The notion of left-right (resp., right-left) \( f \)-derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular \( f \)-derivation, we give characterizations of a \( p \)-semisimple BCI-algebra.

1. Introduction and preliminaries

In the theory of rings and near-rings, the properties of derivations are an important topic to study, see [2, 3, 7, 10]. In [6], Jun and Xin applied the notions in rings and near-rings theory to BCI-algebras, and obtained some related properties. In this paper, the notion of left-right (resp., right-left) \( f \)-derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular \( f \)-derivation, we give characterizations of a \( p \)-semisimple BCI-algebra.

By a BCI-algebra we mean an algebra \((X; *, 0)\) of type (2,0) satisfying the following conditions:

(I) \(((x * y) * (x * z)) * (z * y) = 0\);

(II) \((x * (x * y)) * y = 0\);

(III) \(x * x = 0\);

(IV) \(x * y = 0\) and \(y * x = 0\) imply that \(x = y\);

for all \(x, y, z \in X\).

In any BCI-algebra \(X\), one can define a partial order “\(\leq\)” by putting \(x \leq y\) if and only if \(x * y = 0\).

A subset \(S\) of a BCI-algebra \(X\) is called a subalgebra of \(X\) if \(x * y \in S\) for all \(x, y \in S\). A subset \(I\) of a BCI-algebra \(X\) is called an ideal of \(X\) if it satisfies (i) \(0 \in I\); (ii) \(x * y \in I\) and \(y \in I\) imply that \(x \in I\) for all \(x, y \in X\).

A mapping \(f\) of a BCI-algebra \(X\) into itself is called an endomorphism of \(X\) if \(f(x * y) = f(x) * f(y)\) for all \(x, y \in X\). Note that \(f(0) = 0\). Especially, \(f\) is monic if for any \(x, y \in X\), \(f(x) = f(y)\) implies that \(x = y\).

A BCI-algebra \(X\) has the following properties:

(1) \(x * 0 = x\);

(2) \((x * y) * z = (x * z) * y\);
(3) \( x \leq y \) implies that \( x \ast z \leq y \ast z \) and \( z \ast y \leq z \ast x \);
(4) \( x \ast (x \ast (x \ast y)) = x \ast y \);
(5) \( (x \ast z) \ast (y \ast z) \leq x \ast y \);
(6) \( 0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y) \);
(7) \( x \ast 0 = 0 \) implies that \( x = 0 \).

For a BCI-algebra \( X \), denote by \( X_+ \) (resp., \( G(X) \)) the BCK-part (resp., the BCI-G part) of \( X \), that is, \( X_+ = \{ x \in X \mid 0 \leq x \} \) (resp., \( G(X) = \{ x \in X \mid 0 \ast x = x \} \)). Note that \( G(X) \cap X_+ = \{ 0 \} \). If \( X_+ = \{ 0 \} \), then \( X \) is called a \( p \)-semisimple BCI-algebra.

In a \( p \)-semisimple BCI-algebra \( X \), the following hold:

(8) \( (x \ast z) \ast (y \ast z) = x \ast y \);
(9) \( 0 \ast (0 \ast x) = x \);
(10) \( x \ast (0 \ast y) = y \ast (0 \ast x) \);
(11) \( x \ast y = 0 \) implies that \( x = y \);
(12) \( x \ast a = x \ast b \) implies that \( a = b \);
(13) \( a \ast x = b \ast x \) implies that \( a = b \);
(14) \( a \ast (a \ast x) = x \).

Let \( X \) be a \( p \)-semisimple BCI-algebra. We define addition “+” as \( x + y = x \ast (0 \ast y) \) for all \( x, y \in X \). Then \( (X, +) \) is an abelian group with identity 0 and \( x - y = x \ast y \). Conversely, let \( (X, +) \) be an abelian group with identity 0 and let \( x \ast y = x - y \). Then \( X \) is a \( p \)-semisimple BCI-algebra and \( x + y = x \ast (0 \ast y) \) for all \( x, y \in X \) (see [5]).

For a BCI-algebra \( X \), we denote \( x \wedge y = y \ast (y \ast x) \), in particular, \( 0 \ast (0 \ast x) = a_x \), and \( L_p(X) = \{ a \in X \mid x \ast a = 0 \Rightarrow x = a \) for any \( x \in X \}. \) We call the elements of \( L_p(X) \) the \( p \)-atoms of \( X \). For any \( a \in X \), let \( V(a) = \{ x \in X \mid a \ast x = 0 \} \), which is called the branch of \( X \) with respect to \( a \). It follows that \( x \ast y \in V(a \ast b) \) whenever \( x \in V(a) \) and \( y \in V(b) \) for all \( x, y \in X \) and \( a, b \in L_p(X) \). Note that \( L_p(X) = \{ x \in X \mid a_x = x \} \), which is the \( p \)-semisimple part of \( X \), and \( X \) is a \( p \)-semisimple BCI-algebra if and only if \( L_p(X) = X \) (see [6]). Note also that \( a_x \in L_p(X) \), that is, \( 0 \ast (0 \ast a_x) = a_x \), which implies that \( a_x \ast y \in L_p(X) \) for all \( y \in X \). It is clear that \( G(X) \subseteq L_p(X) \), \( x \ast (x \ast a) = a \), and \( a \ast x \in L_p(X) \) for all \( a \in L_p(X) \) and \( x \in X \). For more details, refer to [1, 8, 11].

**Definition 1.1** [9]. A BCI-algebra \( X \) is said to be commutative if \( x \ast y \) whenever \( x \leq y \) for all \( x, y \in X \).

**Definition 1.2** [4]. A BCI-algebra \( X \) is said to be branchwise commutative if \( x \ast y = y \wedge x \) for all \( x, y \in V(a) \) and all \( a \in L_p(X) \).

**Lemma 1.3** [6]. A BCI-algebra \( X \) is commutative if and only if it is branchwise commutative.

**Definition 1.4** [6]. Let \( X \) be a BCI-algebra. By a left-right derivation (briefly, \( (l, r) \)-derivation) of \( X \), a self-map \( d \) of \( X \) satisfying the identity \( d(x \ast y) = (d(x) \ast y) \wedge (x \ast d(y)) \) for all \( x, y \in X \) is meant. If \( d \) satisfies the identity \( d(x \ast y) = (x \ast d(y)) \wedge (d(x) \ast y) \) for all \( x, y \in X \), then it is said that \( d \) is a right-left derivation (briefly, \( (r, l) \)-derivation) of \( X \). Moreover, if \( d \) is both an \( (r, l) \)- and an \( (l, r) \)-derivation, it is said that \( d \) is a derivation.

### 2. \( f \)-derivations

In what follows, let \( f \) be an endomorphism of \( X \) unless otherwise specified.
Definition 2.1. Let \( X \) be a BCI-algebra. By a left-right \( f \)-derivation (briefly, \((l, r)\)-\( f \)-derivation) of \( X \), a self-map \( d_f \) of \( X \) satisfying the identity
\[
d_f(x \ast y) = (d_f(x) \ast f(y)) \land (f(x) \ast d_f(y))
\]
for all \( x, y \in X \) is meant, where \( f \) is an endomorphism of \( X \). If \( d_f \) satisfies the identity
\[
d_f(x \ast y) = (f(x) \ast d_f(y)) \land (d_f(x) \ast f(y))
\]
for all \( x, y \in X \), then it is said that \( d_f \) is a right-left \( f \)-derivation (briefly, \((r, l)\)-\( f \)-derivation) of \( X \). Moreover, if \( d_f \) is both an \((r, l)\)- and an \((l, r)\)-\( f \)-derivation, it is said that \( d_f \) is an \( f \)-derivation.

Example 2.2. Let \( X = \{0, 1, 2, 3, 4, 5\} \) be a BCI-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
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<td>0</td>
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</tr>
<tr>
<td>5</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a map \( d_f : X \rightarrow X \) by
\[
d_f(x) = \begin{cases} 
2 & \text{if } x = 0, 1, \\
0 & \text{otherwise}, 
\end{cases} \tag{2.1}
\]
and define an endomorphism \( f \) of \( X \) by
\[
f(x) = \begin{cases} 
0 & \text{if } x = 0, 1, \\
2 & \text{otherwise}. 
\end{cases} \tag{2.2}
\]
Then it is easily checked that \( d_f \) is both derivation and \( f \)-derivation of \( X \).

Example 2.3. Let \( X \) be a BCI-algebra as in Example 2.2. Define a map \( d_f : X \rightarrow X \) by
\[
d_f(x) = \begin{cases} 
2 & \text{if } x = 0, 1, \\
0 & \text{otherwise}, 
\end{cases} \tag{2.3}
\]
Then it is easily checked that \( d_f \) is a derivation of \( X \).

Define an endomorphism \( f \) of \( X \) by
\[
f(x) = 0, \quad \forall x \in X. \tag{2.4}
\]
Then \( d_f \) is not an \( f \)-derivation of \( X \) since
\[
d_f(2 \ast 3) = d_f(0) = 2, \tag{2.5}
\]
but
\[
(d_f(2) \ast f(3)) \land (f(2) \ast d_f(3)) = (0 \ast 0) \land (0 \ast 0) = 0 \land 0 = 0, \tag{2.6}
\]
and thus \( d_f(2 \ast 3) \neq (d_f(2) \ast f(3)) \land (f(2) \ast d_f(3)) \).
Remark 2.4. From Example 2.3, we know that there is a derivation of $X$ which is not an $f$-derivation of $X$.

Example 2.5. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a BCI-algebra with the following Cayley table:

```
  | 0 1 2 3 4 5
---|-----|-----|-----|-----|-----|-----|
 0 | 0 0 3 2 3 2
 1 | 1 0 5 4 3 2
 2 | 2 2 0 3 0 3
 3 | 3 3 2 0 2 0
 4 | 4 2 1 5 0 3
 5 | 5 3 4 1 2 0
```

Define a map $d_f : X \to X$ by

$$d_f(x) = \begin{cases} 
0 & \text{if } x = 0, 1, \\
2 & \text{if } x = 2, 4, \\
3 & \text{if } x = 3, 5, 
\end{cases}$$

(2.7)

and define an endomorphism $f$ of $X$ by

$$f(x) = \begin{cases} 
0 & \text{if } x = 0, 1, \\
2 & \text{if } x = 2, 4, \\
3 & \text{if } x = 3, 5. 
\end{cases}$$

(2.8)

Then it is easily checked that $d_f$ is both derivation and $f$-derivation of $X$.

Example 2.6. Let $X$ be a BCI-algebra as in Example 2.5. Define a map $d_f : X \to X$ by

$$d_f(x) = \begin{cases} 
0 & \text{if } x = 0, 1, \\
2 & \text{if } x = 2, 4, \\
3 & \text{if } x = 3, 5, 
\end{cases}$$

(2.9)

Then it is easily checked that $d_f$ is a derivation of $X$.

Define an endomorphism $f$ of $X$ by

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 3, \quad f(3) = 2, \quad f(4) = 5, \quad f(5) = 4.$$  

(2.10)

Then $d_f$ is not an $f$-derivation of $X$ since

$$d_f(2 \ast 3) = d_f(3) = 3,$$

(2.11)

but

$$(d_f(2) \ast f(3)) \land (f(2) \ast d_f(3)) = (2 \ast 2) \land (3 \ast 3) = 0 \land 0 = 0,$$

(2.12)

and thus $d_f(2 \ast 3) \neq (d_f(2) \ast f(3)) \land (f(2) \ast d_f(3))$. 
Example 2.7. Let $X$ be a BCI-algebra as in Example 2.5. Define a map $d_f : X \rightarrow X$ by

$$d_f(0) = 0, \quad d_f(1) = 1, \quad d_f(2) = 3, \quad d_f(3) = 2, \quad d_f(4) = 5, \quad d_f(5) = 4. \quad \text{(2.13)}$$

Then $d_f$ is not a derivation of $X$ since

$$d_f(2 \ast 3) = d_f(3) = 2, \quad \text{(2.14)}$$

but

$$(d_f(2 \ast 3) \land (2 \ast d_f(3))) = (3 \land 2) = 0 \land 0 = 0, \quad \text{(2.15)}$$

and thus $d_f(2 \ast 3) \neq (d_f(2) \ast 3) \land (2 \ast d_f(3))$.

Define an endomorphism $f$ of $X$ by

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 3, \quad f(3) = 2, \quad f(4) = 5, \quad f(5) = 4. \quad \text{(2.16)}$$

Then it is easily checked that $d_f$ is an $f$-derivation of $X$.

Remark 2.8. From Example 2.7, we know that there is an $f$-derivation of $X$ which is not a derivation of $X$.

For convenience, we denote $f_x = 0 \ast (0 \ast f(x))$ for all $x \in X$. Note that $f_x \in L_p(X)$.

Theorem 2.9. Let $d_f$ be a self-map of a BCI-algebra $X$ defined by $d_f(x) = f_x$ for all $x \in X$. Then $d_f$ is an $(l, r)$-$f$-derivation of $X$. Moreover, if $X$ is commutative, then $d_f$ is an $(r, l)$-$f$-derivation of $X$.

Proof. Let $x, y \in X$.

Since

$$0 \ast (0 \ast (f_x \ast f(y))) = 0 \ast (0 \ast ((0 \ast (0 \ast f(x))) \ast f(y)))$$

$$= 0 \ast (0 \ast ((0 \ast f(y)) \ast (0 \ast f(x))))$$

$$= 0 \ast (0 \ast ((0 \ast f(y))) \ast (0 \ast f(x)) = 0 \ast f(y \ast x)$$

$$= 0 \ast (f(y) \ast f(x)) = (0 \ast f(y)) \ast (0 \ast f(x))$$

$$= (0 \ast (0 \ast f(x))) \ast f(y) = f_x \ast f(y), \quad \text{(2.17)}$$

we have $f_x \ast f(y) \in L_p(X)$, and thus

$$f_x \ast f(y) = (f(x) \ast f(y)) \ast ((f(x) \ast f(y)) \ast (f \ast f(y))). \quad \text{(2.18)}$$

It follows that

$$d_f(x \ast y) = f_{x \ast y} = 0 \ast (0 \ast f(x \ast y)) = 0 \ast (0 \ast ((0 \ast f(x))) \ast f(y)))$$

$$= (0 \ast (0 \ast f(x))) \ast (0 \ast f(y)) = f_x \ast f_y$$

$$= (0 \ast (0 \ast f)) \ast (0 \ast f(y))) = 0 \ast (0 \ast (f \ast f(y)))$$

$$= 0 \ast (f \ast f(y)) \ast (f) \ast f(y)) = (f \ast f(y)) \ast (d_f(x) \ast f(y) \ast (f \ast d_f(y))), \quad \text{(2.19)}$$
and so $d_f$ is an $(l,r)$-$f$-derivation of $X$. Now, assume that $X$ is commutative. Using Lemma 1.3, it is sufficient to show that $d_f(x) \ast f(y)$ and $f(x) \ast d_f(y)$ belong to the same branch for all $x, y \in X$, we have

$$d_f(x) \ast f(y) = f_x \ast f(y) = 0 \ast (0 \ast (f_x \ast f(y))) = (0 \ast (0 \ast f_x)) \ast (0 \ast (0 \ast f(y))) = f_x \ast f_y \in V(f_x \ast f_y),$$

and so $f_x \ast f_y = (0 \ast (0 \ast f(x))) \ast (0 \ast (0 \ast f_y)) \ast (0 \ast (0 \ast (f(x) \ast d_f(y)))) \leq f(x) \ast d_f(y)$, which implies that $f(x) \ast d_f(y) \in V(f_x \ast f_y)$. Hence, $d_f(x) \ast f(y)$ and $f(x) \ast d_f(y)$ belong to the same branch, and so

$$d_f(x \ast y) = (d_f(x) \ast f(y)) \land (f(x) \ast d_f(y)) = (f(x) \ast d_f(y)) \land (d_f(x) \ast f(y)).$$

This completes the proof.\hfill \qed

**Proposition 2.10.** Let $d_f$ be a self-map of a BCI-algebra $X$. Then the following hold.

(i) If $d_f$ is an $(l,r)$-$f$-derivation of $X$, then $d_f(x) = d_f(x) \land f(x)$ for all $x \in X$.

(ii) If $d_f$ is an $(r,l)$-$f$-derivation of $X$, then $d_f(x) = f(x) \land d_f(x)$ for all $x \in X$ if and only if $d_f(0) = 0$.

**Proof.**

(i) Let $d_f$ be an $(l,r)$-$f$-derivation of $X$. Then,

$$d_f(x) = d_f(x \ast 0) = (d_f(x) \ast f(0)) \land (f(x) \ast d_f(0)) = (d_f(x) \ast 0) \land (f(x) \ast d_f(0)) = d_f(x) \land (f(x) \ast d_f(0)) = d_f(0) \ast (d_f(x) \ast f(x)) \ast (d_f(x) \ast f(0)) \leq f(x) \ast (f(x) \ast d_f(x)) = d_f(x) \land f(x).$$

But $d_f(x) \land f(x) \leq d_f(x)$ is trivial and so (i) holds.

(ii) Let $d_f$ be an $(r,l)$-$f$-derivation of $X$. If $d_f(x) = f(x) \land d_f(x)$ for all $x \in X$, then for $x = 0$, $d_f(0) = f(0) \land d_f(0) = 0 \land d_f(0) = d_f(0) \ast (d_f(0) \ast 0) = 0$.

Conversely, if $d_f(0) = 0$, then $d_f(x) = d_f(x \ast 0) = (f(x) \ast d_f(0)) \land (d_f(x) \ast f(0)) = (f(x) \ast 0) \land (d_f(x) \ast 0) = f(x) \land d_f(x)$, ending the proof.\hfill \qed

**Proposition 2.11.** Let $d_f$ be an $(l,r)$-$f$-derivation of a BCI-algebra $X$. Then,

(i) $d_f(0) \in L_p(X)$, that is, $d_f(0) = 0 \ast (0 \ast d_f(0))$;

(ii) $d_f(a) = d_f(0) \ast (0 \ast f(a)) = d_f(0) + f(a)$ for all $a \in L_p(X)$;

(iii) $d_f(a) \in L_p(X)$ for all $a \in L_p(X)$;

(iv) $d_f(a + b) = d_f(a) + d_f(b) - d_f(0)$ for all $a, b \in L_p(X)$.
Proof. (i) The proof follows from Proposition 2.10(i).

(ii) Let \( a \in L_p(X) \), then \( a = 0 \ast (0 \ast a) \), and so \( f(a) = 0 \ast (0 \ast f(a)) \), that is, \( f(a) \in L_p(X) \). Hence

\[
\begin{align*}
d_f(a) &= d_f(0 \ast (0 \ast a)) \\
&= (d_f(0) \ast f(0 \ast a)) \land (f(0) \ast d_f(0 \ast a)) \\
&= (d_f(0) \ast f(0 \ast a)) \land (0 \ast d_f(0 \ast a)) \\
&= (0 \ast d_f(0 \ast a)) \ast ((0 \ast d_f(0 \ast a)) \ast (d_f(0) \ast f(0 \ast a))) \\
&= (0 \ast d_f(0 \ast a)) \ast ((0 \ast (d_f(0) \ast f(0 \ast a))) \ast d_f(0 \ast a)) \\
&= 0 \ast (0 \ast (d_f(0) \ast (f(0) \ast f(a)))) \\
&= d_f(0) \ast (0 \ast f(a)) = d_f(0) + f(a).
\end{align*}
\]

(iii) The proof follows directly from (ii).

(iv) Let \( a, b \in L_p(X) \). Note that \( a + b \in L_p(X) \), so from (ii), we note that

\[
\begin{align*}
d_f(a + b) &= d_f(0) + f(a + b) \\
&= d_f(0) + f(a) + d_f(0) + f(b) - d_f(0) = d_f(a) + d_f(b) - d_f(0).
\end{align*}
\]

\[\Box\]

**Proposition 2.12.** Let \( d_f \) be a \((r,l)\)-f-derivation of a BCI-algebra \( X \). Then,

(i) \( d_f(a) \in G(X) \) for all \( a \in G(X) \);

(ii) \( d_f(a) \in L_p(X) \) for all \( a \in G(X) \);

(iii) \( d_f(a) = f(a) \ast d_f(0) = f(a) + d_f(0) \) for all \( a \in L_p(X) \);

(iv) \( d_f(a + b) = d_f(a) + d_f(b) - d_f(0) \) for all \( a, b \in L_p(X) \).

**Proof.** (i) For any \( a \in G(X) \), we have

\[
\begin{align*}
d_f(a) &= d_f(0 \ast (0 \ast a)) = (d_f(0) \ast f(0 \ast a)) \land (d_f(0) \ast f(0)) \\
&= (0 \ast d_f(0 \ast a)) \ast ((0 \ast (d_f(0) \ast f(0 \ast a))) \ast (0 \ast d_f(0 \ast a))) \\
&= 0 \ast d_f(0 \ast a) \in L_p(X).
\end{align*}
\]

(ii) For any \( a \in L_p(X) \), we get

\[
\begin{align*}
d_f(a) &= d_f(a \ast 0) = (f(a) \ast d_f(0)) \land (d_f(a) \ast f(0)) \\
&= d_f(a) \ast (d_f(a) \ast (f(a) \ast d_f(0))) = f(a) \ast d_f(0) \\
&= f(a) \ast (0 \ast d_f(0)) = f(a) + d_f(0).
\end{align*}
\]

(iii) For any \( a \in L_p(X) \), we get

\[
\begin{align*}
d_f(a) &= d_f(a \ast 0) = (f(a) \ast d_f(0)) \land (d_f(a) \ast f(0)) \\
&= d_f(a) \ast (d_f(a) \ast (f(a) \ast d_f(0))) = f(a) \ast d_f(0) \\
&= f(a) \ast (0 \ast d_f(0)) = f(a) + d_f(0).
\end{align*}
\]

(iv) The proof follows from (iii). This completes the proof. \[\Box\]
Using Proposition 2.12, we know there is an \((l,r)\)-\(f\)-derivation which is not an \((r,l)\)-\(f\)-derivation as shown in the following example.

Example 2.13. Let \(\mathbb{Z}\) be the set of all integers and \(\neg\) the minus operation on \(\mathbb{Z}\). Then \((\mathbb{Z},\neg,0)\) is a BCI-algebra. Let \(d_f : X \to X\) be defined by \(d_f(x) = f(x) - 1\) for all \(x \in \mathbb{Z}\). Then,

\[
\begin{align*}
(d_f(x) - f(y)) \land (f(x) - d_f(y)) &= (f(x) - 1 - f(y)) \land (f(x) - (f(y) - 1)) \\
&= (f(x - y) - 1) \land (f(x - y) + 1) \\
&= (f(x - y) + 1) - 2 = f(x - y) - 1 \\
&= d_f(x - y).
\end{align*}
\]

Hence, \(d_f\) is an \((l,r)\)-\(f\)-derivation of \(X\). But \(d_f(0) = f(0) - 1 = -1 \neq 1 = f(0) - d_f(0) = 0 - d_f(0)\), that is, \(d_f(0) \notin G(X)\). Therefore, \(d_f\) is not an \((r,l)\)-\(f\)-derivation of \(X\) by Proposition 2.12(i).

3. Regular \(f\)-derivations

Definition 3.1. An \(f\)-derivation \(d_f\) of a BCI-algebra \(X\) is said to be regular if \(d_f(0) = 0\).

Remark 3.2. We know that the \(f\)-derivations \(d_f\) in Examples 2.5 and 2.7 are regular.

Proposition 3.3. Let \(X\) be a commutative BCI-algebra and let \(d_f\) be a regular \((r,l)\)-\(f\)-derivation of \(X\). Then the following hold.

(i) Both \(f(x)\) and \(d_f(x)\) belong to the same branch for all \(x \in X\).

(ii) \(d_f\) is an \((l,r)\)-\(f\)-derivation of \(X\).

Proof. (i) Let \(x \in X\). Then,

\[
0 = d_f(0) = d_f(ax \ast x) \\
= (f(ax) \ast d_f(x)) \land (d(ax) \ast f(x)) \\
= (d(ax) \ast f(x)) \ast ((d(ax) \ast f(x)) \ast (f(ax) \ast d_f(x))) \\
= (d(ax) \ast f(x)) \ast ((d(ax) \ast f(x)) \ast (f_x \ast d_f(x))) \\
= f_x \ast d_f(x) \quad \text{since } f_x \ast d_f(x) \in L_p(X),
\]

and so \(f_x \leq d_f(x)\). This shows that \(d_f(x) \in V(f_x)\). Clearly, \(f(x) \in V(f_x)\).

(ii) By (i), we have \(f(x) \ast d_f(y) \in V(f_x \ast f_y)\) and \(d_f(x) \ast f(y) \in V(f_x \ast f_y)\). Thus \(d_f(x \ast y) = (f(x) \ast d_f(y)) \land (d_f(x) \ast f(y)) = (d_f(x) \ast f(y)) \land (f(x) \ast d_f(y))\), which implies that \(d_f\) is an \((l,r)\)-\(f\)-derivation of \(X\).

Remark 3.4. The \(f\)-derivations \(d_f\) in Examples 2.5 and 2.7 are regular \(f\)-derivations but we know that the \((l,r)\)-\(f\)-derivation \(d_f\) in Example 2.2 is not regular. In the following, we give some properties of regular \(f\)-derivations.

Definition 3.5. Let \(X\) be a BCI-algebra. Then define \(\ker d_f = \{x \in X \mid d_f(x) = 0 \text{ for all } f\text{-derivations } d_f\}\).
PROPOSITION 3.6. Let $d_f$ be an $f$-derivation of a BCI-algebra $X$. Then the following hold:

(i) $d_f(x) \leq f(x)$ for all $x \in X$;

(ii) $d_f(x) \ast f(y) \leq f(x) \ast d_f(y)$ for all $x, y \in X$;

(iii) $d_f(x \ast y) = d_f(x) \ast f(y) \leq d_f(x) \ast d_f(y)$ for all $x, y \in X$;

(iv) ker $d_f$ is a subalgebra of $X$. Especially, if $f$ is monic, then ker $d_f \subseteq X_+.$

Proof. (i) The proof follows by Proposition 2.10(ii).

(ii) Since $d_f(x) \leq f(x)$ for all $x \in X$, then $d_f(x) \ast f(y) \leq f(x) \ast f(y) \leq f(x) \ast d_f(y)$.

(iii) For any $x, y \in X$, we have

\[
d_f(x \ast y) = (f(x) \ast d_f(y)) \wedge (d_f(x) \ast f(y)) = (d_f(x) \ast f(y)) \ast ((d_f(x) \ast f(y)) \ast (f(x) \ast d_f(y))) = (d_f(x) \ast f(y)) \ast 0 = d_f(x) \ast f(y) \leq d_f(x) \ast d_f(y),
\]

which proves (iii).

(iv) Let $x, y \in \ker d_f$, then $d_f(x) = 0 = d_f(y)$, and so $d_f(x \ast y) \leq d_f(x) \ast d_f(y) = 0 \ast 0 = 0$ by (iii), and thus $d_f(x \ast y) = 0$, that is, $x \ast y \in \ker d_f$. Hence, ker $d_f$ is a subalgebra of $X$. Especially, if $f$ is monic, and letting $x \in \ker d_f$, then $0 = d_f(x) \leq f(x)$ by (i), and so $f(x) \in X_+$, that is, $0 \ast f(x) = 0$, and thus $f(0 \ast x) = f(x)$, which implies that $0 \ast x = x$, and so $x \in X_+$, that is, ker $d_f \subseteq X_+.$

THEOREM 3.7. Let $f$ be monic of a commutative BCI-algebra $X$. Then $X$ is $p$-semisimple if and only if ker $d_f = \{0\}$ for every regular $f$-derivation $d_f$ of $X$.

Proof. Assume that $X$ is $p$-semisimple BCI-algebra and let $d_f$ be a regular $f$-derivation of $X$. Then $X_+ = \{0\}$, and so ker $d_f = \{0\}$ by using Proposition 3.6(iv). Conversely, let ker $d_f = \{0\}$ for every regular $f$- derivation $d_f$ of $X$. Define a self-map $d_f^*$ of $X$ by $d_f^*(x) = f_x$ for all $x \in X$. Using Theorem 2.9, $d_f^*$ is an $f$-derivation of $X$. Clearly, $d_f^*(0) = f_0 = 0 \ast (0 \ast f(0)) = 0$, and so $d_f^*$ is a regular $f$-derivation of $X$. It follows from the hypothesis that ker $d_f^* = \{0\}$. In addition, $d_f^*(x) = f_x = 0 \ast (0 \ast f(x)) = f(0 \ast (0 \ast x)) = f(0) = 0$ for all $x \in X_+$, and thus $x \in \ker d_f^*$, which shows that $X_+ \subseteq \ker d_f^*$. Hence, by Proposition 3.6(iv), $X_+ = \ker d_f^* = \{0\}$. Therefore $X$ is $p$-semisimple.

Definition 3.8. An ideal $A$ of a BCI-algebra $X$ is said to be an $f$-ideal if $f(A) \subseteq A$.

Definition 3.9. Let $d_f$ be a self-map of a BCI-algebra $X$. An $f$-ideal $A$ of $X$ is said to be $d_f$-invariant if $d_f(A) \subseteq A$.

THEOREM 3.10. Let $d_f$ be a regular $(r,l)$-$f$-derivation of a BCI-algebra $X$, then every $f$-ideal $A$ of $X$ is $d_f$-invariant.

Proof. By Proposition 2.10(ii), we have $d_f(x) = f(x) \wedge d_f(x) \leq f(x)$ for all $x \in X$. Let $y \in d_f(A)$. Then $y = d_f(x)$ for some $x \in A$. It follows that $y \ast f(x) = d_f(x) \ast f(x) = 0 \in A$. Since $x \in A$, then $f(x) \in f(A) \subseteq A$ as $A$ is an $f$-ideal. It follows that $y \in A$ since $A$ is an ideal of $X$. Hence $d_f(A) \subseteq A$, and thus $A$ is $d_f$-invariant.

THEOREM 3.11. Let $d_f$ be an $f$-derivation of a BCI-algebra $X$. Then $d_f$ is regular if and only if every $f$-ideal of $X$ is $d_f$-invariant.
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Proof. Let $d_f$ be a derivation of a BCI-algebra $X$ and assume that every $f$-ideal of $X$ is $d_f$-invariant. Then since the zero ideal $\{0\}$ is $f$-ideal and $d_f$-invariant, we have $d_f(\{0\}) \subseteq \{0\}$, which implies that $d_f(0) = 0$. Thus $d_f$ is regular. Combining this and Theorem 3.10, we complete the proof. □

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