ON THE MAXIMUM MODULUS OF A POLYNOMIAL AND ITS DERIVATIVES

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Let $f(z)$ be an arbitrary entire function and $M(f, r) = \max_{|z|=r} |f(z)|$. For a polynomial $P(z)$ of degree n , having no zeros in $|z| < k$, $k \geq 1$, Bidkham and Dewan (1992) proved $\max_{|z|=r} |P'(z)| \leq (n(r+k)^{n-1}/(1+k)^n) \max_{|z|=1} |P(z)|$ for $1 \leq r \leq k$. In this paper, we generalize as well as improve upon the above inequality.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $M(P, r) = \max_{|z|=r} |P(z)|$, then according to Bernstein's inequality

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \quad (1.1)$$

The result is best possible and equality in (1.1) is obtained for $P(z) = \alpha z^n$, $\alpha \neq 0$.

If we restrict ourselves to the class of polynomials not vanishing in $|z| < 1$, then Erdős conjectured and Lax [4] proved

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \quad (1.2)$$

Inequality (1.2) is best possible and the extremal polynomial is $P(z) = \alpha + \beta z^n$ with $|\alpha| = |\beta|$.

As an extension of (1.2), Malik [5] proved the following.

THEOREM 1.1. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \quad (1.3)$$

The result is best possible and equality holds for $P(z) = (z+k)^n$.

Further, as a generalization of (1.3), Bidkham and Dewan [1] proved the following theorem.

THEOREM 1.2. If $P(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $1 \leq \rho \leq k$,

$$\max_{|z|=\rho} |P'(z)| \leq \frac{n(\rho+k)^{n-1}}{(1+k)^n} \max_{|z|=1} |P(z)|. \quad (1.4)$$

The result is best possible and equality in (1.4) holds for $P(z) = (z+k)^n$.

In this paper, we obtain the following result which is a generalization as well as an improvement of Theorem 1.2.

THEOREM 1.3. If $P(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $0 \leq r \leq \rho \leq k$,

$$\begin{aligned} \max_{|z|=\rho} |P'(z)| &\leq \frac{n(\rho+k)^{n-1}}{(k+r)^n} \left\{ 1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(k^2+\rho^2)n|a_0| + 2k^2\rho|a_1|} \left(\frac{\rho-r}{k+\rho} \right) \left(\frac{k+r}{k+\rho} \right)^{n-1} \right\} \times M(P, r). \end{aligned} \quad (1.5)$$

Remark 1.4. Since it is well known that if $P(z) = \sum_{v=0}^n a_v z^v$, $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then $|a_1|/|a_0| \leq n/k$, the above theorem with $r = 1$ gives a bound that is much better than obtainable from Theorem 1.2.

If we assume $P'(0) = 0$ in the above theorem, we get the following result.

COROLLARY 1.5. If $P(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$ and $P'(0) = 0$, then for $0 \leq r \leq \rho \leq k$,

$$\max_{|z|=\rho} |P'(z)| \leq \frac{n(\rho+k)^{n-1}}{(k+r)^n} \left\{ 1 - \frac{k(k-\rho)(\rho-r)n}{(k^2+\rho^2)(k+\rho)} \left(\frac{k+r}{k+\rho} \right)^{n-1} \right\} M(P, r). \quad (1.6)$$

2. Lemmas

We require the following lemmas for the proof of the theorem. The first lemma is due to Govil et al. [2].

LEMMA 2.1. If $P(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \geq k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq n \frac{n|a_0| + k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|} \max_{|z|=1} |P(z)|. \quad (2.1)$$

LEMMA 2.2. If $P(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k > 0$, then for $0 \leq r \leq \rho \leq k$,

$$M(P, r) \geq \left(\frac{r+k}{\rho+k} \right)^n M(P, \rho). \quad (2.2)$$

There is equality in (2.2) for $P(z) = (z+k)^n$.

The above lemma is due to Jain [3].

LEMMA 2.3. If $P(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $0 \leq r \leq \rho \leq k$,

$$M(P, r) \geq \left(\frac{k+r}{k+\rho} \right)^n \left\{ 1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(k^2 + \rho^2)n|a_0| + 2k^2\rho|a_1|} \left(\frac{\rho-r}{k+\rho} \right) \left(\frac{k+r}{k+\rho} \right)^{n-1} \right\}^{-1} \times M(P, \rho). \quad (2.3)$$

Proof. Since $P(z)$ has no zeros in $|z| < k$, $k \geq 1$, therefore, the polynomial $T(z) = P(tz)$ where $0 \leq t \leq k$ has no zeros in $|z| < k/t$, where $k/t \geq 1$. Using Lemma 2.1 with the polynomial $T(z)$, we get

$$\max_{|z|=1} |T'(z)| \leq n \left\{ \frac{n|a_0| + k^2/t^2|ta_1|}{(1 + k^2/t^2)n|a_0| + 2(k^2/t^2)|ta_1|} \right\} \max_{|z|=1} |T(z)|, \quad (2.4)$$

which implies

$$\max_{|z|=t} |P'(z)| \leq n \left\{ \frac{n|a_0|t + k^2|a_1|}{(t^2 + k^2)n|a_0| + 2k^2t|a_1|} \right\} \max_{|z|=t} |P(z)|. \quad (2.5)$$

Now for $0 \leq r \leq \rho \leq k$ and $0 \leq \theta < 2\pi$, we have

$$\begin{aligned} |P(\rho e^{i\theta}) - P(re^{i\theta})| &\leq \int_r^\rho |P'(te^{i\theta})| dt \\ &\leq \int_r^\rho n \left\{ \frac{n|a_0|t + k^2|a_1|}{(t^2 + k^2)n|a_0| + 2k^2t|a_1|} \right\} \max_{|z|=t} |P(z)| dt \quad (\text{by (2.5)}), \end{aligned} \quad (2.6)$$

which implies on using inequality (2.2) of Lemma 2.2,

$$\begin{aligned} |P(\rho e^{i\theta}) - P(re^{i\theta})| &\leq \int_r^\rho n \left\{ \frac{n|a_0|t + k^2|a_1|}{(t^2 + k^2)n|a_0| + 2k^2t|a_1|} \right\} \left(\frac{k+t}{k+r} \right)^n M(P, r) dt \\ &\leq \frac{nM(P, r)}{(k+r)^n} \int_r^\rho \left\{ \frac{n|a_0|t + k^2|a_1|}{(t^2 + k^2)n|a_0| + 2k^2t|a_1|} \right\} (k+t)^n dt, \end{aligned} \quad (2.7)$$

which gives, for $0 \leq r \leq \rho \leq k$,

$$\begin{aligned}
 & M(P, \rho) \\
 & \leq \left[1 + \frac{n}{(k+r)^n} \int_r^\rho \left\{ \frac{n|a_0|t+k^2|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|} \right\} (k+t)^n dt \right] M(P, r) \\
 & \leq \left[1 + \frac{n(k+\rho)}{(k+r)^n} \left\{ \frac{n|a_0|\rho+k^2|a_1|}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|} \right\} \int_r^\rho (k+t)^{n-1} dt \right] M(P, r) \\
 & = \left[1 - \left\{ \frac{(k+\rho)(n|a_0|\rho+k^2|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|} \right\} + \left\{ \frac{(k+\rho)(n|a_0|\rho+k^2|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|} \right\} \left(\frac{k+\rho}{k+r} \right)^n \right] M(P, r) \\
 & = \left[\frac{k(k-\rho)(n|a_0|-k|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|} + \left\{ 1 - \frac{k(k-\rho)(n|a_0|-k|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|} \right\} \left(\frac{k+\rho}{k+r} \right)^n \right] M(P, r) \\
 & = \left(\frac{k+\rho}{k+r} \right)^n \left[1 - \frac{k(k-\rho)(n|a_0|-k|a_1|)}{(k^2+\rho^2)n|a_0|+2k^2\rho|a_1|} \left\{ 1 - \left(\frac{k+r}{k+\rho} \right)^n \right\} \right] M(P, r) \\
 & = \left(\frac{k+\rho}{k+r} \right)^n \left[1 - \frac{k(k-\rho)(n|a_0|-k|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|} \times \frac{\rho-r}{(k+\rho)\{1-((k+r)/(k+\rho))\}} \right. \\
 & \quad \left. \times \left\{ 1 - \left(\frac{k+r}{k+\rho} \right)^n \right\} \right] M(P, r) \\
 & \leq \left(\frac{k+\rho}{k+r} \right)^n \left[1 - \frac{k(k-\rho)(n|a_0|-k|a_1|)n}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|} \left(\frac{\rho-r}{k+\rho} \right) \left(\frac{k+r}{k+\rho} \right)^{n-1} \right] M(P, r),
 \end{aligned} \tag{2.8}$$

from which inequality (2.3) follows. $\square$

3. Proof of theorem

Since the polynomial $P(z) = \sum_{v=0}^n a_v z^v$ has no zero in $|z| < k$, where $k \geq 1$, therefore, it follows that $F(z) = P(\rho z)$ has no zeros in $|z| < k/\rho$ where $k/\rho \geq 1$. Applying inequality (1.3) to the polynomial $F(z)$, we get

$$\max_{|z|=1} |F'(z)| \leq \frac{n}{1+k/\rho} \max_{|z|=1} |F(z)|, \tag{3.1}$$

which gives

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{\rho+k} \max_{|z|=\rho} |F(z)|. \tag{3.2}$$

Now if $0 \leq r \leq \rho \leq k$, then applying inequality (2.3) of Lemma 2.3 to (3.2), it follows that

$$\begin{aligned}
 \max_{|z|=\rho} |P'(z)| & \leq \frac{n(k+\rho)^{n-1}}{(k+r)^n} \left[1 - \frac{k(k-\rho)(n|a_0|-k|a_1|)n}{(k^2+\rho^2)n|a_0|+2k^2\rho|a_1|} \left(\frac{\rho-r}{k+\rho} \right) \left(\frac{k+r}{k+\rho} \right)^{n-1} \right] \\
 & \quad \times \max_{|z|=r} |P(z)|,
 \end{aligned} \tag{3.3}$$

which is (1.5) and the theorem is proved.

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