COMMON FIXED POINTS OF SINGLE-VALUED AND MULTIVALUED MAPS

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We define a new property which contains the property (EA) for a hybrid pair of single- and multivalued maps and give some new common fixed point theorems under hybrid contractive conditions. Our results extend previous ones. As an application, we give a partial answer to the problem raised by Singh and Mishra.

1. Introduction and preliminaries

Let \((X,d)\) be a metric space. Then, for \(x \in X, A \subset X, \ d(x,A) = \inf \{d(x,y),\ y \in A\}\). We denote \(CB(X)\) as the class of all nonempty bounded closed subsets of \(X\). Let \(H\) be the Hausdorff metric with respect to \(d\), that is,

\[
H(A,B) = \max \left\{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A) \right\},
\]

for every \(A,B \in CB(X)\). A self-map \(T\) defined on \(X\) satisfies Rhoades’ contractive definition in following sense: (see [19]) for all \(x,y \in X, x \neq y,\)

\[
d(Tx,Ty) < \max \{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}. 
\]

The fixed points theorems for Rhoades-type contraction mapping were investigated by many authors \([1, 5, 8, 10, 13, 16, 22]\) and the more results on this fields can be found in \([2, 4, 9, 11, 15, 23]\). Hybrid fixed point theory for nonlinear single-valued and multivalued maps is a new development in the domain of contraction-type multivalued theory (see \([3, 7, 10, 12, 14, 17, 18, 20]\) and references therein). In 1998, Jungck and Rhoades \([12]\) introduced the notion of weak compatibility to the setting of single-valued and multivalued maps. In \([21]\), Singh and Mishra introduced the notion of (IT)-commutativity for hybrid pair of single-valued and multivalued maps which need not be weakly compatible. Recently, Aamri and El Moutawakil \([1]\) defined a property (EA) for self-maps which contained the class of noncompatible maps. More recently, Kamran \([13]\) extended the property (EA) for a hybrid pair of single- and multivalued maps and generalized the notion of (IT)-commutativity for such pair.
The aim of this paper is to define a new property which contains the property (EA) for a hybrid pair of single- and multivalued maps and give some new common fixed point theorems under hybrid contractive conditions. As an application, we give an affirmative (half-) answer (Theorem 2.8) to the open problem in [21].

Now we state some known definitions and facts.

**Definition 1.1** [12]. Maps \( f : X \to X \) and \( T : X \to CB(X) \) are weakly compatible if they commute at their coincidence points, that is, if \( fTx = Tf x \) whenever \( fx \in Tx \).

**Definition 1.2** [21]. Maps \( f : X \to X \) and \( T : X \to CB(X) \) are said to be (IT)-commuting at \( x \in X \) if \( fTx \subset Tfx \) whenever \( fx \in Tx \).

**Definition 1.3** [1]. Maps \( f, g : X \to X \) are said to satisfy the property (EA) if there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} g x_n = t \in X \).

**Definition 1.4** [13]. Maps \( f : X \to X \) and \( T : X \to CB(X) \) are said to satisfy the property (EA) if there exist a sequence \( \{x_n\} \) in \( X \), some \( t \in X \), and \( A \) in \( CB(X) \) such that

\[
\lim_{n \to \infty} fx_n = t \in A = \lim_{n \to \infty} T x_n. \tag{1.3}
\]

**Definition 1.5** [13]. Let \( T : X \to CB(X) \). The map \( f : X \to X \) is said to be \( T \)-weakly commuting at \( x \in X \) if \( ffx \in Tfx \).

For the rest of the introduction, we state the following theorem as the prototype in this paper.

**Theorem 1.6** (see [13]). Let \( f \) be a self-map of the metric space \((X,d)\) and let \( F \) be a map from \( X \) into \( CB(X) \) such that

1. \((f,F)\) satisfies the property (EA);
2. for all \( x \neq y \) in \( X \),

\[
H(Fx,Fy) < \max \left\{ d(fx, fy), \frac{d(fx,Fx) + d(fy,Fy)}{2}, \frac{d(fx,Fy) + d(fy,Fx)}{2} \right\}. \tag{1.4}
\]

If \( fX \) is closed subset of \( X \), then

(a) \( f \) and \( F \) have a coincidence point;
(b) \( f \) and \( F \) have a common fixed point provided that \( f \) is \( F \)-weakly commuting at \( v \) and \( ffv = fv \) for \( v \in C(f,F) \), where \( C(f,F) = \{x : x \) is a coincidence point of \( f \) and \( F\} \).

2. Main results

We begin with the following definition.

**Definition 2.1.** (1) Let \( f, g, F, G : X \to X \). The maps pair \((f,F)\) and \((g,G)\) are said to satisfy the common property (EA) if there exist two sequences \( \{x_n\}, \{y_n\} \) in \( X \) and some \( t \) in \( X \) such that

\[
\lim_{n \to \infty} G y_n = \lim_{n \to \infty} F x_n = \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g y_n = t \in X. \tag{2.1}
\]
(2) Let \( f, g : X \rightarrow X \) and \( F, G : X \rightarrow CB(X) \). The maps pair \( (f, F) \) and \( (g, G) \) are said to satisfy the common property (EA) if there exist two sequences \( \{x_n\} \), \( \{y_n\} \) in \( X \), some \( t \) in \( X \), and \( A, B \) in \( CB(X) \) such that
\[
\lim_{n \rightarrow \infty} Fx_n = A, \quad \lim_{n \rightarrow \infty} Gy_n = B, \quad \lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g y_n = t \in A \cap B. \quad (2.2)
\]

**Example 2.2.** Let \( X = [1, +\infty) \) with the usual metric. Define \( f, g : X \rightarrow X \) and \( F, G : X \rightarrow CB(X) \) by \( f(x) = 2 + x/3 \), \( g(x) = 2 + x/2 \), and \( F(x) = [1, 2 + x], G(x) = [3, 3 + x/2] \) for all \( x \in X \). Consider the sequences \( \{x_n\} = \{3 + 1/n\}, \{y_n\} = \{2 + 1/n\} \). Clearly, \( \lim_{n \rightarrow \infty} Fx_n = [1, 5] = A, \lim_{n \rightarrow \infty} Gy_n = [3, 4] = B, \lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g y_n = 3 \in A \cap B \). Therefore, \( (f, F) \) and \( (g, G) \) are said to satisfy the common property (EA).

**Theorem 2.3.** Let \( f, g \) be two self-maps of the metric space \( (X, d) \) and let \( F, G \) be two maps from \( X \) into \( CB(X) \) such that
(1) \( (f, F) \) and \( (g, G) \) satisfy the common property (EA);
(2) for all \( x \neq y \) in \( X \),
\[
H(Fx, Gy) < \max \left\{ \frac{d(fx, gy)}{2}, \frac{d(fx, Fx) + d(gy, Gy)}{2}, \frac{d(fx, Gy) + d(gy, Fx)}{2} \right\}. \quad (2.3)
\]

If \( FX \) and \( GX \) are closed subsets of \( X \), then
(a) \( f \) and \( F \) have a coincidence point;
(b) \( g \) and \( G \) have a coincidence point;
(c) \( f \) and \( F \) have a common fixed point provided that \( f \) is \( F \)-weakly commuting at \( v \) and \( f v = f v \) for \( v \in C(f, F) \);
(d) \( g \) and \( G \) have a common fixed point provided that \( g \) is \( G \)-weakly commuting at \( v \) and \( g v = g v \) for \( v \in C(g, G) \);
(e) \( f, g, F, \) and \( G \) have a common fixed point provided that both (c) and (d) are true.

**Proof.** Since \( (f, F) \) and \( (g, G) \) satisfy the common property (EA), there exist two sequences \( \{x_n\}, \{y_n\} \) in \( X \) and \( u \in X, A, B \in CB(X) \) such that
\[
\lim_{n \rightarrow \infty} Fx_n = A, \quad \lim_{n \rightarrow \infty} Gy_n = B,
\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g y_n = u \in A \cap B. \quad (2.4)
\]

By virtue of \( FX \) and \( GX \) being closed, we have \( u = f v \) and \( u = g w \) for some \( v, w \in X \). We claim that \( f v \in Fv \) and \( g w \in Gw \). Indeed, condition (2) implies that
\[
H(Fx_n, Gw) < \max \left\{ d(fx_n, gw), \frac{d(fx_n, Fx_n) + d(gw, Gw)}{2}, \frac{d(fx_n, Gw) + d(gw, Fx_n)}{2} \right\}. \quad (2.5)
\]
Taking the limit as \( n \to \infty \), we obtain
\[
H(A, Gw) < \max \left\{ d(fv, gw), \frac{d(fv, A) + d(gw, Gw)}{2}, \frac{d(fv, Gw) + d(gw, A)}{2} \right\}
\]
\[
= \frac{d(gw, Gw)}{2}.
\]
(2.6)

Since \( gw = fv \in A \), it follows from the definition of Hausdorff metric that
\[
d(gw, Gw) \leq H(A, Gw) \leq \frac{d(gw, Gw)}{2},
\]
which implies that \( gw \in Gw \).

On the other hand, by condition (2) again, we have
\[
H(Fv, Gyn) < \max \left\{ d(fv, gyn), \frac{d(fv, Fv) + d(gyn, Gyn)}{2}, \frac{d(fv, Gyn) + d(gyn, Fv)}{2} \right\}
\]
(2.7)

Similarly, we obtain
\[
d(fv, Fv) \leq H(Fv, B) \leq \frac{d(fv, Fv)}{2}.
\]
(2.8)

Hence \( fv \in Fv \). Thus \( f \) and \( F \) have a coincidence point \( v, g \) and \( G \) have a coincidence point \( w \). This ends the proofs of part (a) and part (b).

Furthermore, by virtue of condition (c), we obtain \( ffv = fv \) and \( ffv \in Ffv \). Thus \( u = fu \in Fu \). This proves (c). A similar argument proves (d). Then (e) holds immediately.

\[ \square \]

Remark 2.4. In Theorem 2.3, if \( F, G \) are two maps from \( K \) into \( CB(X) \), where \( K \) is a closed subset of \( X \). In this case, it is necessary to assume that \( (X, d) \) is a metrically convex metric space. In this direction, many excellent works have appeared (see [5, 21]).

Corollary 2.5 (see [13, Theorem 3.10]). Let \( f \) be a self-map of the metric space \( (X, d) \) and let \( F \) be a map from \( X \) into \( CB(X) \) such that
(1) \( (f, F) \) satisfies the property (EA);
(2) for all \( x \neq y \) in \( X \),
\[
H(Fx, Fy) < \max \left\{ d(fx, fy), \frac{d(fx, Fx) + d(fy, Fy)}{2}, \frac{d(fx, Fy) + d(fy, Fx)}{2} \right\}.
\]
(2.10)

If \( fx \) is closed subset of \( X \), then
(a) \( f \) and \( F \) have a coincidence point;
(b) \( f \) and \( F \) have a common fixed point provided that \( f \) is \( F \)-weakly commuting at \( v \) and \( ffv = fv \) for \( v \in C(f, F) \).
Proof. Let \( F = G \) and \( f = g \), then the results follow from Theorem 2.3 immediately. □

If \( f = g \), we can conclude the following corollary.

Corollary 2.6. Let \( f \) be a self-map of the metric space \((X,d)\) and let \( F, G \) be two maps from \( X \) into \( \text{CB}(X) \) such that

1. \((f,F)\) and \((f,G)\) satisfy the common property (EA);
2. for all \( x \neq y \) in \( X \),

\[
H(Fx,Gy) < \max \left\{ \frac{d(fx,fy)}{2}, \frac{d(fx,Fx) + d(fy,Gy)}{2}, \frac{d(fx,Gy) + d(fy,Fx)}{2} \right\}. \tag{2.11}
\]

If \( fX \) is closed subset of \( X \), then

(a) \( f, G \) and \( F \) have a coincidence point;
(b) \( f, G \) and \( F \) have a common fixed point provided that \( f \) is both \( F \)-weakly commuting and \( G \)-weakly commuting at \( v \) and \( ffv = f v \) for \( v \in C(f,F) \).

If both \( F \) and \( G \) are single-valued maps in Theorem 2.3, then we have the following corollary.

Corollary 2.7. Let \( f, g, F, \) and \( G \) be four self-maps of the metric space \((X,d)\) such that

1. \((f,F)\) and \((g,G)\) satisfy the common property (EA);
2. for all \( x \neq y \) in \( X \),

\[
d(Fx,Gy) < \max \left\{ \frac{d(fx,gy)}{2}, \frac{d(fx,Fx) + d(gy,Gy)}{2}, \frac{d(fx,Gy) + d(gy,Fx)}{2} \right\}. \tag{2.12}
\]

If \( fX \) and \( gX \) are closed subsets of \( X \), then

(a) \( f \) and \( F \) have a coincidence point;
(b) \( g \) and \( G \) have a coincidence point;
(c) \( f \) and \( F \) have a common fixed point provided that \( f \) is \( F \)-weakly commuting at \( v \) and \( ffv = f v \) for \( v \in C(f,F) \);
(d) \( g \) and \( G \) have a common fixed point provided that \( g \) is \( G \)-weakly commuting at \( v \) and \( ggv = gv \) for \( v \in C(g,G) \);
(e) \( f, g, F, \) and \( G \) have a common fixed point provided that both (c) and (d) are true.

Theorem 2.8. Let \( f, g \) be two self-maps of the complete metric space \((X,d)\), let \( \lambda \in (0,1) \) be a constant, and let \( F, G \) be two maps from \( X \) into \( \text{CB}(X) \) such that for all \( x \neq y \) in \( X \),

\[
H(Fx,Gy) \leq \lambda \max \left\{ d(fx,gy), \frac{d(fx,Fx) + d(gy,Gy)}{2}, \frac{d(fx,Gy) + d(gh,Fx)}{2} \right\}. \tag{2.13}
\]

If \( fX \) and \( gX \) are closed subsets of \( X \) and \( FX \subset gX \), \( GX \subset fX \), then

(a) \( f \) and \( F \) have a coincidence point;
(b) \( g \) and \( G \) have a coincidence point;
(c) \( f \) and \( F \) have a common fixed point provided that \( f \) is \( F \)-weakly commuting at \( v \) and \( ffv = f v \) for \( v \in C(f,F) \);
Hence, we obtain

\[ d(y_1, y_2) \leq H(Fx_0, Gx_1) + \lambda. \]  \tag{2.14}

Since \( GX \subset fX \), there exists \( x_2 \) such that \( f x_2 = y_2 \in Gx_1 \), then we choose \( y_3 \in Fx_2 \) satisfying

\[ d(y_2, y_3) \leq H(Gx_1, Fx_2) + \lambda^2, \]  \tag{2.15}

and \( y_3 = g x_3 \) for some \( x_3 \in X \).

We continue this process to obtain a sequence \( \{y_n\} \) in \( X \) such that

\[ y_{2n} = f x_{2n} \in Gx_{2n-1}, \quad y_{2n+1} = g x_{2n+1} \in Fx_{2n}, \]

\[ d(y_{2n}, y_{2n+1}) \leq H(Gx_{2n-1}, Fx_{2n}) + \lambda^{2n}, \]  \tag{2.16}

\[ d(y_{2n-1}, y_{2n}) \leq H(Fx_{2n-2}, Gx_{2n-1}) + \lambda^{2n-1}, \quad n = 1, 2, \ldots. \]

Let \( a_n = d(y_n, y_{n+1}) \), then

\[ a_{2n} = d(y_{2n}, y_{2n+1}) \leq H(Gx_{2n-1}, Fx_{2n}) + \lambda^{2n} \]

\[ \leq \lambda \max \left\{ d(f x_{2n}, g x_{2n-1}), d(f x_{2n}, Fx_{2n}), d(g x_{2n-1}, Gx_{2n-1}), \right\} + \lambda^{2n}. \]  \tag{2.17}

By \( f x_{2n} \in Gx_{2n-1} \), we have

\[ d(g x_{2n-1}, Gx_{2n-1}) \leq d(g x_{2n-1}, f x_{2n}), \quad d(f x_{2n}, Fx_{2n}) \leq H(Gx_{2n-1}, Fx_{2n}). \]  \tag{2.18}

Thus, we rewrite (2.17) as

\[ a_{2n} \leq \lambda \max \left\{ d(f x_{2n}, g x_{2n-1}), \frac{d(g x_{2n-1}, Fx_{2n})}{2} \right\} + \lambda^{2n}. \]  \tag{2.19}

Hence, we obtain

\[ a_{2n} \leq \lambda \max \left\{ a_{2n-1}, \frac{a_{2n-1} + a_{2n}}{2} \right\} + \lambda^{2n}. \]  \tag{2.20}

If \( a_{2n-1} \leq a_{2n} \) for some \( n \), we have \( a_{2n} \leq \lambda^{2n}/(1 - \lambda) \). Otherwise, we get

\[ a_{2n} \leq \lambda a_{2n-1} + \lambda^{2n}. \]  \tag{2.21}
Therefore, by (2.20), we achieve
\[
a_{2n} \leq \max \left\{ \lambda a_{2n-1} + \lambda^{2n}, \frac{\lambda^{2n}}{1 - \lambda} \right\}.
\] (2.22)

On the other hand,
\[
a_{2n-1} \leq H(Gx_{2n-1}, Fx_{2n-2}) + \lambda^{2n-1}
\leq \lambda \max \left\{ d(fx_{2n-2}, gx_{2n-1}), d(fx_{2n-2}, Fx_{2n-2}), d(gx_{2n-1}, Gx_{2n-1}) \right\},
\]
\[
d(fx_{2n-2}, Gx_{2n-1}) + d(gx_{2n-1}, Fx_{2n-2}) \right\} + \lambda^{2n-1}.
\] (2.23)

Since \( gx_{2n-1} \in Fx_{2n-2} \), we have
\[
d(gx_{2n-1}, Gx_{2n-1}) \leq H(Gx_{2n-1}, Fx_{2n-2})
\]
\[
d(fx_{2n-2}, Fx_{2n-2}) \leq d(gx_{2n-1}, fx_{2n-2}).
\] (2.24)

Thus, we obtain
\[
a_{2n-1} \leq \lambda \max \left\{ a_{2n-2}, \frac{a_{2n-2} + a_{2n-1}}{2} \right\} + \lambda^{2n-1}.
\] (2.25)

Similarly, we get
\[
a_{2n-1} \leq \max \left\{ \lambda a_{2n-2} + \lambda^{2n-1}, \frac{\lambda^{2n-1}}{1 - \lambda} \right\}.
\] (2.26)

By (2.22) and (2.26), we obtain
\[
a_n \leq \max \left\{ \lambda a_{n-1} + \lambda^n, \frac{\lambda^n}{1 - \lambda} \right\}, \quad n = 1, 2, \ldots.
\] (2.27)

It is easy to see that
\[
a_n \leq \max \left\{ \lambda^n(a_0 + n), \frac{\lambda^n}{1 - \lambda} \right\}, \quad n = 1, 2, \ldots.
\] (2.28)

Thus, there exists \( n_0 > 0 \) such that for \( n \geq n_0 \),
\[
a_n \leq \lambda^n(a_0 + n).
\] (2.29)

Hence \( \lim_{n \to \infty} a_n = 0 \).

In order to prove that \( \{ y_n \} \) is Cauchy sequence, for any \( \epsilon > 0 \), we choose a sufficiently large number \( N \) such that
\[
\lambda^N(a_0 + N) \leq \frac{\epsilon(1 - \lambda)}{2}, \quad \lambda^N \leq \frac{\epsilon(1 - \lambda)^2}{4}.
\] (2.30)
Thus, for any positive integer $k$, we obtain

$$d(\gamma N, \gamma N + k) \leq \sum_{i=0}^{k-1} a_{N+i} \leq \sum_{i=0}^{k-1} \lambda^N (a_0 + N + i)$$

$$< \lambda^N (a_0 + N) \frac{1}{1-\lambda} + \lambda^N \sum_{i=0}^{k-1} i \lambda^i$$

$$< \lambda^N (a_0 + N) \frac{1}{1-\lambda} + \lambda^N \frac{2}{(1-\lambda)^2} \leq \epsilon.$$  

This implies that $\{y_n\}$ is a Cauchy sequence. Thus there is $u$ satisfying

$$\lim_{n \to \infty} y_n = u = \lim_{n \to \infty} f x_{2n} = \lim_{n \to \infty} g x_{2n+1}.$$  

(2.32)

Since $fX$ and $gX$ are closed, there exist $a, b$ such that $fa = u = gb$. A similar argument proves that

$$\lim_{n \to \infty} F x_{2n} = \lim_{n \to \infty} G x_{2n+1},$$

$$u \in \lim_{n \to \infty} F x_{2n} = \lim_{n \to \infty} G x_{2n+1}.  

(2.33)$$

Then $(f, F)$ and $(g, G)$ satisfy the common property (EA). The rest of the proof follows Theorem 2.3 immediately, then the proof of Theorem 2.8 is complete.  

\[ \square \]

**Corollary 2.9.** Let $f, g$ be two self-maps of the complete metric space $(X, d)$, let $\lambda \in (0, 1)$ be a constant, and let $F$, $G$ be two maps from $X$ into $\text{CB}(X)$ such that for all $x \neq y$ in $X$,

$$H(Fx, Gy) \leq \alpha d(fx, gy) + \beta \max \{d(fx, Fx), d(gy, Gy)\}$$

$$+ \gamma \max \{d(fx, Gy) + d(gy, Fx), d(fx, Fx) + d(gy, Gy)\},$$

(2.34)

and $\alpha + \beta + 2\gamma < 1$. If $fX$ and $gX$ are closed subsets of $X$ and $FX \subset gX, GX \subset fX$, then

(a) $f$ and $F$ have a coincidence point;

(b) $g$ and $G$ have a coincidence point;

(c) $f$ and $F$ have a common fixed point provided that $f$ is $F$-weakly commuting at $v$ and $f \ast v = f v$ for $v \in C(f,F)$;

(d) $g$ and $G$ have a common fixed point provided that $g$ is $G$-weakly commuting at $v$ and $g \ast v = g v$ for $v \in C(g,G)$;

(e) $f, g, F,$ and $G$ have a common fixed point provided that both (c) and (d) are true.

**Proof.** Let $\lambda = \alpha + \beta + 2\gamma$. Following (2.34) and $\max \{d(fx, Fx), d(gy, Gy)\} \geq (d(fx, Fx) + d(gy, Gy))/2$, it is easy to see that

$$H(Fx, Gy) \leq \lambda \max \left\{d(fx, gy), d(fx, Fx), d(gy, Gy), \frac{d(fx, Gy) + d(gy, Fx)}{2}\right\}.  

(2.35)$$

Thus by Theorem 2.8, we arrive to the conclusion in Corollary 2.9.  

\[ \square \]
The next theorem involves a function $\varphi$. Various conditions on $\varphi$ have been investigated by different authors [4, 6, 15, 16]. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ continue and satisfy the following conditions:

(A1) $\varphi$ is nondecreasing on $\mathbb{R}^+$,
(A2) $0 < \varphi(t) < t$, for each $t \in (0, +\infty)$.

**Theorem 2.10.** Let $f, g$ be two self-maps of the metric space $(X, d)$ and let $F, G : X \to X$ be two maps from $X$ into $\text{CB}(X)$ such that

1. $(f, F)$ and $(g, G)$ satisfy the common property (EA);
2. for all $x \neq y$ in $X$,

$$H(Fx, Gy) \leq \varphi(\max\{d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)\}).$$  \hspace{1cm} (2.36)

If $fX$ and $gX$ are closed subsets of $X$, then

(a) $f$ and $F$ have a coincidence point;
(b) $g$ and $G$ have a coincidence point;
(c) $f$ and $F$ have a common fixed point provided that $f$ is $F$-weakly commuting at $v$ and $ffv = f v$ for $v \in C(f, F)$;
(d) $g$ and $G$ have a common fixed point provided that $g$ is $G$-weakly commuting at $v$ and $ggv = gv$ for $v \in C(g, G)$;
(e) $f$, $g$, $F$, and $G$ have a common fixed point provided that both (c) and (d) are true.

**Proof.** Since $(f, F)$ and $(g, G)$ satisfy the common property (EA), there exist two sequences $\{x_n\}, \{y_n\}$ in $X$ and $u \in X, A, B \in \text{CB}(X)$ such that

$$\lim_{n \to \infty} Fx_n = A, \lim_{n \to \infty} Gy_n = B,$$

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gy_n = u \in A \cap B. \hspace{1cm} (2.37)$$

By virtue of $fX$ and $gX$ being closed, we have $u = fv$ and $u = gw$ for some $v, w \in X$. We claim that $fv \in Fv$ and $gw \in Gw$. Indeed, condition (2) implies that

$$H(Fx_n, Gw) \leq \varphi(\max\{d(fx_n, gw), d(fx_n, Fx_n), d(gw, Gw), d(fx_n, Gw), d(gw, Fx_n)\}).$$ \hspace{1cm} (2.38)

Taking the limit as $n \to \infty$, we obtain

$$H(A, Gw) \leq \varphi(\max\{d(fv, gw), d(fv, A), d(gw, Gw), d(fv, Gw), d(gw, A)\})$$

$$\leq \varphi(d(gw, Gw)) < d(gw, Gw). \hspace{1cm} (2.39)$$

Since $gw = fv \in A$, it follows from the definition of Hausdorff metric that

$$d(gw, Gw) \leq H(A, Gw) < d(gw, Gw), \hspace{1cm} (2.40)$$

which implies that $gw \in Gw$. 
Common fixed points of hybrid maps

On the other hand, by condition (2) again, we have

\[
H(Fv, Gy_n) \leq \varphi(\max\{d(fv, gy_n), d(fv, Fv), d(gy_n, Gy_n), d(fv, Gy_n), d(gy_n, Fv)\}). \tag{2.41}
\]

Similarly, we obtain

\[
d(fv, Fv) \leq H(Fv, B) < d(fv, Fv). \tag{2.42}
\]

Hence \(fv \in Fv\). Thus \(f\) and \(F\) have a coincidence point \(v\), \(g\) and \(G\) have a coincidence point \(w\). This ends the proofs of part (a) and part (b). The rest of proof is similar to the argument of Theorem 2.3.

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References


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