We study integral properties of two classes of functions with negative coefficients defined using differential operators. The obtained results are sharp and they improve known results.

1. Introduction

Let \( \mathbb{N} \) denote the set of nonnegative integers \( \{0, 1, \ldots, n, \ldots\} \), \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \), and let \( \mathcal{N}_j, j \in \mathbb{N}^* \), be the class of functions of the form

\[
f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad a_k \geq 0, \quad k \in \mathbb{N}, \quad k \geq j + 1,
\]

that are analytic in the open unit disc \( U = \{z : |z| < 1\} \).

Definition 1.1 [11]. The operator \( D^n : \mathcal{N}_j \to \mathcal{N}_j, n \in \mathbb{N} \), is defined by (a) \( D^0 f(z) = f(z) \); (b) \( D^1 f(z) = D f(z) = zf'(z) \); (c) \( D^n f(z) = D(D^{n-1} f(z)) \), \( z \in U \).

Definition 1.2 [4]. Let \( \alpha, \lambda \in [0, 1), n \in \mathbb{N}, j, m \in \mathbb{N}^* \); a function \( f \) belonging to \( \mathcal{N}_j \) is said to be in the class \( T_j(n, m, \lambda, \alpha) \) if and only if

\[
\Re \frac{D^{n+m} f(z)/D^n f(z)}{\lambda(D^{n+m} f(z)/D^n f(z)) + 1 - \lambda} > \alpha, \quad z \in U.
\]

Remark 1.3. The classes \( T_j(n, m, \lambda, \alpha) \) are generalizations of the classes

(i) \( T_1(0, 1, 0, \alpha) \) and \( T_1(1, 1, 0, \alpha) \) defined and studied by Silverman [12] (these classes are the class of starlike functions with negative coefficients and the class of convex functions with negative coefficients, resp.),

(ii) \( T_j(0, 1, 0, \alpha) \) and \( T_j(1, 1, 0, \alpha) \) studied by Chatterjea [7] and Srivastava et al. [13],

(iii) \( T_1(n, 1, 0, \alpha) \) studied by Hur and Oh [10],

(iv) \( T_1(0, 1, \lambda, \alpha) \) and \( T_1(1, 1, \lambda, \alpha) \) studied by Altintas and Owa [2],

(v) \( T_1(n, 1, \lambda, \alpha) \) studied by Aouf and Cho [3, 8],

(vi) \( T_1(n, m, 0, \alpha) \) studied by Hossen et al. [9].
Theorem 1.7. Let \( n \in \mathbb{N} \), \( j, m \in \mathbb{N}^* \), \( \alpha, \lambda \in [0, 1) \), and let \( f \in \mathcal{N}_j \); then \( f \in T_j(n, m, \lambda, \alpha) \) if and only if
\[
\sum_{k=j+1}^{\infty} k^n [(m-1)\alpha] a_k \leq 1 - \alpha. \tag{1.3}
\]

The result is sharp and the extremal functions are
\[
f(z) = z - \frac{1 - \alpha}{k^n[(m-1)\alpha]} z^k, \quad k \in \mathbb{N}, \quad k \geq j + 1. \tag{1.4}
\]

Definition 1.5 [5]. Let \( m, n \in \mathbb{N} \), \( j \in \mathbb{N}^* \), \( \alpha \in [0, 1) \), \( \lambda \in [0, 1] \); a function \( f \) belonging to \( \mathcal{N}_j \) is said to be in the class \( L_j(n, m, \lambda, \alpha) \) if and only if
\[
\text{Re} \left[ \frac{(1 - \lambda)D^{n+1}f(z) + \lambda D^{n+m+1}f(z)}{(1 - \lambda)D^nf(z) + \lambda D^{n+m}f(z)} \right] > \alpha, \quad z \in U. \tag{1.5}
\]

Remark 1.6. The classes \( L_j(n, m, \lambda, \alpha) \) are generalizations of the classes
(1) \( L_1(0, 0, 0, \alpha) = T_1(0, 1, 0, \alpha) \) and \( L_1(1, 0, 1, \alpha) = T_1(1, 1, 0, \alpha) \) (the classes defined and studied by Silverman [12]),
(2) \( L_j(0, 0, 0, \alpha) = T_j(0, 1, 0, \alpha) \) and \( L_j(0, 1, 1, \alpha) = T_j(1, 1, 0, \alpha) \) (the classes studied by Chatterjea [7] and Srivastava et al. [13]),
(3) \( L_j(0, 1, \lambda, \alpha) \) studied by Altintas [1],
(4) \( L_j(n, 1, \lambda, \alpha) \), \( L_j(n, m, 0, \alpha) \), and \( L_j(n, 1, 1, \alpha) \) studied by Aouf and Srivastava [6].

In [5], the next characterization theorem of the class \( L_j(n, m, \lambda, \alpha) \) is given.

Theorem 1.7. Let \( n, m \in \mathbb{N} \), \( j \in \mathbb{N}^* \), \( \alpha \in [0, 1) \), \( \lambda \in [0, 1] \), and let \( f \in \mathcal{N}_j \); then \( f \in L_j(n, m, \lambda, \alpha) \) if and only if
\[
\sum_{k=j+1}^{\infty} k^n [(k - 1)\alpha] a_k \leq 1 - \alpha. \tag{1.6}
\]

The result is sharp and the extremal functions are
\[
f(z) = z - \frac{1 - \alpha}{k^n[(k - 1)\alpha]} z^k, \quad k \in \mathbb{N}, \quad k \geq j + 1. \tag{1.7}
\]

Let \( I_c : \mathcal{N}_j \rightarrow \mathcal{N}_j \) be the integral operator defined by \( g = I_c(f) \), where \( c \in (-1, \infty) \), \( f \in \mathcal{N}_j \), and
\[
g(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) dt. \tag{1.8}
\]

We note that if \( f \in \mathcal{N}_j \) is a function of the form (1.1), then
\[
g(z) = I_c(f)(z) = z - \sum_{k=j+1}^{\infty} \frac{c + 1}{c + k} a_k z^k. \tag{1.9}
\]
By using Theorem 1.4, in [4] it is proved that $I_c(T_j(n,m,\lambda,\alpha)) \subset T_j(n,m,\lambda,\alpha)$ and by using Theorem 1.7, in [5] it is proved that $I_c(L_j(n,m,\lambda,\alpha)) \subset L_j(n,m,\lambda,\alpha)$. In this note, these results are improved.

2. Integral properties of the class $T_j(n,m,\lambda,\alpha)$

**Theorem 2.1.** Let $n \in \mathbb{N}$, $j, m \in \mathbb{N}^*$, $\alpha, \lambda \in [0,1)$, and let $c \in (-1, \infty)$; if $f \in T_j(n,m,\lambda,\alpha)$ and $g = I_c(f)$, then $g \in T_j(n,m,\lambda,\beta)$, where

$$\beta = \beta(m,\lambda,\alpha,c; j+1) = 1 - \frac{[(j+1)^m - 1](1 - \alpha)(1 - \lambda)(c + 1)}{[(j+1)^m - 1][((1 - \alpha \lambda)(c + j + 1) - \lambda(c + 1)(1 - \alpha)] + (1 - \alpha)j}$$

(2.1)

and $\alpha < \beta(m,\lambda,\alpha,c; j + 1) < 1$. The result is sharp.

**Proof.** From Theorem 1.4 and from (1.9) we have $g \in T_j(n,m,\lambda,\beta)$ if and only if

$$\sum_{k=j+1}^{\infty} k^n \frac{k^m(1 - \beta \lambda) - \beta(1 - \lambda)}{1 - \beta} \frac{a_k}{(c + k)} \leq 1.$$  

(2.2)

We find the largest $\beta$ such that (2.2) holds. We note that the inequalities

$$\frac{k^m(1 - \beta \lambda) - \beta(1 - \lambda)}{1 - \beta} \frac{c + 1}{c + k} \leq \frac{k^m(1 - \alpha \lambda) - \alpha(1 - \lambda)}{1 - \alpha}, \quad k \geq j + 1,$$

(2.3)

imply (2.2), because $f \in T_j(n,m,\lambda,\alpha)$ and it satisfies (1.3). But the inequalities (2.3) are equivalent to

$$A(m,\lambda,\alpha,c;k)\beta \leq B(m,\lambda,\alpha,c;k),$$

(2.4)

where

$$A(m,\lambda,\alpha,c;k) = (k^m - 1)[(1 - \alpha \lambda)(c + k) - \lambda(c + 1)(1 - \alpha)] + (1 - \alpha)(k - 1),$$

$$B(m,\lambda,\alpha,c;k) = A(m,\lambda,\alpha,c;k) - (k^m - 1)(c + 1)(1 - \alpha)(1 - \lambda).$$

(2.5)

Since $1 - \alpha \lambda > 1 - \alpha$ and $c + k > c + 1$, we have $A(m,\lambda,\alpha,c;k) > 0$ and from (2.4) we obtain

$$\beta \leq \frac{B(m,\lambda,\alpha,c;k)}{A(m,\lambda,\alpha,c;k)} \forall k \geq j + 1.$$  

(2.6)
We define $\beta(m, \lambda, c ; k) := B(m, \lambda, c ; k)/A(m, \lambda, c ; k)$. We show now that $\beta(m, \lambda, c ; k)$ is an increasing function of $k$, $k \geq j + 1$. Indeed

$$\beta(m, \lambda, c ; k) = 1 - (1 - \alpha)(1 - \lambda)(c + 1) \frac{k^m - 1}{A(m, \lambda, c ; k)}$$

$$= 1 - (1 - \alpha)(1 - \lambda)(c + 1) \frac{1}{E(m, \lambda, c ; k)},$$

where $E(m, \lambda, c ; k) = A(m, \lambda, c ; k)/(k^m - 1)$ and $\beta(m, \lambda, c ; k)$ increases when $k$ increases if and only if $E(m, \lambda, c ; k)$ is also an increasing function of $k$. Let $h(x) = E(m, \lambda, c ; x)$, $x \in [j + 1, \infty) \subset [2, \infty)$; we have

$$h'(x) = 1 - \alpha \lambda + (1 - \alpha) \frac{x^m - 1 - mx^m + x^{m-1}}{(x^m - 1)^2}$$

$$= 1 - \alpha \lambda + (1 - \alpha) \left[ \frac{1 - m}{x^m - 1} + \frac{m(x^{m-1} - 1)}{(x^m - 1)^2} \right]$$

$$> 1 - \alpha \lambda - (1 - \alpha) = \alpha (1 - \lambda) \geq 0, \quad x \in [j + 1, \infty),$$

where we used the fact that

$$\frac{1 - m}{x^m - 1} + \frac{m(x^{m-1} - 1)}{(x^m - 1)^2} \geq \frac{1 - m}{x^m - 1} > -1.$$

We obtained $h(j + 1) \leq h(k)$, $k \geq j + 1$, and this implies

$$\beta = \beta(m, \lambda, c ; j + 1) \leq \beta(m, \lambda, c ; k), \quad k \geq j + 1. \quad (2.10)$$

The result is sharp because

$$I_c(f_\alpha) = f_{\beta}, \quad (2.11)$$

where

$$f_\alpha(z) = z - \frac{1 - \alpha}{(j + 1)^n[(j + 1)^m(1 - \alpha \lambda) - \alpha(1 - \lambda)]} z^{j + 1},$$

$$f_\beta(z) = z - \frac{1 - \beta}{(j + 1)^n[(j + 1)^m(1 - \beta \lambda) - \beta(1 - \lambda)]} z^{j + 1} \quad (2.12)$$

are the extremal functions of $T_j(n, m, \lambda, \alpha)$ and $T_j(n, m, \lambda, \beta)$, respectively, and $\beta = \beta(m, \lambda, \alpha, c ; j + 1)$.

Indeed, we have

$$I_c(f_\alpha)(z) = z - \frac{(1 - \alpha)(c + 1)}{(j + 1)^n(c + j + 1)[(j + 1)^m(1 - \alpha \lambda) - \alpha(1 - \lambda)]} z^{j + 1}. \quad (2.13)$$
But if we use the notations \( A = A(m, \lambda, \alpha, c; j + 1) \) and \( B = B(m, \lambda, \alpha, c; j + 1) \), we deduce

\[
\frac{1 - \beta}{(j + 1)^m(1 - \beta \lambda) - \beta(1 - \lambda)} = \frac{A - B}{(j + 1)^m(A - B \lambda) - B(1 - \lambda)} = \frac{[(j + 1)^m - 1](1 - \alpha)(1 - \lambda)(c + 1)}{(1 - \lambda)[A(j + 1)^m + [(j + 1)^m - 1]\lambda(j + 1)^m(1 - \alpha)(c + 1) - B]}
\]

(2.14)

\[
= \frac{[(j + 1)^m - 1](1 - \alpha)(c + 1)}{[(j + 1)^m - 1][(j + 1)^m \lambda(1 - \alpha)(1 + c) + A + (1 - \alpha)(c + 1)(1 - \lambda)]}
\]

and this implies (2.11).

From \( \beta = 1 - [(j + 1)^m - 1](1 - \alpha)(1 - \lambda)(c + 1)/A \) and because \( A > 0 \), we obtain \( \beta < 1 \). We also have \( \beta > \alpha \); indeed

\[
\beta - \alpha = (1 - \alpha)\left\{1 - \frac{[(j + 1)^m - 1](c + 1)(1 - \lambda)}{[(j + 1)^m - 1][(1 - \alpha \lambda)(c + j + 1) - \lambda(c + 1)(1 - \alpha)] + (1 - \alpha)j}\right\}
\]

\[
> (1 - \alpha)\left\{1 - \frac{(c + 1)(1 - \lambda)}{(1 - \alpha \lambda)(c + j + 1) - \lambda(c + 1)(1 - \alpha)}\right\}
\]

\[
= \frac{(1 - \alpha)(1 - \alpha \lambda)j}{j(1 - \alpha \lambda) + (c + 1)(1 - \lambda)} > 0.
\]

(2.15)  \( \square \)

3. Integral properties of the class \( L_j(n, m, \lambda, \alpha) \)

**Theorem 3.1.** Let \( n, m \in \mathbb{N} \), \( j \in \mathbb{N}^* \), \( \alpha \in [0, 1) \), \( \lambda \in [0, 1) \), and let \( c \in (-1, \infty) \); if \( f \in L_j(n, m, \lambda, \alpha) \) and \( g = I_c(f) \), then \( g \in L_j(n, m, \lambda, \gamma) \), where

\[
\gamma = \gamma(\alpha, c; j + 1) = 1 - \frac{(1 - \alpha)(c + 1)}{2 - \alpha + c + j}
\]

(3.1)

and \( \alpha < \gamma(\alpha, c; j + 1) < 1 \). The result is sharp.

**Proof.** From Theorem 1.7 and from (1.9) we have \( g \in L_j(n, m, \lambda, \beta) \) if and only if

\[
\sum_{k=j+1}^{\infty} k^n (k - \gamma)[1 + (k^n - 1)\lambda(1 - \gamma)](1 - \gamma)(c + k) a_k \leq 1.
\]

(3.2)

We find the largest \( \gamma \) such that (3.2) holds. We note that the inequalities

\[
\frac{(k - \gamma)(c + 1)}{(1 - \gamma)(c + k)} \leq \frac{k - \alpha}{1 - \alpha}, \quad k \geq j + 1,
\]

(3.3)
imply (3.2), because \( f \in L_j(n, m, \lambda, \alpha) \). But the inequalities (3.3) are equivalent to

\[
(k - 1)(k + c + 1 - \alpha) \gamma \leq (k - 1)(k + \alpha c), \quad k \geq j + 1.
\]  

(3.4)

Since \((k + c + 1 - \alpha) > 0\) and \(k - 1 \geq j \geq 1\), we deduce

\[
\gamma \leq \frac{k + \alpha c}{k + c + 1 - \alpha} \quad \forall k \geq j + 1.
\]  

(3.5)

We define \(\gamma(\alpha, c; k) := 1 - (1 - \alpha)(c + 1)/(k + c + 1 - \alpha)\). Obviously, \(\gamma(\alpha, c; j + 1) \leq \gamma(\alpha, c; j + 1)\).

We have \(\gamma < 1\) because \((1 - \alpha)(c + 1)/(k + c + 1 - \alpha) > 0\) and \(\gamma > \alpha\) because

\[
\gamma - \alpha = (1 - \alpha) \frac{1 - \alpha + j}{2 - \alpha + c + j} > 0.
\]  

(3.6)

The result is sharp. Indeed, we consider the function

\[
\varphi_\alpha(z) = z - \frac{1 - \alpha}{(j + 1)^n(j + 1 - \alpha)[1 - \lambda + \lambda(j + 1)^m]} z^{j+1}
\]  

(3.7)

that belongs to \(L_j(n, m, \lambda, \alpha)\). Then

\[
I_c(\varphi_\alpha)(z) = z - \frac{(1 - \alpha)(c + 1)}{(j + 1)^n(j + 1 - \alpha)[1 - \lambda + \lambda(j + 1)^m](c + j + 1)} z^{j+1},
\]  

(3.8)

and because

\[
\frac{(1 - \alpha)(c + 1)}{(j + 1 - \alpha)(c + j + 1)} = \frac{1 - \gamma}{j + 1 - \gamma},
\]  

(3.9)

we deduce that \(I_c(\varphi_\alpha) = \varphi_\gamma\) belongs to \(L_j(n, m, \lambda, \gamma)\).

\[\square\]

References


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