \(x \in G, x \geq 1\) be such that \(w'(x) > 1\). Because \(w'(x) = v(x)H\), we have \(v(x) \notin H, v(x) > 1\). Then there exists \(n \in \mathbb{N}\) such that \(v(x)^n \geq \alpha\). In fact, if \(v(x)^n < \alpha\) for all \(n \in \mathbb{N}\), then the convex subgroup \(H'\) generated by \(H \cup \{ v(x) \}\) does not contain \(\alpha\) and \(H \leq H'\). On the other hand, we have \(v(x) \in H' \setminus H\), a contradiction. Then \(x^n \in J\) for some \(n \in \mathbb{N}\) and according to the assumption, we have \(w(x)^n > 1\). Hence \(w(x) > 1\) and we proved the implication

\[
x \in G, \quad x \geq 1, \quad w'(x) > 1 \implies w(x) > 1.
\]
Let $\rho : G_w \to G_{w'}$ be defined by $\rho(w(g)) = w'(g)$. Then $\rho$ is well defined. In fact, let $w(x) = w(y)$. Since $w$ is essential, there exists $t \in \ker w$ such that $t \geq 1, xy^{-1}$. If $w'(x) \neq w'(y)$, we have, for example, $w'(xy^{-1}) > 1$. Then $w'(t) \geq w'(xy^{-1}) > 1$. According to (2.3), we have $w(t) > 1$, a contradiction with $t \in \ker w$. Thus $w' = \rho \circ w$ and $w' \leq w$. Then, we have also $w' \leq w \land v$. For any $b \in G$ such that $\alpha = v(b)$, we obtain $w'(b) = v(b)H = aH \neq 1$ and $v \land w(b) = d_{vw} > v(b) = d_{vw}(\alpha) = 1$, a contradiction, because $v \land w \geq w'$.

**Lemma 2.5.** Let $w_1, \ldots, w_n$ be essential $t$-valuations of $G$ and let $(\alpha_1, \ldots, \alpha_n) \in \prod_{i=1}^n G_{w_i}^+$ be compatible elements. Then there exists $a_1 \in G$, $a_1 \geq 1$, such that

$$\forall j \neq 1, \quad w_1(a_1) = \alpha_1, \quad w_j(a_1) > \alpha_j. \quad (2.4)$$

**Proof.** The proof will be done by the induction with respect to $n$. For $n = 1$, the proof is trivial. Let us assume that the statement is true for any compatible set of $n - 1$ elements. Let us assume firstly that $w_1 < w_k$ for some $k \neq 1$. According to the induction assumption, there exists $a \in G_+$ such that

$$\forall j \neq k, 1, \quad w_k(a) = \alpha_k, \quad w_j(a) > \alpha_j. \quad (2.5)$$

Because $w_1 < w_k$, there exists an $o$-epimorphism $\sigma : G_{w_k} \to G_{w_1}$ such that $w_1 = \sigma \circ w_k$. Since $(\alpha_1, \alpha_k)$ is compatible, we have $\sigma(\alpha_k) = \alpha_1$. Since $\ker \sigma \neq \{1\}$, there exists $\delta \in \ker \sigma$, $\delta > 1$. From the fact that $w_k$ is essential, it follows that there exists $g \in G$, $g > 1$, such that $w_k(g) = \delta$. We set $a_1 = ga$. Then, we have

$$w_1(a_1) = \sigma \circ w_k(ga) = \sigma(\delta) \circ \sigma(\alpha_k) = \alpha_1,$$

$$w_k(a_1) = \delta \circ \alpha_k > \alpha_k,$$

$$\forall i \neq k, i \geq 2, \quad w_i(a_1) \geq w_i(a) > \alpha_i. \quad (2.6)$$

Let us assume now that $w_1 \parallel w_j$, $j \geq 2$. Then $w_j \neq w_1 \land w_j$ and for any $j \geq 2$, there exists $\delta_j \in \ker d_{ji}, \delta_j > 1$. According to Lemma 2.4, for any $j \geq 2$, there exists $g_j \in G_+$ such that $w_1(g_j) = 1$, $w_j(g_j) \geq \delta_j$. We set $g_1 = \prod_{j \geq 2} g_j$. Then

$$\forall j \geq 2, \quad w_1(g_1) = 1, \quad w_j(g_1) \geq w_j(g_j) \geq \delta_j > 1. \quad (2.7)$$

According to the induction assumption, there exists $a_1 \in G_+$ such that

$$\forall 2 \leq j \leq n - 1, \quad w_1(a_1) = \alpha_1, \quad w_j(a_1) > \alpha_j. \quad (2.8)$$

Without the loss of generality, we may assume that

$$\forall 2 \leq j, \quad w_1(a_1) = \alpha_1, \quad w_j(a_1) \geq \alpha_j. \quad (2.9)$$
In fact, if \( w_n(a_1) < \alpha_n \), then \( d_{n1}(\alpha_n \cdot w_n^{-1}(a_1)) = d_{1n}(\alpha_1) \cdot d_{1n}(w_1^{-1}(a_1)) = 1 \) and according to Lemma 2.4, there exists \( a_1' \in G_1 \) such that \( w_1(a_1') = 1, w_n(a_1') \geq \alpha \cdot w_n^{-1}(a_1) \). Then for \( a_1'' = a_1a_1' \), we have

\[
\begin{align*}
w_1(a_1'') &= w_1(a_1a_1') = \alpha_1, \\
\forall n > j \geq 2, \quad w_j(a_1'') &\geq w_j(a_1) > \alpha_j, \quad (2.10) \\
w_n(a_1'') &\geq \alpha_n.
\end{align*}
\]

We set \( c_1 = a_1g_1 \), where \( a_1 \) satisfies the relation (2.9). Then we have

\[
\begin{align*}
w_1(c_1) &= w_1(a_1) = \alpha_1, \\
w_j(c_1) &> w_j(a_1) \geq \alpha_j, \quad j \geq 2. \quad \Box
\end{align*}
\]

If \( G \) admits a quasi-divisor property of finite character, the existence of a map

\[
\sigma : \mathcal{H}(W) \rightarrow \mathcal{J}_1^f(G)
\]

follows immediately from Proposition 2.2. Between the \( l \)-group of compatible elements \( \mathcal{H}(W) \) and a semigroup \( \mathcal{J}_1^f(G) \) of finitely generated \( t \)-ideals of any \( po \)-group \( G \), there exists another naturally defined map, namely,

\[
\tau : \mathcal{J}_1^f(G) \rightarrow \mathcal{H}(W)
\]

such that \( \tau(X_t) = (\wedge w(X))_{w \in W} = (\wedge w(X_t))_{w \in W} \in \mathcal{H}(W) \). \( \tau \) is well defined and it can be proved easily that \( \tau \) is a semigroup monomorphism (because \( t \)-ideals are defined by \( W \)). If \( G \) admits a quasi-divisor property of finite character, then \( \sigma \) and \( \tau \) are mutually inverse \( o \)-isomorphisms (see Corollary 2.3). Moreover, if \( h : G \rightarrow \mathcal{J}_1^f(G) \) and \( d : G \rightarrow \mathcal{H}(W) \) are natural embedding maps such that \( h(g) = (g)_t \) and \( d(g) = (w(g))_{w \in W} \), then the following diagram commutes:

\[
\begin{tikzcd}
\mathcal{J}_1^f(G) \arrow{r}{\tau} \arrow[swap]{d}{h} & \mathcal{H}(W) \arrow{d}{d} & \mathcal{J}_1^f(G) \arrow{l}{\sigma} \arrow[swap]{d}{h} \\
G & G & G
\end{tikzcd}
\]

In the group \( \mathcal{H}(W) \), a group topology \( \mathcal{T}_W \) can be defined such that \( \ker \hat{w} = \{(\alpha_v)_v \in \mathcal{H}(W) : \alpha_w = 1\} \) is a subbase of neighborhoods of 1 for any \( w \in W \) (clearly, \( \hat{w} : \mathcal{H}(W) \rightarrow G_w \) is the projection map). Then the semigroup monomorphism \( \tau : \mathcal{J}_1^f(G) \rightarrow \mathcal{H}(W) \) induces a semigroup topology \( \mathcal{T}_W \) on \( \mathcal{J}_1^f(G) \). If for \( w \in W \), we define a map \( \hat{w} : \mathcal{J}_1^f(G) \rightarrow G_w \) such that \( \hat{w}(X_t) = \wedge w(X) = (\wedge w(X_t)) \), then for any finite \( F \subseteq W \), we obtain

\[
\tau^{-1}
\left( \bigcap_{w \in F} \ker \hat{w} \right) = \bigcap_{w \in F} \ker \hat{w}.
\]
Hence, the topology $\mathcal{T}_W$ can be defined by maps $\tilde{\omega}, w \in W$. Moreover, in the ordered semigroup $(\mathcal{J}_t^f(G), \times_t, \leq_t)$, where $X_t \subseteq Y_t$ if $Y_t \subseteq X_t$, a $t$-ideals structure can be defined analogously as in any po-group. The following lemma shows that the topology $\mathcal{T}_W$ is defined also by $t$-valuations.

**Lemma 2.6.** For any $w \in W$, $\tilde{\omega}$ is a $(t,t)$-morphism from $(\mathcal{J}_t^f(G), \times_t, \leq_t)$ to $G_w$.

**Proof.** Let $\mathcal{X}_t$ be a $t$-ideal in $\mathcal{J}_t^f(G)$ generated by a lower bounded subset $\mathcal{X}$ and let $X_t \in \mathcal{X}_t$. Then there exists a finite set $\mathcal{F} \subseteq \mathcal{X}$ such that $X_t \in \mathcal{F}_t \subseteq \mathcal{T}_W$. We set $S = \cup_{F_t \in \mathcal{F}} F_t$. Then, $S$ is a finite subset in $G$ and $S_t \subseteq F_t$ for any $F_t \in \mathcal{F}$. Hence, $X_t$ is a $t$-ideal such that $\wedge w(X_t) = \wedge w(S_t) = \wedge w(S)$. Thus $\tilde{\omega}(X_t) \in (\tilde{\omega}(S_t))_t = (\wedge_{F_t \in \mathcal{F}} \tilde{\omega}(F_t))_t = (\tilde{\omega}(\mathcal{F}))_t$. □

**Theorem 2.7.** Let $G$ be defined by a family of $t$-valuations of finite character. Then the following statements are equivalent.

1. $G$ admits a strong quasi-divisor property.
2. $G$ admits a quasi-divisor property and for any $(\alpha_w)_w \in \mathcal{H}(W)$ and $a \in G$ such that $\alpha_w \leq w(a)$ for all $w \in W$, there exists $b \in G$ such that $\alpha_w = w(a) \wedge w(b)$ for all $w \in W$.
3. $G$ admits a quasi-divisor property and for any $X_t \in \mathcal{J}_t^f(G)$ and $a \in X_t$, there exists $b \in G$ such that $X_t = (a, b)_t$.

If $W$ is an infinite set, then these statements are equivalent to the following equivalent statements.

4. $G$ admits a quasi-divisor property and $h(G)$ is dense in $(\mathcal{J}_t^f(G), \mathcal{T}_W)$.
5. $d(G)$ is dense in $(\mathcal{H}(W), \mathcal{T}_W)$.

**Proof.** (1) $\Rightarrow$ (2) Let $(\alpha_w)_w \in \mathcal{H}(W), a \in G$ such that $w(a) \geq \alpha_w$ for all $w \in W$. Let $W_1 = \{w \in W : \alpha_w \neq 1\} \cup \{v \in W : v(a) \neq 1\}$. According to Lemma 2.1, $(\alpha_w)_{w \in W_1}$ is compatible and $W_1$-complete and according to AT, there exists $b \in G$ such that

$$w(b) = \alpha_w, \quad w \in W_1,$$

$$w(b) \geq 1, \quad w \in W \setminus W_1. \tag{2.16}$$

Then for $w \in W_1$, we have $w(a) \wedge w(b) = w(a) \wedge \alpha_w = \alpha_w$, and for $w \in W \setminus W_1$,

$$w(a) \wedge w(b) = 1 \wedge w(b) = 1 = \alpha_w.$$

(2) $\Rightarrow$ (3) Let $a \in X_t \in \mathcal{J}_t^f(G)$. Because $t$-system is defined by $W$, we have $X_t = \{g \in G : w(g) \geq \wedge w(X), w \in W\}$. According to [3, Lemma 2.9], $(\wedge w(X))_w \in \mathcal{H}(W)$ and there exists $b \in G$ such that $\wedge w(X) = w(a) \wedge w(b)$, for all $w \in W$. Then we have $X_t = \{g \in G : w(g) \in (w(a), w(b))_t, w \in W\} = (a, b)_t$.

(3) $\Rightarrow$ (1) We show that $G$ satisfies the positive weak approximation theorem (PWAT). Let $(\alpha_1, \ldots, \alpha_n) \in \prod_{i=1}^n G_{w_i}^+$ be compatible. According to Lemma 2.5, there exist $a_1, \ldots, a_n \in G_+$ such that

$$\forall i, \forall j \neq i, \quad w_i(a_i) = \alpha_i, \quad w_j(a_i) > \alpha_j. \tag{2.17}$$
We set $b = a_1 \cdots a_n$. Then $b \in (a_1, \ldots, a_n)_t$. Hence, there exists $a \in G_+$ such that $(a_1, \ldots, a_n)_t = (a,b)_t$. Then for any $i$, we have

$$w_i(b) = \alpha_i \cdot \prod_{j \neq i} w_i(a_j) > \alpha_i^n \geq \alpha_i.$$  \hfill (2.18)

Let us assume that there exists $i$ such that $w_i(b) < w_i(a)$. Since $a_i \in (a,b)_t$, we have $\alpha_i = w_i(a_i) \geq w_i(a) \wedge w_i(b) = w_i(b)$, a contradiction. Then we have $\alpha_i = w_i(a_i) \geq w_i(a) \wedge w_i(b) = w_i(a_i)$. Since $a \in (a_1, \ldots, a_n)_t$, we have $w_i(a) \geq w_i(a_1) \wedge \cdots \wedge w_i(a_n) = \alpha_i \wedge \bigwedge_{j \neq i} w_i(a_j) = \alpha_i$. Thus $w_i(a) = \alpha_i$, $i = 1, \ldots, n$ and $G$ satisfies the PWAT. According to [7, Theorem 3.5], $G$ admits a strong quasi-divisor property.

Now let $W$ be an infinite set.

(1)⇒(4) Since $G$ admits a quasi-divisor property, $(\mathcal{F}_G(G), \times_t)$ is a group and the sub-base of neighborhoods of unity in topology $\mathcal{T}_W$ is $\{ \ker \hat{w} : w \in W \}$. We show that a map $\sigma : \mathcal{H}(W) \to \mathcal{F}_G(G)$ is a homeomorphism. Let $a, b \in \mathcal{H}(W)$. Then there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in G$ such that $a = d(a_1) \wedge \cdots \wedge d(a_n)$, $b = d(b_1) \wedge \cdots \wedge d(b_m)$ and we have $\sigma(a) = (a_1, \ldots, a_n)$, $\sigma(b) = (b_1, \ldots, b_m)$. Then $a \cdot b = (a_1 b_1) \wedge \cdots \wedge (a_n b_m)$ and $\sigma(a \cdot b) = (a_1 b_1, \ldots, a_n b_m) = \sigma(a) \cdot \sigma(b)$. If $\sigma(a) = (1)$, then $(a_1, \ldots, a_n) = (1)$, and it follows easily that $a = 1$. It is clear that $\sigma$ is also homeomorphism. According to [8, Theorem 2.6], there exists an $o$-isomorphism $\psi$ such that the following diagram commutes:

$$\begin{array}{ccc}
\Lambda_t(G) & \xrightarrow{\psi} & \mathcal{H}(W) \\
\downarrow \phi & & \downarrow \hat{w} \\
G_w & = & G_w
\end{array}$$  \hfill (2.19)

where $\hat{w}$ is a canonical extension of $w$. Since $G \to \Lambda_t(G)$ is a strong quasi-divisor morphism, it follows that $d : G \to \mathcal{H}(W)$ is a strong quasi-divisor morphism as well. Then, according to [5, Theorem 2.9], $d(G)$ is dense in $(\mathcal{H}(W), \mathcal{T}_W)$ and it follows that $h(G)$ is also dense in $(\mathcal{F}_G(G), \mathcal{T}_W)$.

(4)⇒(5) If $G$ admits a quasi-divisor property, then $\mathcal{F}_G(G)$ is $o$-isomorphic to $\Lambda_t(G)$ and according to [8, Theorem 6], $d(G)$ is dense in $(\mathcal{H}(W), \mathcal{T}_W)$ and it can be proved easily that $(\mathcal{F}_G(G), \mathcal{T}_W)$ is also homeomorphic to $(\mathcal{H}(W), \mathcal{T}_W)$.

(5)⇒(1) It follows directly from [5, Theorem 2.9].

References


Compatible elements in partly ordered groups


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