INTEGRAL PROPERTIES OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

G. S. SĂLĂGEAN

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We study integral properties of two classes of functions with negative coefficients defined using differential operators. The obtained results are sharp and they improve known results.

1. Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers $\{0,1,\dots,n,\dots\}$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, and let $\mathcal{N}_j$, $j \in \mathbb{N}^*$, be the class of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad a_k \geq 0,\ k \in \mathbb{N},\ k \geq j+1, \tag{1.1}$$

that are analytic in the open unit disc $U = \{z : |z| < 1\}$.

Definition 1.1 [11]. The operator $D^n : \mathcal{N}_j \to \mathcal{N}_j$, $n \in \mathbb{N}$, is defined by (a) $D^0 f(z) = f(z)$; (b) $D^1 f(z) = Df(z) = zf'(z)$; (c) $D^n f(z) = D(D^{n-1} f(z))$, $z \in U$.

Definition 1.2 [4]. Let $\alpha, \lambda \in [0,1)$, $n \in \mathbb{N}$, $j, m \in \mathbb{N}^*$; a function f belonging to $\mathcal{N}_j$ is said to be in the class $T_j(n,m,\lambda,\alpha)$ if and only if

$$\operatorname{Re} \frac{D^{n+m} f(z)/D^n f(z)}{\lambda(D^{n+m} f(z)/D^n f(z)) + 1 - \lambda} > \alpha, \quad z \in U. \tag{1.2}$$

Remark 1.3. The classes $T_j(n,m,\lambda,\alpha)$ are generalizations of the classes

(i) $T_1(0,1,0,\alpha)$ and $T_1(1,1,0,\alpha)$ defined and studied by Silverman [12] (these classes are the class of starlike functions with negative coefficients and the class of convex functions with negative coefficients, resp.),
(ii) $T_j(0,1,0,\alpha)$ and $T_j(1,1,0,\alpha)$ studied by Chatterjea [7] and Srivastava et al. [13],
(iii) $T_1(n,1,0,\alpha)$ studied by Hur and Oh [10],
(iv) $T_1(0,1,\lambda,\alpha)$ and $T_1(1,1,\lambda,\alpha)$ studied by Altintas and Owa [2],
(v) $T_1(n,1,\lambda,\alpha)$ studied by Aouf and Cho [3, 8],
(vi) $T_1(n,m,0,\alpha)$ studied by Hossen et al. [9].

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In [4], the next characterization theorem of the class $T_j(n,m,\lambda,\alpha)$ is given.

Theorem 1.4. *Let $n \in \mathbb{N}$, $j,m \in \mathbb{N}^*$, $\alpha,\lambda \in [0,1)$, and let $f \in \mathcal{N}_j$; then $f \in T_j(n,m,\lambda,\alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n\left[k^m(1-\alpha\lambda)-\alpha(1-\lambda)\right]a_k \le 1-\alpha. \tag{1.3}$$

The result is sharp and the extremal functions are

$$f(z) = z - \frac{1-\alpha}{k^n\left[k^m(1-\alpha\lambda)-\alpha(1-\lambda)\right]}z^k, \quad k \in \mathbb{N},\ k \ge j+1. \tag{1.4}$$

Definition 1.5 [5]. Let $m,n \in \mathbb{N}$, $j \in \mathbb{N}^*$, $\alpha \in [0,1)$, $\lambda \in [0,1]$; a function f belonging to $\mathcal{N}_j$ is said to be in the class $L_j(n,m,\lambda,\alpha)$ if and only if

$$\operatorname{Re}\frac{(1-\lambda)D^{n+1}f(z)+\lambda D^{n+m+1}f(z)}{(1-\lambda)D^n f(z)+\lambda D^{n+m}f(z)} > \alpha, \quad z \in U. \tag{1.5}$$

Remark 1.6. The classes $L_j(n,m,\lambda,\alpha)$ are generalizations of the classes

(1) $L_1(0,0,0,\alpha) = T_1(0,1,0,\alpha)$ and $L_1(1,0,1,\alpha) = T_1(1,1,0,\alpha)$ (the classes defined and studied by Silverman [12]),
(2) $L_j(0,0,0,\alpha) = T_j(0,1,0,\alpha)$ and $L_j(0,1,1,\alpha) = L_j(1,0,1,\alpha) = T_j(1,1,0,\alpha)$ (the classes studied by Chatterjea [7] and Srivastava et al. [13]),
(3) $L_j(0,1,\lambda,\alpha)$ studied by Altintas [1],
(4) $L_j(n,1,\lambda,\alpha)$, $L_j(n,m,0,\alpha)$, and $L_j(n,1,1,\alpha)$ studied by Aouf and Srivastava [6].

In [5], the next characterization theorem of the class $L_j(n,m,\lambda,\alpha)$ is given.

Theorem 1.7. *Let $n,m \in \mathbb{N}$, $j \in \mathbb{N}^*$, $\alpha \in [0,1)$, $\lambda \in [0,1]$, and let $f \in \mathcal{N}_j$; then $f \in L_j(n,m,\lambda,\alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)\left[1+(k^m-1)\lambda\right]a_k \le 1-\alpha. \tag{1.6}$$

The result is sharp and the extremal functions are

$$f(z) = z - \frac{1-\alpha}{k^n(k-\alpha)\left[1+(k^m-1)\lambda\right]}z^k, \quad k \in \mathbb{N},\ k \ge j+1. \tag{1.7}$$

Let $I_c : \mathcal{N}_j \to \mathcal{N}_j$ be the integral operator defined by $g = I_c(f)$, where $c \in (-1,\infty)$, $f \in \mathcal{N}_j$, and

$$g(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt. \tag{1.8}$$

We note that if $f \in \mathcal{N}_j$ is a function of the form (1.1), then

$$g(z) = I_c(f)(z) = z - \sum_{k=j+1}^{\infty}\frac{c+1}{c+k}a_k z^k. \tag{1.9}$$

By using Theorem 1.4, in [4] it is proved that $I_c(T_j(n,m,\lambda,\alpha)) \subset T_j(n,m,\lambda,\alpha)$ and by using Theorem 1.7, in [5] it is proved that $I_c(L_j(n,m,\lambda,\alpha)) \subset L_j(n,m,\lambda,\alpha)$. In this note, these results are improved.

2. Integral properties of the class $T_j(n,m,\lambda,\alpha)$

Theorem 2.1. *Let $n \in \mathbb{N}$, $j,m \in \mathbb{N}^*$, $\alpha,\lambda \in [0,1)$, and let $c \in (-1,\infty)$; if $f \in T_j(n,m,\lambda,\alpha)$ and $g = I_c(f)$, then $g \in T_j(n,m,\lambda,\beta)$, where*

$$\begin{aligned}\beta &= \beta(m,\lambda,\alpha,c;j+1)\\ &= 1 - \frac{[(j+1)^m - 1](1-\alpha)(1-\lambda)(c+1)}{[(j+1)^m - 1][(1-\alpha\lambda)(c+j+1) - \lambda(c+1)(1-\alpha)] + (1-\alpha)j}\end{aligned} \tag{2.1}$$

and $\alpha < \beta(m,\lambda,\alpha,c;j+1) < 1$. The result is sharp.

Proof. From Theorem 1.4 and from (1.9) we have $g \in T_j(n,m,\lambda,\beta)$ if and only if

$$\sum_{k=j+1}^{\infty} \frac{k^n[k^m(1-\beta\lambda) - \beta(1-\lambda)](c+1)}{(1-\beta)(c+k)} a_k \le 1. \tag{2.2}$$

We find the largest β such that (2.2) holds. We note that the inequalities

$$\frac{k^m(1-\beta\lambda) - \beta(1-\lambda)}{1-\beta}\frac{c+1}{c+k} \le \frac{k^m(1-\alpha\lambda) - \alpha(1-\lambda)}{1-\alpha}, \quad k \ge j+1, \tag{2.3}$$

imply (2.2), because $f \in T_j(n,m,\lambda,\alpha)$ and it satisfies (1.3). But the inequalities (2.3) are equivalent to

$$A(m,\lambda,\alpha,c;k)\beta \le B(m,\lambda,\alpha,c;k), \tag{2.4}$$

where

$$\begin{aligned}A(m,\lambda,\alpha,c;k) &= (k^m - 1)[(1-\alpha\lambda)(c+k) - \lambda(c+1)(1-\alpha)] + (1-\alpha)(k-1),\\ B(m,\lambda,\alpha,c;k) &= A(m,\lambda,\alpha,c;k) - (k^m - 1)(c+1)(1-\alpha)(1-\lambda).\end{aligned} \tag{2.5}$$

Since $1-\alpha\lambda > 1-\alpha$ and $c+k > c+1$, we have $A(m,\lambda,\alpha,c;k) > 0$ and from (2.4) we obtain

$$\beta \le \frac{B(m,\lambda,\alpha,c;k)}{A(m,\lambda,\alpha,c;k)} \quad \forall k \ge j+1. \tag{2.6}$$

We define $\beta(m,\lambda,\alpha,c;k) := B(m,\lambda,\alpha,c;k)/A(m,\lambda,\alpha,c;k)$. We show now that $\beta(m,\lambda,\alpha,c;k)$ is an increasing function of k, $k \geq j+1$. Indeed

$$\begin{aligned}\beta(m,\lambda,\alpha,c;k) &= 1-(1-\alpha)(1-\lambda)(c+1)\frac{k^m-1}{A(m,\lambda,\alpha,c;k)}\\ &= 1-(1-\alpha)(1-\lambda)(c+1)\frac{1}{E(m,\lambda,\alpha,c;k)},\end{aligned} \tag{2.7}$$

where $E(m,\lambda,\alpha,c;k) = A(m,\lambda,\alpha,c;k)/(k^m-1)$ and $\beta(m,\lambda,\alpha,c;k)$ increases when k increases if and only if $E(m,\lambda,\alpha,c;k)$ is also an increasing function of k.

Let $h(x) = E(m,\lambda,\alpha,c;x)$, $x \in [j+1,\infty) \subset [2,\infty)$; we have

$$\begin{aligned}h'(x) &= 1-\alpha\lambda+(1-\alpha)\frac{x^m-1-mx^m+x^{m-1}}{(x^m-1)^2}\\ &= 1-\alpha\lambda+(1-\alpha)\left[\frac{1-m}{x^m-1}+\frac{m(x^{m-1}-1)}{(x^m-1)^2}\right]\\ &> 1-\alpha\lambda-(1-\alpha) = \alpha(1-\lambda) \geq 0, \quad x \in [j+1,\infty),\end{aligned} \tag{2.8}$$

where we used the fact that

$$\frac{1-m}{x^m-1}+\frac{m(x^{m-1}-1)}{(x^m-1)^2} \geq \frac{1-m}{x^m-1} > -1. \tag{2.9}$$

We obtained $h(j+1) \leq h(k)$, $k \geq j+1$, and this implies

$$\beta = \beta(m,\lambda,\alpha,c;j+1) \leq \beta(m,\lambda,\alpha,c;k), \quad k \geq j+1. \tag{2.10}$$

The result is sharp because

$$I_c(f_\alpha) = f_\beta, \tag{2.11}$$

where

$$\begin{aligned}f_\alpha(z) &= z-\frac{1-\alpha}{(j+1)^n[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}z^{j+1},\\ f_\beta(z) &= z-\frac{1-\beta}{(j+1)^n[(j+1)^m(1-\beta\lambda)-\beta(1-\lambda)]}z^{j+1}\end{aligned} \tag{2.12}$$

are the extremal functions of $T_j(n,m,\lambda,\alpha)$ and $T_j(n,m,\lambda,\beta)$, respectively, and $\beta = \beta(m,\lambda,\alpha,c;j+1)$.

Indeed, we have

$$I_c(f_\alpha)(z) = z-\frac{(1-\alpha)(c+1)}{(j+1)^n(c+j+1)[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}z^{j+1}. \tag{2.13}$$

But if we use the notations $A = A(m,\lambda,\alpha,c;j+1)$ and $B = B(m,\lambda,\alpha,c;j+1)$, we deduce

$$\begin{aligned}
&\frac{1-\beta}{(j+1)^m(1-\beta\lambda)-\beta(1-\lambda)} \\
&\quad = \frac{A-B}{(j+1)^m(A-B\lambda)-B(1-\lambda)} \\
&\quad = \frac{[(j+1)^m-1](1-\alpha)(1-\lambda)(c+1)}{(1-\lambda)\{A(j+1)^m+[(j+1)^m-1]\lambda(j+1)^m(1-\alpha)(c+1)-B\}} \\
&\quad = \frac{[(j+1)^m-1](1-\alpha)(c+1)}{[(j+1)^m-1][(j+1)^m\lambda(1-\alpha)(1+c)+A+(1-\alpha)(c+1)(1-\lambda)]} \\
&\quad = \frac{(1-\alpha)(c+1)}{(c+j+1)[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}
\end{aligned} \tag{2.14}$$

and this implies (2.11).

From $\beta = 1-[(j+1)^m-1](1-\alpha)(1-\lambda)(c+1)/A$ and because $A>0$, we obtain $\beta<1$. We also have $\beta>\alpha$; indeed

$$\begin{aligned}
\beta-\alpha &= (1-\alpha)\left\{1-\frac{[(j+1)^m-1](c+1)(1-\lambda)}{[(j+1)^m-1][(1-\alpha\lambda)(c+j+1)-\lambda(c+1)(1-\alpha)]+(1-\alpha)j}\right\} \\
&> (1-\alpha)\left\{1-\frac{(c+1)(1-\lambda)}{(1-\alpha\lambda)(c+j+1)-\lambda(c+1)(1-\alpha)}\right\} \\
&= \frac{(1-\alpha)(1-\alpha\lambda)j}{j(1-\alpha\lambda)+(c+1)(1-\lambda)} > 0.
\end{aligned} \tag{2.15}$$

□

3. Integral properties of the class $L_j(n,m,\lambda,\alpha)$

Theorem 3.1. *Let $n,m\in\mathbb{N}$, $j\in\mathbb{N}^*$, $\alpha\in[0,1)$, $\lambda\in[0,1]$, and let $c\in(-1,\infty)$; if $f\in L_j(n,m,\lambda,\alpha)$ and $g=I_c(f)$, then $g\in L_j(n,m,\lambda,\gamma)$, where*

$$\gamma = \gamma(\alpha,c;j+1) = 1-\frac{(1-\alpha)(c+1)}{2-\alpha+c+j} \tag{3.1}$$

and $\alpha<\gamma(\alpha,c;j+1)<1$. The result is sharp.

Proof. From Theorem 1.7 and from (1.9) we have $g\in L_j(n,m,\lambda,\beta)$ if and only if

$$\sum_{k=j+1}^{\infty}\frac{k^n(k-\gamma)[1+(k^m-1)\lambda](c+1)}{(1-\gamma)(c+k)}a_k \le 1. \tag{3.2}$$

We find the largest γ such that (3.2) holds. We note that the inequalities

$$\frac{(k-\gamma)(c+1)}{(1-\gamma)(c+k)} \le \frac{k-\alpha}{1-\alpha}, \quad k\ge j+1, \tag{3.3}$$

imply (3.2), because $f \in L_j(n,m,\lambda,\alpha)$. But the inequalities (3.3) are equivalent to

$$(k-1)(k+c+1-\alpha)\gamma \le (k-1)(k+\alpha c), \quad k \ge j+1. \tag{3.4}$$

Since $(k+c+1-\alpha) > 0$ and $k-1 \ge j \ge 1$, we deduce

$$\gamma \le \frac{k+\alpha c}{k+c+1-\alpha} \quad \forall k \ge j+1. \tag{3.5}$$

We define $\gamma(\alpha,c;k) := 1-(1-\alpha)(c+1)/(k+c+1-\alpha)$. Obviously, $\gamma(\alpha,c;j+1) \le \gamma(\alpha,c;k)$ for $k \ge j+1$, hence we obtain that $\gamma = \gamma(\alpha,c;j+1)$.

We have $\gamma < 1$ because $(1-\alpha)(c+1)/(k+c+1-\alpha) > 0$ and $\gamma > \alpha$ because

$$\gamma - \alpha = (1-\alpha)\frac{1-\alpha+j}{2-\alpha+c+j} > 0. \tag{3.6}$$

The result is sharp. Indeed, we consider the function

$$\varphi_\alpha(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)\left[1-\lambda+\lambda(j+1)^m\right]}z^{j+1} \tag{3.7}$$

that belongs to $L_j(n,m,\lambda,\alpha)$. Then

$$I_c(\varphi_\alpha)(z) = z - \frac{(1-\alpha)(c+1)}{(j+1)^n(j+1-\alpha)\left[1-\lambda+\lambda(j+1)^m\right](c+j+1)}z^{j+1}, \tag{3.8}$$

and because

$$\frac{(1-\alpha)(c+1)}{(j+1-\alpha)(c+j+1)} = \frac{1-\gamma}{j+1-\gamma}, \tag{3.9}$$

we deduce that $I_c(\varphi_\alpha) = \varphi_\gamma$ belongs to $L_j(n,m,\lambda,\gamma)$. □

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G. S. Sălăgean: Faculty of Mathematics and Computer Science, Babes-Bolyai University, 1 M. Kogalniceanu Street, 3400 Cluj-Napoca, Romania

E-mail address: salagean@math.ubbcluj.ro