ON FUNCTIONS WITH THE CAUCHY DIFFERENCE BOUNDED BY A FUNCTIONAL. PART II

WŁODZIMIERZ FECHNER

Received 10 December 2004 and in revised form 17 May 2005

We are going to consider the functional inequality $f(x+y) - f(x) - f(y) \geq \phi(x, y)$, $x, y \in X$, where $(X, +)$ is an abelian group, and $\phi : X \times X \rightarrow \mathbb{R}$ and $f : X \rightarrow \mathbb{R}$ are unknown mappings. In particular, we will give conditions which force biadditivity and symmetry of ϕ and the representation $f(x) = (1/2)\phi(x, x) + a(x)$ for $x \in X$, where a is an additive function. In the present paper, we continue and develop our earlier studies published by the author (2004).

Let $(X, +)$ be an abelian group. We consider the functional inequality

$$f(x+y) - f(x) - f(y) \geq \phi(x, y), \quad x, y \in X, \quad (1)$$

where $\phi : X \times X \rightarrow \mathbb{R}$ and $f : X \rightarrow \mathbb{R}$ are unknown mappings.

First, we quote [3, Proposition].

PROPOSITION 1. *If $f : X \rightarrow \mathbb{R}$, $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1) and*

$$\phi(x, -x) \geq -\phi(x, x), \quad x \in X, \quad (2)$$

then, (a) $f(0) \leq 0$; (b) $f(x) + f(-x) \leq \phi(x, x)$ for $x \in X$; (c) $f(2x) \geq 3f(x) + f(-x)$ for $x \in X$.

One can see that an even function $f : X \rightarrow \mathbb{R}$ which fulfills assumptions of Proposition 1 satisfies $f(2x) \geq 4f(x)$ for $x \in X$. This observation was used in [3], where a new function $Q : X \rightarrow \mathbb{R}$ was defined by the formula $Q(x) := \lim_{k \rightarrow +\infty} f(2^k x)/4^k$ for $x \in X$. The resulted equality $Q(2x) = 4Q(x)$ for $x \in X$ played a crucial role.

The main idea of the present paper is to drop the assumption that f is even and use Proposition 1(c) to get a limit function $\varphi : X \rightarrow \mathbb{R}$ satisfying the equality $\varphi(2x) = 3\varphi(x) + \varphi(-x)$ for $x \in X$ (see Theorems 14 and 16).

It is assumed that $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Let us quote here [3, Lemma 1].

LEMMA 2. Assume that $f : X \rightarrow \mathbb{R}$ and $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1). If

$$\phi(x, -y) \geq -\phi(x, y), \quad x, y \in X, \quad (3)$$

$$f(2x) \leq 4f(x), \quad x \in X, \quad (4)$$

then

$$f(x) = \frac{1}{2}\phi(x, x), \quad x \in X. \quad (5)$$

Moreover, ϕ is biadditive and symmetric.

The foregoing result was the main tool in [3]. In fact, this lemma, in slightly different version, was first proved by K. Baron (see [4]). In the present paper, we need to state a more general lemma, which works for maps satisfying $f(2x) \leq 3f(x) + f(-x)$ for $x \in X$.

LEMMA 3. Assume that $f : X \rightarrow \mathbb{R}$ and $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1) and (3). If

$$f(2x) \leq 3f(x) + f(-x), \quad x \in X, \quad (6)$$

then there exists an additive function $a : X \rightarrow \mathbb{R}$ such that

$$f(x) = \frac{1}{2}\phi(x, x) + a(x), \quad x \in X. \quad (7)$$

Moreover, ϕ is biadditive and symmetric.

Proof. Setting $-y$ instead of y in (1), we obtain

$$f(x - y) - f(x) - f(-y) \geq \phi(x, -y) \geq -\phi(x, y), \quad x, y \in X. \quad (8)$$

Adding this to (1) leads to

$$f(x + y) + f(x - y) \geq 2f(x) + f(y) + f(-y), \quad x, y \in X. \quad (9)$$

Fix arbitrarily $u, v \in X$. Applying this inequality with $x = u + v$ and $y = u - v$ and using (6), we infer that

$$\begin{aligned} & 3f(u) + f(-u) + 3f(v) + f(-v) \\ & \geq f(2u) + f(2v) \geq 2f(u + v) + f(u - v) + f(v - u), \quad u, v \in X. \end{aligned} \quad (10)$$

The last two inequalities imply that f satisfies the equality

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y), \quad x, y \in X. \quad (11)$$

Now, define $q : X \rightarrow \mathbb{R}$ and $a : X \rightarrow \mathbb{R}$ by the formulas

$$a(x) := \frac{f(x) - f(-x)}{2}, \quad q(x) := \frac{f(x) + f(-x)}{2}, \quad x \in X. \quad (12)$$

It is clear that

$$a(x+y) + a(x-y) = 2a(x), \quad x, y \in X, \quad (13)$$

thus a is additive. Moreover,

$$q(x+y) + q(x-y) = 2q(x) + 2q(y), \quad x, y \in X, \quad (14)$$

that is, q is quadratic. There exists a biadditive and symmetric functional $B : X \times X \rightarrow \mathbb{R}$ such that $q(x) = B(x, x)$ for $x \in X$ (see, e.g., Aczél and Dhombres [1, Chapter 11, Proposition 1]). Moreover, we have

$$q(x+y) - q(x) - q(y) = 2B(x, y), \quad x, y \in X. \quad (15)$$

This implies that $2B(x, y) \geq \phi(x, y)$ for $x, y \in X$. By the use of this, (3), and the biadditivity of B , we get that $\phi(x, y) \geq -\phi(x, -y) \geq -2B(x, -y) = 2B(x, y)$ for $x, y \in X$. So $2B = \phi$. This completes the proof. $\square$

Our next step is to drop the assumption of the evenness of function f in [3, Lemma 3]. We have the following generalization of this result.

Recall that a group X is called uniquely 2-divisible if and only if the map $X \ni x \rightarrow x+x \in X$ is bijective.

LEMMA 4. Assume X to be uniquely 2-divisible, $f : X \rightarrow \mathbb{R}$, $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1), (2), and

$$\phi(2x, 2x) \leq 4\phi(x, x), \quad x \in X. \quad (16)$$

If f is nonnegative, then f is even and $f(x) = (1/2)\phi(x, x)$ for $x \in X$.

Proof. By Proposition 1(c) and nonnegativity of f , we get that for $x \in X$, the sequence $(2^n f(x/2^n))_{n \in \mathbb{N}}$ is nonincreasing and nonnegative and thus convergent. So, the formula

$$A(x) := \lim_{n \rightarrow +\infty} 2^n f\left(\frac{x}{2^n}\right), \quad x \in X, \quad (17)$$

correctly defines a map $A : X \rightarrow \mathbb{R}$. Moreover, $A(x) \geq 0$ and $A(2x) = 2A(x)$ for $x \in X$.

Proposition 1(c) implies that

$$2^n f\left(\frac{x}{2^{n-1}}\right) \geq 3 \cdot 2^n f\left(\frac{x}{2^n}\right) + 2^n f\left(\frac{-x}{2^n}\right), \quad x \in X, \quad n \in \mathbb{N}. \quad (18)$$

So

$$2A(x) = A(2x) \geq 3A(x) + A(-x), \quad x \in X, \quad (19)$$

and we can easily observe that $A = 0$.

Now, we will follow the original proof of [3, Lemma 3]. Fix an $x \in X$. From (1) and (16), we derive inductively the estimations

$$2^k f\left(\frac{x}{2^{k-1}}\right) - 2^{k+1} f\left(\frac{x}{2^k}\right) \geq 2^k \phi\left(\frac{x}{2^k}, \frac{x}{2^k}\right) \geq \frac{1}{2^k} \phi(x, x), \quad (20)$$

for all $k \in \mathbb{N}$. Summing up these inequalities side by side for $k \in \{1, \dots, n\}$, we get that

$$2f(x) - 2^{n+1}f\left(\frac{x}{2^n}\right) \geq \sum_{k=1}^n \frac{1}{2^k} \phi(x, x), \quad n \in \mathbb{N}. \quad (21)$$

Letting n tend to $+\infty$ yields the inequality $2f(x) \geq \phi(x, x)$.

On the other hand, Proposition 1(b) states that $f(x) + f(-x) \leq \phi(x, x)$ for $x \in X$. So, f is even and $f(x) = (1/2)\phi(x, x)$ for $x \in X$. This completes the proof. $\square$

In the next lemma, we will provide a certain property of the inequality from Proposition 1(c).

LEMMA 5. Assume X to be uniquely 2-divisible. If $f : X \rightarrow \mathbb{R}$ satisfies

$$f(2x) \geq 3f(x) + f(-x), \quad x \in X, \quad (22)$$

$$\forall_{x \in X} \exists_{k_0 \in \mathbb{N}} \forall_{k \geq k_0} f\left(\frac{x}{2^k}\right) \geq 0, \quad (23)$$

then $f \geq 0$.

Proof. Define a sequence $(\varphi_k)_{k \in \mathbb{N}_0}$ of real mappings on X by the formula

$$\varphi_k(x) := \frac{4^k + 2^k}{2} f\left(\frac{x}{2^k}\right) + \frac{4^k - 2^k}{2} f\left(-\frac{x}{2^k}\right), \quad x \in X, k \in \mathbb{N}_0. \quad (24)$$

We will show that this sequence is nonincreasing. Fix an $x \in X$ and $k \in \mathbb{N}_0$. We have

$$\begin{aligned} \varphi_k(x) &= \frac{4^k + 2^k}{2} f\left(\frac{x}{2^k}\right) + \frac{4^k - 2^k}{2} f\left(-\frac{x}{2^k}\right) \\ &\geq \frac{4^k + 2^k}{2} \left[3f\left(\frac{x}{2^{k+1}}\right) + f\left(-\frac{x}{2^{k+1}}\right) \right] \\ &\quad + \frac{4^k - 2^k}{2} \left[3f\left(-\frac{x}{2^{k+1}}\right) + f\left(\frac{x}{2^{k+1}}\right) \right] \\ &= \frac{4^{k+1} + 2^{k+1}}{2} f\left(\frac{x}{2^{k+1}}\right) + \frac{4^{k+1} - 2^{k+1}}{2} f\left(-\frac{x}{2^{k+1}}\right) \\ &= \varphi_{k+1}(x). \end{aligned} \quad (25)$$

The assumption (23) implies that the sequence $(\varphi_k(x))_{k \in \mathbb{N}_0}$ is nonnegative for $x \in X$. In particular $f(x) = \varphi_0(x) \geq 0$ for $x \in X$. This completes the proof. $\square$

Now, we may join this lemma with our Lemmas 4, 2 and Proposition 1(c) to get the following result.

COROLLARY 6. Assume X to be uniquely 2-divisible, $f : X \rightarrow \mathbb{R}$, $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1), (2), (16), and (23). Then f is nonnegative, even, and $f(x) = (1/2)\phi(x, x)$ for $x \in X$. Moreover, if (3) is also satisfied, then ϕ is biadditive and symmetric.

Next, we will quote [3, Theorem 2].

THEOREM 7. *Assume X to be uniquely 2-divisible and that $f : X \rightarrow \mathbb{R}$, $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1), (3), (16) jointly with*

$$f(x) + f(-x) \geq 0, \quad x \in X. \quad (26)$$

Then there exists an additive function $a : X \rightarrow \mathbb{R}$ such that

$$f(x) = \frac{1}{2}\phi(x, x) + a(x), \quad x \in X. \quad (27)$$

Moreover, ϕ is biadditive and symmetric.

This result together with Lemma 5 applied for a map $x \mapsto f(x) + f(-x)$ leads to the following corollary.

COROLLARY 8. *Assume X to be uniquely 2-divisible and that $f : X \rightarrow \mathbb{R}$, $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1), (3), (16) jointly with*

$$\forall_{x \in X} \exists_{k_0 \in \mathbb{N}} \forall_{k \geq k_0} f\left(\frac{x}{2^k}\right) + f\left(-\frac{x}{2^k}\right) \geq 0. \quad (28)$$

Then there exists an additive function $a : X \rightarrow \mathbb{R}$ such that

$$f(x) = \frac{1}{2}\phi(x, x) + a(x), \quad x \in X. \quad (29)$$

Moreover, ϕ is biadditive and symmetric.

Now, we quote [2, Corollary 2].

COROLLARY 9. *Assume X to be a real linear space and that $f : X \rightarrow \mathbb{R}$, $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1), f is nonnegative, and $\phi(x, \cdot)$ is homogeneous for $x \in X$. Then ϕ is bilinear and symmetric and $f(x) = (1/2)\phi(x, x)$ for $x \in X$.*

In the light of Lemma 5, we get then the following corollary.

COROLLARY 10. *Assume X to be a real linear space, $f : X \rightarrow \mathbb{R}$, $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1), (23), and $\phi(x, \cdot)$ is homogeneous for $x \in X$. Then ϕ is bilinear and symmetric and $f(x) = (1/2)\phi(x, x) \geq 0$ for $x \in X$.*

We recall also the following corollary.

COROLLARY 11 [2, Corollary 1]. *Assume X to be a real linear space and that $f : X \rightarrow \mathbb{R}$, $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1). If for every $x \in X$ the function $\mathbb{R} \ni t \mapsto f(tx) \in \mathbb{R}$ has the property that its Jensen convexity implies its convexity and f satisfies (26) with $\phi(x, \cdot)$ being homogeneous for $x \in X$, then there exists a linear functional $L : X \rightarrow \mathbb{R}$ such that*

$$f(x) = \frac{1}{2}\phi(x, x) + L(x), \quad x \in X. \quad (30)$$

Moreover, ϕ is bilinear and symmetric.

A similar reasoning as above allows us to derive the following fact.

COROLLARY 12. Assume X to be a real linear space and that $f : X \rightarrow \mathbb{R}$, $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1). If for every $x \in X$ the function $\mathbb{R} \ni t \mapsto f(tx) \in \mathbb{R}$ has the property that its Jensen convexity implies its convexity and f satisfies (28) with $\phi(x, \cdot)$ being homogeneous for $x \in X$, then there exists a linear functional $L : X \rightarrow \mathbb{R}$ such that

$$f(x) = \frac{1}{2}\phi(x, x) + L(x), \quad x \in X. \quad (31)$$

Moreover, ϕ is bilinear and symmetric.

Remark 13. If X is a real linear topological Hausdorff space, then (23) is satisfied if f is nonnegative in a certain neighborhood of zero.

Now, we state and prove our next result.

THEOREM 14. Assume X to be uniquely 2-divisible, $f : X \rightarrow \mathbb{R}$, $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1), (3),

$$\phi(2x, 2y) \leq 4\phi(x, y), \quad x, y \in X, \quad (32)$$

$$\begin{aligned} & \forall_{x \in X} \left(\liminf_{k \rightarrow +\infty} \left[4^k f\left(\frac{x}{2^k}\right) + 4^k f\left(\frac{-x}{2^k}\right) \right] > -\infty \right), \\ & \forall_{x \in X} \left(\liminf_{k \rightarrow +\infty} 2^k f\left(\frac{x}{2^k}\right) > -\infty \vee \limsup_{k \rightarrow +\infty} 2^k f\left(\frac{-x}{2^k}\right) < +\infty \right). \end{aligned} \quad (33)$$

Then there exists an additive function $a : X \rightarrow \mathbb{R}$ such that

$$f(x) = \frac{1}{2}\phi(x, x) + a(x), \quad x \in X. \quad (34)$$

Moreover, ϕ is biadditive and symmetric.

Proof. Define a sequence $(\varphi_k)_{k \in \mathbb{N}_0}$ of real mappings on X by the formula (24). We have already checked (proof of Lemma 5) that this sequence is nonincreasing. We will show that it is pointwise bounded. Fix an $x \in X$ and observe that

$$\begin{aligned} \varphi_k(x) &= \frac{4^k + 2^k}{2 \cdot 4^k} \left[4^k f\left(\frac{x}{2^k}\right) + 4^k f\left(\frac{-x}{2^k}\right) \right] - 2^k f\left(\frac{-x}{2^k}\right), \quad k \in \mathbb{N}_0, \\ \varphi_k(x) &= \frac{4^k - 2^k}{2 \cdot 4^k} \left[4^k f\left(\frac{x}{2^k}\right) + 4^k f\left(\frac{-x}{2^k}\right) \right] + 2^k f\left(\frac{x}{2^k}\right), \quad k \in \mathbb{N}_0. \end{aligned} \quad (35)$$

So, by (33), the sequence $(\varphi_k)_{k \in \mathbb{N}_0}$ is pointwise convergent. Define $\varphi : X \rightarrow \mathbb{R}$ by $\varphi(x) := \lim_{k \rightarrow +\infty} \varphi_k(x)$ for $x \in X$. Observe that

$$\varphi_{k+1}(2x) = 3\varphi_k(x) + \varphi_k(-x), \quad x \in X, k \in \mathbb{N}_0, \quad (36)$$

and thus

$$\varphi(2x) = 3\varphi(x) + \varphi(-x), \quad x \in X. \quad (37)$$

Next, by the definition of φ and φ_k , (1) and (32), we have

$$\begin{aligned}\varphi(x+y) - \varphi(x) - \varphi(y) &= \lim_{k \rightarrow +\infty} [\varphi_k(x+y) - \varphi_k(x) - \varphi_k(y)] \\ &\geq \limsup_{k \rightarrow +\infty} \frac{4^k + 2^k}{2} \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) + \limsup_{k \rightarrow +\infty} \frac{4^k - 2^k}{2} \phi\left(\frac{-x}{2^k}, \frac{-y}{2^k}\right) \\ &\geq \frac{1}{2} \phi(x, y) + \frac{1}{2} \phi(-x, -y), \quad x, y \in X.\end{aligned}\quad (38)$$

Define $\phi_1 : X \times X \rightarrow \mathbb{R}$ by $\phi_1(x, y) := (1/2)[\phi(x, y) + \phi(-x, -y)]$ for $x, y \in X$. Now, we may apply Lemma 3 with φ and ϕ_1 to get that ϕ_1 is biadditive and symmetric and $\varphi = q + a$, where q is a quadratic mapping and a is an additive one. Moreover,

$$\varphi(x+y) - \varphi(x) - \varphi(y) = q(x+y) - q(x) - q(y) = \phi_1(x, y), \quad x, y \in X. \quad (39)$$

Now, put $f_1 := f - \varphi$ and $\phi_2 := \phi - \phi_1$. We have $f_1 \geq 0$ and

$$f_1(x+y) - f_1(x) - f_1(y) \geq \phi_2(x, y), \quad x, y \in X. \quad (40)$$

Lemma 4 applied for $f = f_1$ and $\phi = \phi_2$ implies that f_1 is even and $f_1(2x) = 4f_1(x)$ for $x \in X$. By Proposition 1(c), we have

$$\begin{aligned}3\varphi(x) + \varphi(-x) + 4f_1(x) &= \varphi(2x) + f_1(2x) = f(2x) \\ &\geq 3f(x) + f(-x) = 3\varphi(x) + \varphi(-x) + 4f_1(x), \quad x \in X.\end{aligned}\quad (41)$$

So $f(2x) = 3f(x) + f(-x)$ for $x \in X$. This means that $f = \varphi$, and as a consequence $\phi_2 = 0$. This completes the proof. $\square$

Remark 15. The assumption (33) is fulfilled if f satisfies the condition (26), which appears (among others) in Theorem 7. But Theorem 14 does not generalize Theorem 7 or Corollary 8, unless we are able to replace the assumption (32) by (16) in Theorem 14 (note that (32) in its whole strength was used only to prove that $\varphi(x+y) - \varphi(x) - \varphi(y) \geq \phi_1(x, y)$ for $x, y \in X$).

Now, we will state and prove our last result, which yields a generalization to [3, Theorem 1].

THEOREM 16. Assume that $f : X \rightarrow \mathbb{R}$ and $\phi : X \times X \rightarrow \mathbb{R}$ satisfy (1), (3) and

$$\begin{aligned}\limsup_{k \rightarrow +\infty} \frac{1}{4^k} \phi(2^k x, 2^k x) &< +\infty, \quad x \in X, \\ \liminf_{k \rightarrow +\infty} \frac{1}{4^k} \phi(2^k x, 2^k y) &\geq \phi(x, y), \quad x, y \in X.\end{aligned}\quad (42)$$

If the sequence $(2^{-k}[f(2^k x) - f(-2^k x)])_{k \in \mathbb{N}}$ is pointwise convergent to a superadditive function, then there exists a subadditive function $A : X \rightarrow \mathbb{R}$ such that

$$f(x) = \frac{1}{2} \phi(x, x) - A(x), \quad x \in X. \quad (43)$$

Moreover, ϕ is biadditive and symmetric.

Proof. Define a sequence $(\hat{\varphi}_k)_{k \in \mathbb{N}_0}$ of real mappings on X by the formula

$$\hat{\varphi}_k(x) := \frac{4^{-k} + 2^{-k}}{2} f(2^k x) + \frac{4^{-k} - 2^{-k}}{2} f(-2^k x), \quad x \in X, k \in \mathbb{N}_0. \quad (44)$$

We will show that this sequence is convergent. Fix an $x \in X$. We have

$$\hat{\varphi}_k(x) = \frac{f(2^k x) + f(-2^k x)}{2 \cdot 4^k} + \frac{f(2^k x) - f(-2^k x)}{2^{k+1}}, \quad k \in \mathbb{N}_0. \quad (45)$$

Observe that by Proposition 1(c), the first summand is nondecreasing and (by Proposition 1(b)) pointwise upper bounded by $4^{-k} \phi(2^k x, 2^k x)$, whereas the second one is convergent by the assumption. Thus the sequence $(\hat{\varphi}_k)_{k \in \mathbb{N}}$ is convergent. Therefore, the formula

$$\hat{\varphi}(x) := \lim_{k \rightarrow +\infty} \hat{\varphi}_k(x), \quad x \in X, \quad (46)$$

correctly defines a map $\hat{\varphi} : X \rightarrow \mathbb{R}$. Moreover, $\hat{\varphi}(2x) = 3\hat{\varphi}(x) + \hat{\varphi}(-x)$ for $x \in X$ and the following inequality is satisfied:

$$\begin{aligned} & \hat{\varphi}(x+y) - \hat{\varphi}(x) - \hat{\varphi}(y) \\ &= \lim_{k \rightarrow +\infty} \frac{1}{2} \cdot 4^{-k} [f(2^k x + 2^k y) - f(2^k x) - f(2^k y)] \\ & \quad + \frac{1}{2} \cdot 4^{-k} [f(-2^k x - 2^k y) - f(-2^k x) - f(-2^k y)] \\ & \quad + 2^{-k-1} [f(2^k x + 2^k y) - f(2^k x) - f(2^k y)] \\ & \quad - 2^{-k-1} [f(-2^k x - 2^k y) - f(-2^k x) - f(-2^k y)] \\ & \geq \liminf_{k \rightarrow +\infty} \frac{1}{2} \cdot 4^{-k} [\phi(2^k x, 2^k y) + \phi(-2^k x, -2^k y)] \\ & \quad + \frac{1}{2} [p(x+y) - p(x) - p(y)] \\ & \geq \frac{1}{2} [\phi(x, y) + \phi(-x, -y)], \quad x, y \in X, \end{aligned} \quad (47)$$

where $p : X \rightarrow \mathbb{R}$ is defined by

$$p(x) := \lim_{k \rightarrow +\infty} \frac{1}{2^k} [f(2^k x) - f(-2^k x)], \quad x \in X. \quad (48)$$

Lemma 3 states that the map $\phi_1 : X \times X \rightarrow \mathbb{R}$, defined by $\phi_1(x, y) = (1/2)[\phi(x, y) + \phi(-x, -y)]$ for $x, y \in X$, is biadditive and symmetric and $\hat{\varphi}(x) = (1/2)\phi_1(x, x) + a(x)$ for $x \in X$, where a is an additive mapping. It implies that

$$\hat{\varphi}(x+y) - \hat{\varphi}(x) - \hat{\varphi}(y) = \phi_1(x, y), \quad x, y \in X, \quad (49)$$

that is, the foregoing estimation holds with the equality. In particular,

$$\lim_{k \rightarrow +\infty} 4^{-k} [f(2^k x + 2^k y) - f(2^k x) - f(2^k y)] = \phi(x, y), \quad x, y \in X. \quad (50)$$

Moreover, observe that $\hat{\varphi}_k(x) - \hat{\varphi}_k(-x) = (1/2^k)(f(2^k x) - f(-2^k x))$ for $x \in X$ and $k \in \mathbb{N}_0$, whence $2a = p$.

Now, put $f_1 := f - \hat{\varphi}$ and $\phi_2 := \phi - \phi_1$. Clearly, ϕ_2 satisfies (3), (42), and

$$f_1(x+y) - f_1(x) - f_1(y) \geq \phi_2(x, y), \quad x, y \in X. \quad (51)$$

Moreover, one has

$$\lim_{k \rightarrow +\infty} 4^{-k} [f_1(2^k x + 2^k y) - f_1(2^k x) - f_1(2^k y)] = \phi_2(x, y), \quad x, y \in X, \quad (52)$$

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \frac{1}{2^k} [f_1(2^k x) - f_1(-2^k x)] \\ &= \lim_{k \rightarrow +\infty} \frac{1}{2^k} [f(2^k x) - f(-2^k x)] - \lim_{k \rightarrow +\infty} \frac{1}{2^k} [\hat{\varphi}(2^k x) - \hat{\varphi}(-2^k x)] \\ &= p(x) - 2a(x) = 0, \quad x \in X. \end{aligned} \quad (53)$$

Split f_1 into its even and odd parts, that is, define $P, g : X \rightarrow \mathbb{R}$ by $P(x) := (1/2)[f_1(x) + f_1(-x)]$ and $g(x) := (1/2)[f_1(x) - f_1(-x)]$ for $x \in X$. Next, fix $x, y \in X$ and apply (51) twice: for x and y and then for $-x$ and $-y$. Summing up side by side the two inequalities obtained and using the definition of ϕ_1 and ϕ_2 , we get

$$f_1(x+y) + f_1(-x-y) - f_1(x) - f_1(-x) - f_1(y) - f_1(-y) \geq 0, \quad (54)$$

that is, P is superadditive. In particular, due to its evenness, P is nonpositive and $P(2x) \geq 2P(x)$ for $x \in X$. Thus, the sequence $(2^{-k}P(2^k x))_{k \in \mathbb{N}}$ is convergent, whence

$$\lim_{k \rightarrow +\infty} 4^{-k} P(2^k x) = 0, \quad x \in X. \quad (55)$$

This, jointly with (52), implies that

$$\lim_{k \rightarrow +\infty} 4^{-k} [g(2^k x + 2^k y) - g(2^k x) - g(2^k y)] = \phi_2(x, y), \quad x, y \in X. \quad (56)$$

On the other hand, we have

$$\lim_{k \rightarrow +\infty} 2^{-k} g(2^k x) = \lim_{k \rightarrow +\infty} \frac{1}{2^{k+1}} [f_1(2^k x) - f_1(-2^k x)] = 0, \quad x \in X. \quad (57)$$

From the last two equalities, it follows that $\phi_2 = 0$. So $\phi = \phi_1$ is biadditive and symmetric. It remains to define $A : X \rightarrow \mathbb{R}$ by $A(x) := (1/2)\phi(x, x) - f(x)$ for $x \in X$. This completes the proof. $\square$

Remark 17. The convergence assumption spoken of in Theorem 16 is weaker than the supposition of the evenness of f , used in [3, Theorem 1]. However, we do not know definitely whether or not it could be omitted.

References

- [1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia of Mathematics and Its Applications, vol. 31, Cambridge University Press, Cambridge, 1989.
- [2] K. Baron and Z. Kominek, *On functionals with the Cauchy difference bounded by a homogeneous functional*, Bull. Polish Acad. Sci. Math. **51** (2003), no. 3, 301–307.
- [3] W. Fechner, *On functions with the Cauchy difference bounded by a functional*, Bull. Polish Acad. Sci. Math. **52** (2004), no. 3, 265–271.
- [4] S. Rolewicz, *Φ -convex functions defined on metric spaces*, J. Math. Sci. (New York) **115** (2003), no. 5, 2631–2652.

Włodzimierz Fechner: Institute of Mathematics, Faculty of Mathematics, Physics and Chemistry,
University of Silesia, 14 Bankowa Street, 40-007 Katowice, Poland
E-mail address: fechner@ux2.math.us.edu.pl