ON STRONG COMMUTATIVITY-PRESERVING MAPS

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We identify some strong commutativity-preserving maps on semiprime rings. Among other results, we prove the following. (i) A centralizing homomorphism f of a semiprime ring R onto itself is strong commutativity preserving. (ii) A centralizing antihomomorphism f of a 2-torsion-free semiprime ring R onto itself is strong commutativity preserving.

1. Introduction and preliminaries

Let R be a ring with center $Z(R)$. We write the commutator $[x, y] = xy - yx$, $(x, y \in R)$. The following commutator identities hold: $[xy, z] = x[y, z] + [x, z]y$; $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$. We recall that R is *prime* if $aRb = (0)$ implies that $a = 0$ or $b = 0$; it is *semiprime* if $aRa = (0)$ implies that $a = 0$. A prime ring is clearly a semiprime ring. A mapping $f : R \rightarrow R$ is called *centralizing* if $[f(x), x] \in Z(R)$ for all $x \in R$; in particular if $[f(x), x] = 0$ for all $x \in R$, then it is called *commuting*. A commuting map is centralizing but the converse is not true, in general. It is easy to see that if $f : R \rightarrow R$ is an additive and commuting map, then $[f(x), y] = [x, f(y)]$ for all $x, y \in R$.

A mapping $f : R \rightarrow R$ is called *commutativity preserving* if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$. Commutativity-preserving maps have been extensively studied on operator algebras (see [7, 9, 11, 12, 13] and the references therein). Many authors have also worked on commutativity-preserving maps on rings (see [1, 2, 6, 8], where further references are also given).

There has also been considerable interest in strong commutativity-preserving maps. A mapping $f : R \rightarrow R$ is called *strong commutativity preserving* if $[f(x), f(y)] = [x, y]$ for all $x, y \in R$. A strong commutativity-preserving map is commutativity preserving but the converse does not hold, in general.

We recall that an additive map f from a ring R into itself is called an *antihomomorphism* if $f(xy) = f(y)f(x)$ for all $x, y \in R$. We will follow Herstein [10] for other undefined notations and terminology used here.

In this paper, we mainly study commutativity-preserving and strong commutativity-preserving properties of homomorphisms and antihomomorphisms of certain rings. We

show (Proposition 2.1) that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing. Furthermore, we prove that if R is a 2-torsion-free semiprime ring and f is a centralizing antihomomorphism of R onto itself, then f is in fact strong commutativity preserving (Proposition 2.4). These and some other related results are proved in Section 2.

2. The results

PROPOSITION 2.1. *Let R be a semiprime ring and f an epimorphism of R . Then f is centralizing if and only if it is strong commutativity preserving.*

Proof. Assume that f is centralizing. Then, by [3, Lemma 2], f is commuting and hence $[f(x), y] = [x, f(y)]$ for all $x, y \in R$. So,

$$[f(xy), x] = [xy, f(x)] = x[y, f(x)] + [x, f(x)]y = x[y, f(x)] = x[f(y), x]. \quad (2.1)$$

That is,

$$[f(xy), x] = x[f(y), x] \quad \forall x, y \in R. \quad (2.2)$$

Also, $[f(xy), x] = [f(x)f(y), x] = f(x)[f(y), x] + [f(x), x]f(y) = f(x)[f(y), x]$. That is,

$$[f(xy), x] = f(x)[f(y), x] \quad \forall x, y \in R. \quad (2.3)$$

By (2.2) and (2.3), we get $f(x)[f(y), x] = x[f(y), x]$. Since f is onto, therefore we have $f(x)[y, x] = x[y, x]$ for all $x, y \in R$. That is,

$$(f(x) - x)[y, x] = 0 \quad \forall x, y \in R. \quad (2.4)$$

Replacing y by uy in (2.4) and using (2.4) again, we get

$$0 = (f(x) - x)[uy, x] = (f(x) - x)u[y, x] + (f(x) - x)[u, x]y = (f(x) - x)u[y, x]. \quad (2.5)$$

So,

$$(f(x) - x)u[y, x] = 0 \quad \forall x, y, u \in R. \quad (2.6)$$

Replacing x by $x + z$ in (2.4), we get

$$\begin{aligned} 0 &= (f(x) - x)[y, x] + (f(x) - x)[y, z] + (f(z) - z)[y, x] + (f(z) - z)[y, z] \\ &= (f(x) - x)[y, z] + (f(z) - z)[y, x]. \end{aligned} \quad (2.7)$$

So,

$$(f(x) - x)[y, z] = -(f(z) - z)[y, x] \quad \forall x, y, z \in R. \quad (2.8)$$

Equation (2.8) implies that for all $x, y, z, v \in R$, we have

$$(f(x) - x)[y, z]v(f(x) - x)[y, z] = -(f(x) - x)[y, z]v(f(z) - z)[y, x]. \quad (2.9)$$

Putting $u = [y, z]v(f(z) - z)$ in (2.6) and using (2.9), we get

$$(f(x) - x)[y, z]v(f(x) - x)[y, z] = 0 \quad \forall v \in R. \quad (2.10)$$

R being semiprime implies that

$$(f(x) - x)[y, z] = 0 \quad \forall x, y, z \in R. \quad (2.11)$$

Replacing y by wy in (2.11), we get

$$0 = (f(x) - x)[wy, z] = (f(x) - x)w[y, z] + (f(x) - x)[w, z]y = (f(x) - x)w[y, z]. \quad (2.12)$$

Thus,

$$(f(x) - x)w[y, z] = 0 \quad \forall x, y, z, w \in R. \quad (2.13)$$

Multiplying (2.13) on the left by $[y, z]$ and on the right by $(f(x) - x)$, we get $[y, z](f(x) - x)w[y, z](f(x) - x) = 0$ for all $w \in R$. By the semiprimeness of R , we get $[y, z](f(x) - x) = 0$ and hence by (2.11), we have $(f(x) - x)[y, z] = [y, z](f(x) - x) = 0$ for all $x, y, z \in R$. So, by Herstein [10, Lemma 1.1.8], $(f(x) - x) \in Z(R)$. Therefore, $[f(x) - x, y] = 0$ for all $x, y \in R$. That is,

$$[f(x), y] = [x, y] \quad \forall x, y \in R. \quad (2.14)$$

Replacing y by $f(y)$ in (2.14), and using (2.14) again, we get $[f(x), f(y)] = [x, f(y)] = [x, y]$ for all $x, y \in R$. This proves that f is strong commutativity preserving.

Conversely, assume that f is strong commutativity preserving. Then,

$$[f(x), f(y)] - [x, y] = 0 \quad \forall x, y \in R. \quad (2.15)$$

Replacing y by xy in (2.15) and using the strong commutativity-preserving property of f , we get

$$\begin{aligned} 0 &= [f(x), f(xy)] - [x, xy] = [f(x), f(x)f(y)] - [x, xy] \\ &= f(x)[f(x), f(y)] + [f(x), f(x)]f(y) - x[x, y] - [x, x]y \\ &= f(x)[x, y] - x[x, y] = (f(x) - x)[x, y]. \end{aligned} \quad (2.16)$$

So,

$$(f(x) - x)[x, y] = 0 \quad \forall x, y \in R. \quad (2.17)$$

Replacing y by zy in (2.17) and using (2.17) again, we get

$$0 = (f(x) - x)[x, zy] = (f(x) - x)z[x, y] + (f(x) - x)[x, z]y = (f(x) - x)z[x, y]. \quad (2.18)$$

That is,

$$(f(x) - x)z[x, y] = 0 \quad \forall x, y, z \in R. \quad (2.19)$$

Replacing y by $f(x)$ in (2.19), we get

$$(f(x) - x)z[f(x), x] = 0 \quad \forall x \in R. \quad (2.20)$$

Replacing z by xz in (2.20), we get

$$(f(x)x - x^2)z[f(x), x] = 0 \quad \forall x \in R. \quad (2.21)$$

Multiplying (2.20) on the left by x , we get

$$(xf(x) - x^2)z[f(x), x] = 0 \quad \forall x \in R. \quad (2.22)$$

Subtracting (2.22) from (2.21), we get $[f(x), x]z[f(x), x] = 0$ for all $x, z \in R$. Since R is semiprime, therefore, $[f(x), x] = 0$ for all $x \in R$. So, f is commuting and hence centralizing. $\square$

Remark 2.2. In Proposition 2.1, the implication that f is strong commutativity preserving implying that it is centralizing also follows from Brešar and Miers [7, Theorem 1]; however, the proof in the case of homomorphisms is simple and we have included it here for the sake of completeness. Furthermore, it may be of independent interest.

Remark 2.3. Let R be a ring and $f : R \rightarrow R$ an antihomomorphism. Then clearly, f is commutativity preserving.

The following proposition shows that under some additional assumptions, an antihomomorphism must be strong commutativity preserving.

PROPOSITION 2.4. *Let R be a 2-torsion-free semiprime ring and f a centralizing antihomomorphism of R onto itself. Then f is strong commutativity preserving.*

Proof. By [5, Proposition 3.1], f is commuting and hence, $[f(x), y] = [x, f(y)]$ for all $x, y \in R$. So, $[f(xy), x] = [xy, f(x)] = x[y, f(x)] + [x, f(x)]y = x[y, f(x)]$. That is,

$$[f(xy), x] = x[y, f(x)] \quad \forall x, y \in R. \quad (2.23)$$

Also, $[f(xy), x] = [f(y)f(x), x] = f(y)[f(x), x] + [f(y), x]f(x) = [f(y), x]f(x)$. That is,

$$[f(xy), x] = [f(y), x]f(x) \quad \forall x, y \in R. \quad (2.24)$$

From (2.23) and (2.24), we get $[f(y), x]f(x) = x[y, f(x)]$; that is, $[f(y), x]f(x) = x[f(y), x]$ for all $x, y \in R$. Now f being onto implies that $[y, x]f(x) = x[y, x]$. So,

$$[y, x]f(x) = x[y, x] \quad \forall x, y \in R. \quad (2.25)$$

Replacing y by uy in (2.25), we get $[uy, x]f(x) = x[uy, x]$. That is,

$$u[y, x]f(x) + [u, x]yf(x) = xu[y, x] + x[u, x]y \quad \forall x, y \in R. \quad (2.26)$$

By (2.25) and (2.26), we get $ux[y, x] + [u, x]yf(x) = xu[y, x] + x[u, x]y$. That is, $ux[y, x] + [u, x]yf(x) = xu[y, x] + [u, x]f(x)y$. This implies that

$$ux[y, x] - xu[y, x] + [u, x]yf(x) - [u, x]f(x)y = 0. \quad (2.27)$$

That is,

$$[u, x][y, x] + [u, x][y, f(x)] = 0. \quad (2.28)$$

Using the fact that f is commuting, we get

$$0 = [u, x][y, x] + [u, x][y, f(x)] = [u, x]([y, x] + [f(y), x]) = [u, x][y + f(y), x]. \quad (2.29)$$

So,

$$[u, x][y + f(y), x] = 0 \quad \forall x, y, u \in R. \quad (2.30)$$

Replacing u by uz in (2.30) and using (2.30) again, we get

$$\begin{aligned} 0 &= [uz, x][y + f(y), x] = [u, x]z[y + f(y), x] + u[z, x][y + f(y), x] \\ &= [u, x]z[y + f(y), x]. \end{aligned} \quad (2.31)$$

That is,

$$[u, x]z[y + f(y), x] = 0 \quad \forall x, y, u, z \in R. \quad (2.32)$$

Replacing u by $y + f(y)$ in (2.32), we get $[y + f(y), x]z[y + f(y), x] = 0$ for all $x, y, z \in R$. Since R is semiprime, we get

$$[y + f(y), x] = 0 \quad \forall x, y \in R. \quad (2.33)$$

Rewriting (2.33), we get $0 = [y, x] + [f(y), x] = [y, x] + [y, f(x)] = [y, x] - [f(x), y]$. So,

$$[f(x), y] = [y, x] \quad \forall x, y \in R. \quad (2.34)$$

That f is strong commutativity preserving follows from (2.34). Indeed, $[f(x), f(y)] = [f(y), x] = [x, y]$ for all $x, y \in R$. $\square$

Remark 2.5. Brešar [4, Proposition 4.1] has proved the following result.

THEOREM 2.6. *Let R be a 2-torsion-free semiprime ring and let $f : R \rightarrow R$ be a centralizing antihomomorphism. Then,*

- (a) $S = \{x \in R : f(x) = x\} \subseteq Z(R)$,
- (b) *if R is prime and f does not map R into $Z(R)$, then $S = Z(R)$.*

We note that Theorem 2.6 can also be obtained as an application of Proposition 2.4 if f is onto. Thus our proof (below) can be regarded as an alternate argument for Theorem 2.6 which may also be of independent interest.

Proof. (a) By (2.33), $f(y) + y \in Z(R)$ for all $y \in R$. Therefore, for z in S , $f(z) + z = 2z \in Z(R)$. So, $[2z, x] = 2[z, x] = 0$ for all $x \in R$. As R is 2-torsion-free, so $[z, x] = 0$ for all $x \in R$. Therefore, $z \in Z(R)$ and hence $S \subseteq Z(R)$.

(b) Assume that R is prime and let $z \in Z(R)$. If $z = 0$, then $f(0) = 0$ implies that $0 \in S$. So, assume that $z \neq 0$. Then $f(z) + z \in Z(R)$, $z \in Z(R)$. So, $f(z) \in Z(R)$. Now replacing x by zx in (2.25), we get $[y, zx]f(zx) = (zx)[y, zx]$. That is,

$$z[y, x]f(x)f(z) + [y, z]xf(x)f(z) = zxz[y, x] + zx[y, z]x. \quad (2.35)$$

As $z \in Z(R)$, by (2.35), we get $z[y, x]f(x)f(z) = zxz[y, x]$. That is,

$$[y, x]f(x)f(z)z = x[y, x]z^2 \quad \forall x, y \in R, z \in Z(R). \quad (2.36)$$

By (2.25) and (2.36), we get $[y, x]f(x)f(z)z = [y, x]f(x)z^2$. That is,

$$[y, x]f(x)(f(z)z - z^2) = 0 \quad \forall x, y \in R, z \in Z(R). \quad (2.37)$$

Since R is prime, then any nonzero central element is not a zero divisor. Hence, if $f(z)z - z^2 \neq 0$, then $[y, x]f(x) = 0$ for all $x, y \in R$. Then by [10, corollary, page 8], either $f(x) = 0$ or $x \in Z(R)$. In any case, $f(x) \in Z(R)$ for all $x \in R$, a contradiction. So, $0 = f(z)z - z^2 = (f(z) - z)z$. As $z \neq 0$, therefore by the above argument, $f(z) - z = 0$ and hence $z \in S$. So, $Z(R) \subseteq S$ and by (a), we have $Z(R) = S$. $\square$

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