A new representation of remainder of Lagrange interpolating polynomial is derived. Error inequalities of Ostrowski-Grüss type for the Lagrange interpolating polynomial are established. Some similar inequalities are also obtained.

1. Introduction

Many error inequalities in polynomial interpolation can be found in [1, 7]. These error bounds for interpolating polynomials are usually expressed by means of the norms $\| \cdot \|_p$, $1 \leq p \leq \infty$. Some new error inequalities (for corrected interpolating polynomials) are given in [10, 11]. The last mentioned inequalities are similar to error inequalities obtained in recent years in numerical integration and they are known in the literature as inequalities of Ostrowski (or Ostrowski-like, Ostrowski-Grüss) type. For example, in [9] we can find inequalities of Ostrowski-Grüss type for the well-known Simpson’s quadrature rule,

$$\left| \int_{x_0}^{x_2} f(t)dt - \frac{h}{3} \left[f(x_0) + 4f(x_1) + f(x_2)\right] \right| \leq C_n (\Gamma_n - \gamma_n) h^{n+1}, \quad (1.1)$$

where $x_i = x_0 + ih$, for $h > 0$, $i = 1, 2$, $\gamma_n$, $\Gamma_n$ are real numbers such that $\gamma_n \leq f^{(n)}(t) \leq \Gamma_n$, for all $t \in [x_0, x_2]$, and $C_n$ are constants, $n \in \{1, 2, 3\}$.

The inequalities of Ostrowski type can be also found in [2, 3, 4, 5, 6, 12]. In some of the mentioned papers, we can find estimations for errors of quadrature formulas which are expressed by means of the differences $\Gamma_k - \gamma_k$, $S - \gamma_k$, $\Gamma_k - S$, where $\Gamma_k$, $\gamma_k$ are real numbers such that $\gamma_k \leq f^{(k)}(t) \leq \Gamma_k$, $t \in [a, b]$ ($k$ is a positive integer while $[a, b]$ is an interval of integration) and $S = \frac{f^{(k-1)}(b) - f^{(k-1)}(a)}{(b - a)}$. It is shown that the estimations expressed in such a way can be much better than the estimations expressed by means of the norms $\| f^{(k)} \|_p$, $1 \leq p \leq \infty$.

As we know there is a close relationship between interpolation polynomials and quadrature rules. Thus, it is a natural try to establish similar error inequalities in polynomial interpolation.
We first establish general error inequalities, expressed by means of \( \| f^{(k)} - P_m \| \), where \( P_m \) is any polynomial of degree \( m \) and then we obtain inequalities of the above mentioned types. For that purpose, we derive a new representation of remainder of the interpolating polynomial. This is done in Section 2. In Section 3, we obtain the error inequalities of the above-mentioned types. In Section 4, we give some results for derivatives.

Finally, we emphasize that the usual error inequalities in polynomial interpolation (for the Lagrange interpolating polynomial \( L_n(x) \)) are given by means of the \( (n+1) \)th derivative while in this paper we can find these error inequalities expressed by means of the \( k \)th derivative for \( k = 1, 2, \ldots, n \).

2. Representation of remainder

Let \( D = \{a = x_0 < x_1 < \cdots < x_n = b\} \) be a given subdivision of the interval \([a, b]\) and let \( f : [a, b] \to \mathbb{R} \) be a given function. The Lagrange interpolation polynomial is given by

\[
L_n(x) = \sum_{i=0}^{n} p_{ni}(x) f(x_i),
\]

where

\[
p_{ni}(x) = \frac{(x-x_0) \cdots (x-x_{i-1})(x-x_{i+1}) \cdots (x-x_n)}{(x_i-x_0) \cdots (x_i-x_{i-1})(x_i-x_{i+1}) \cdots (x_i-x_n)},
\]

for \( i = 0, 1, \ldots, n \). We have the Cauchy relations [7, pages 160-161],

\[
\sum_{i=0}^{n} p_{ni}(x) = 1,
\]

(2.3)

\[
\sum_{i=0}^{n} p_{ni}(x)(x-x_i)^j = 0, \quad j = 1, 2, \ldots, n.
\]

(2.4)

Let \( \bar{D} = \{x_0 = a < x_1 < \cdots < x_n = b\} \) be a given uniform subdivision of the interval \([a, b]\), that is, \( x_i = x_0 + ih, h = (b-a)/n, i = 0, 1, 2, \ldots, n \). Then the Lagrange interpolating polynomial is given by

\[
L_n(x) = L_n(x_0 + th) = (-1)^n \frac{t(t-1) \cdots (t-n)}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{f(x_i)}{t-i},
\]

where \( t \notin \{0, 1, 2, \ldots, n\}, 0 < t < n \).

**Lemma 2.1.** Let \( P_m(t) \) be an arbitrary polynomial of degree \( \leq m \) and let \( p_{ni}(x) \) be defined by (2.2). Then

\[
\sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} P_m(t)(t-x_i)^k dt = 0,
\]

(2.6)

for \( 0 \leq k + m \leq n - 1 \) and \( x \in [a, b] \).
Proof. Let $x$ be a given real number. Then we have

$$P_m(t) = \sum_{j=0}^{m} c_j(x-t)^j,$$

for some coefficients $c_j = c_j(x)$, $j = 0, 1, 2, \ldots, m$. (This is a consequence of the Taylor formula.) Thus,

$$\sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} P_m(t)(t-x_i)^k dt = \sum_{j=0}^{m} c_j \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} (x-t)^j(t-x_i)^k dt. \quad (2.8)$$

Let $\beta(\cdot, \cdot)$ and $\Gamma(\cdot)$ denote the beta and gamma functions, respectively. We now calculate

$$\int_{x_i}^{x} (x-t)^j(t-x_i)^k dt = \int_{0}^{x-x_i} (x-x_i-u)^j u^k du$$

$$= (x-x_i)^j \int_{0}^{1} \left(1 - \frac{u}{x-x_i}\right)^j u^k du$$

$$= (x-x_i)^{j+k+1} \int_{0}^{1} (1-v)^j v^k dv$$

$$= \beta(j+1,k+1)(x-x_i)^{j+k+1}$$

$$= \frac{\Gamma(k+1)\Gamma(j+1)}{\Gamma(k+j+2)}(x-x_i)^{j+k+1}$$

$$= \frac{k!j!}{(k+j+1)!}(x-x_i)^{j+k+1}. \quad (2.9)$$

From (2.8) and (2.9) it follows that

$$\sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} P_m(t)(t-x_i)^k dt = \sum_{j=0}^{m} c_j \frac{k!j!}{(k+j+1)!} \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^{j+k+1}. \quad (2.10)$$

From (2.10) and (2.4) we conclude that (2.6) holds. \hfill \Box

Theorem 2.2. Let $f \in C^{n+1}(a, b)$ and let the assumptions of Lemma 2.1 hold. Then

$$f(x) = L_n(x) + R_{k,m}(x),$$

where $L_n(x)$ is given by (2.1) and

$$R_{k,m}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} \left[f^{(k+1)}(t) - P_m(t)\right](t-x_i)^k dt. \quad (2.12)$$

Proof. We have

$$R_{k,m}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} f^{(k+1)}(t)(t-x_i)^k dt - \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} P_m(t)(t-x_i)^k dt. \quad (2.13)$$
From (2.13) and (2.6) it follows that
\[ R_{k,m}(x) = R_k(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} f^{(k+1)}(t)(t-x_i)^k dt. \] (2.14)

For \( k = 0 \) we have
\[ R_0(x) = \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} f'(t)dt = \sum_{i=0}^{n} p_{ni}(x) [f(x) - f(x_i)] = f(x) - L_n(x), \] (2.15)

since (2.3) holds.

We now suppose that \( k \geq 1 \). Integrating by parts, we obtain
\[ \frac{(-1)^k}{k!} \int_{x_i}^{x} f^{(k+1)}(t)(t-x_i)^k dt = \frac{(-1)^k}{k!} f^{(k)}(x)(x-x_i)^k + \frac{(-1)^{k-1}}{(k-1)!} \int_{x_i}^{x} f^{(k)}(t)(t-x_i)^{k-1} dt. \] (2.16)

In a similar way we get
\[ \frac{(-1)^{k-1}}{(k-1)!} \int_{x_i}^{x} f^{(k)}(t)(t-x_i)^{k-1} dt = \frac{(-1)^{k-1}}{(k-1)!} f^{(k-1)}(x)(x-x_i)^{k-1} \frac{(-1)^{k-2}}{(k-2)!} \int_{x_i}^{x} f^{(k-1)}(t)(t-x_i)^{k-2} dt. \] (2.17)

Continuing in this way, we get
\[ \frac{(-1)^k}{k!} \int_{x_i}^{x} f^{(k+1)}(t)(t-x_i)^k dt = \sum_{j=1}^{k} \frac{(-1)^j}{j!} f^{(j)}(x)(x-x_i)^j + \int_{x_i}^{x} f'(t)dt = f(x) - f(x_i) + \sum_{j=1}^{k} \frac{(-1)^j}{j!} f^{(j)}(x)(x-x_i)^j. \] (2.18)

From (2.14) and (2.18) it follows that
\[ R_k(x) = \sum_{i=0}^{n} p_{ni}(x) \left[ f(x) - f(x_i) + \sum_{j=1}^{k} \frac{(-1)^j}{j!} f^{(j)}(x)(x-x_i)^j \right] \]
\[ = f(x) - L_n(x) + \sum_{j=1}^{k} \frac{(-1)^j}{j!} f^{(j)}(x) \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^j \]
\[ = f(x) - L_n(x), \quad k = 1, 2, \ldots, n, \] (2.19)

since (2.3) and (2.4) hold. From (2.14), (2.15), and (2.19) we see that (2.11) holds. \( \square \)
3. Error inequalities

We now introduce the notations

\[ \omega_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n), \quad (3.1) \]
\[ C_k(x) = \sum_{i=0}^{n} \frac{|x - x_i|^k}{|x_i - x_{i-1}| \cdots |x_{i+1} - x_n|}, \quad (3.2) \]
\[ B_k(x) = \sum_{i=0}^{n} \frac{(S_{ki} - \gamma_{k+1}) |x - x_i|^k}{|x_i - x_{i-1}| \cdots |x_{i+1} - x_n|}, \quad (3.3) \]
\[ D_k(x) = \sum_{i=0}^{n} \frac{(\Gamma_{k+1} - S_{ki}) |x - x_i|^k}{|x_i - x_{i-1}| \cdots |x_{i+1} - x_n|}, \quad (3.4) \]

where \( S_{ki} = [f^{(k)}(x) - f^{(k)}(x_i)]/(x - x_i), i = 0, 1, \ldots, n, \) and \( \gamma_{k+1}, \Gamma_{k+1} \) are real numbers such that \( \gamma_{k+1} \leq f^{(k+1)}(t) \leq \Gamma_{k+1}, t \in [a, b], k = 0, 1, \ldots, n - 1. \)

Let \( g \in C(a, b). \) As we know among all algebraic polynomials of degree \( \leq m \) there exists the only polynomial \( P_m(t) \) having the property that

\[ ||g - P_m||_\infty \leq ||g - P_m||_\infty, \quad (3.5) \]

where \( P_m \in \Pi_m \) is an arbitrary polynomial of degree \( \leq m. \) We define

\[ E_m(g) = ||g - P_m|| = \inf_{P_m \in \Pi_m} ||g - P_m||_\infty. \quad (3.6) \]

**Theorem 3.1.** Under the assumptions of Theorem 2.2,

\[ |f(x) - L_n(x)| \leq \frac{E_m(f^{(k+1)})}{(k+1)!} C_k(x) \omega_n(x), \quad (3.7) \]

where \( C_k(\cdot) \) and \( E_m(\cdot) \) are defined by (3.2) and (3.6), respectively.

**Proof.** Let \( P_m(t) = P_m^*(t) \), where \( P_m^*(t) \) is defined by (3.6) for the function \( g(t) = f^{(k+1)}(t). \)

We have

\[ |R_{k,m}(x)| = \left| \frac{(-1)^k}{k!} \sum_{i=0}^{n} \left[ f^{(k+1)}(t) - P_m^*(t) \right] (t - x_i)^k \right| dt \]
\[ \leq \left| \frac{f^{(k+1)} - P_m^*}{(k+1)!} \right| C_k(x) \omega_n(x) \]
\[ = \frac{E_m(f^{(k+1)})}{(k+1)!} C_k(x) \omega_n(x), \quad (3.8) \]

since

\[ \left| \int_{x_i}^{x} (t - x_i)^k \right| = \left| \frac{x - x_i}{k+1} \right| (k+1). \quad (3.9) \]
Remark 3.2. The above estimate has only theoretical importance, since it is difficult to find the polynomial \( P^* \). In fact, we can find \( P^* \) only for some special cases of functions. However, we can use the estimate to obtain some practical estimations—see Theorem 3.3.

**Theorem 3.3.** Let the assumptions of Theorem 2.2 hold. If \( y_{k+1}, \Gamma_{k+1} \) are real numbers such that \( y_{k+1} \leq f^{(k+1)}(t) \leq \Gamma_{k+1}, t \in [a, b], k = 0, 1, \ldots, n - 1 \), then

\[
| f(x) - L_n(x) | \leq \frac{\Gamma_{k+1} - y_{k+1}}{2(k+1)!} C_k(x) | \omega_n(x) |,
\]

(3.10)

where \( \omega_n \) and \( C_k(\cdot) \) are defined by (3.1) and (3.2), respectively. Also

\[
| f(x) - L_n(x) | \leq \frac{| \omega_n(x) |}{k!} B_k(x),
\]

(3.11)

\[
| f(x) - L_n(x) | \leq \frac{| \omega_n(x) |}{k!} D_k(x),
\]

(3.12)

where \( B_k(\cdot) \) and \( D_k(\cdot) \) are defined by (3.3) and (3.4), respectively.

**Proof.** We set \( P_m(t) = (\Gamma_{k+1} + y_{k+1})/2 \) in (2.12). Then we have

\[
| f(x) - L_n(x) | = | R_k(x) | \leq \frac{1}{k!} \sum_{i=0}^{n} | p_m(x) | \left| f^{(k+1)} - \frac{\Gamma_{k+1} + y_{k+1}}{2} \right| \left( \int_{x_i}^{x} (t - x_i)^k dt \right).
\]

(3.13)

We also have

\[
\left| f^{(k+1)} - \frac{\Gamma_{k+1} + y_{k+1}}{2} \right| \leq \frac{\Gamma_{k+1} - y_{k+1}}{2},
\]

(3.14)

\[
\left| \int_{x_i}^{x} (t - x_i)^k dt \right| = \frac{| x - x_i |^{k+1}}{k+1}.
\]

From the above three relations we get

\[
| f(x) - L_n(x) | \leq \frac{\Gamma_{k+1} - y_{k+1}}{2(k+1)!} \sum_{i=0}^{n} | p_m(x) | \left| x - x_i \right|^{k+1} \leq \frac{\Gamma_{k+1} - y_{k+1}}{2(k+1)!} C_k(x) | \omega_n(x) |.
\]

The first inequality is proved.

We now set \( P_m(t) = y_{k+1} \) in (2.12). Then we have

\[
| f(x) - L_n(x) | = | R_k(x) | \leq \frac{1}{k!} \sum_{i=0}^{n} | p_m(x) | \left| \int_{x_i}^{x} \left[ f^{(k+1)}(t) - y_{k+1} \right] (t - x_i)^k dt \right|.
\]

(3.15)
We also have
\[
\left| \int_{x_i}^x \left[ f^{(k+1)}(t) - \gamma_{k+1} \right] (t - x_i)^k \, dt \right| \leq \left| x - x_i \right|^k \left| f^{(k)}(x) - f^{(k)}(x_i) - \gamma_{k+1} (x - x_i) \right| \\
= \left| x - x_i \right|^{k+1} (S_{ki} - \gamma_{k+1}).
\] (3.16)

Thus,
\[
\left| f(x) - L_n(x) \right| \leq \frac{1}{k!} \sum_{j=0}^n \left| p_{nj}(x) \right| \left| x - x_i \right|^{k+1} (S_{ki} - \gamma_{k+1})
\] (3.17)

The second inequality is proved. In a similar way we prove that the third inequality holds.

\[\square\]

**Lemma 3.4.** Let \( D = \{ x_0 = a < x_1 < \cdots < x_n = b \} \) be a given uniform subdivision of the interval \([a,b]\), that is, \( x_i = x_0 + ih, h = (b - a)/n, i = 0,1,2,\ldots,n \). If \( x \in (x_{j-1},x_j) \), for some \( j \in \{1,2,\ldots,n\} \), then
\[
\left| \omega_n(x) \right| \leq j!(n-j+1)!h^{n+1},
\] (3.18)
\[
C_k(x) \leq \frac{2^n}{n!} \left\{ \frac{1}{2} \left[ n+1 + |n-2j+1| \right] \right\}^k h^{k-n},
\] (3.19)
\[
C_k(x) \left| \omega_n(x) \right| \leq \alpha_{jnk} \frac{n-j+1}{n} \frac{2^n (b-a)^{k+1}}{\binom{n}{j}},
\] (3.20)

where
\[
\alpha_{jnk} = \left[ \frac{1}{2n} (n+1 + |2j-n-1|) \right]^k.
\] (3.21)

This lemma is proved in [10].

**Remark 3.5.** Note that
\[
\alpha_{jnk} \leq 1
\] (3.22)
and \( \alpha_{jnk} = 1 \) if and only if \( j = 1 \) or \( j = n \). If we choose \( x \in [x_j,x_{j+1}] \), \( j = 0,1,\ldots,n-1 \), then we get the factor \((j+1)/n\) instead of the factor \((n-j+1)/n\) in (3.20).

**Theorem 3.6.** Under the assumptions of Lemma 3.4 and Theorem 3.3,
\[
\left| f(x) - L_n(x) \right| \leq \frac{\Gamma_{k+1} - \gamma_{k+1}}{(k+1)!} \alpha_{jnk} \frac{n-j+1}{n} \frac{2^n (b-a)^{k+1}}{\binom{n}{j}}.
\] (3.23)

**Proof.** The proof follows immediately from Theorem 3.3 and Lemma 3.4. \[\square\]
4. Results for derivatives

**Lemma 4.1.** Let \(1 \leq j \leq n-1\) and \(j+1 \leq r \leq n\). Then

\[
\sum_{i=0}^{n} p_{ni}^{(j)}(x)(x-x_i)^r = 0. \tag{4.1}
\]

**Proof.** We have (see (2.4))

\[
A(x) = \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^r = 0, \quad \text{for } 1 \leq r \leq n. \tag{4.2}
\]

Thus,

\[
A'(x) = \sum_{i=0}^{n} p_{ni}'(x)(x-x_i)^r + r \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^{r-1} = 0, \tag{4.3}
\]

if \(1 \leq r \leq n\). If \(n \geq r-1 \geq 1\), that is, \(n+1 \geq r \geq 2\), then

\[
r \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^{r-1} = 0. \tag{4.4}
\]

From (4.3) and (4.4) we get

\[
\sum_{i=0}^{n} p_{ni}'(x)(x-x_i)^r = 0, \quad \text{for } 2 \leq r \leq n. \tag{4.5}
\]

(Note that \(\{r : 1 \leq r \leq n\} \cap \{r : 2 \leq r \leq n+1\} = \{r : 2 \leq r \leq n\}.\) Here we use this fact and similar facts without a special mentioning.)

We now suppose that

\[
\sum_{i=0}^{n} p_{ni}^{(j)}(x)(x-x_i)^r = 0, \tag{4.6}
\]

for \(j = 1,2,\ldots,m, m < n-1\) and \(j+1 \leq r \leq n\). We wish to prove that

\[
\sum_{i=0}^{n} p_{ni}^{(m+1)}(x)(x-x_i)^r = 0, \quad \text{for } m+2 \leq r \leq n. \tag{4.7}
\]

For that purpose, we first calculate

\[
A^{(m)}(x) = \sum_{i=0}^{n} \left[ p_{ni}(x)(x-x_i)^r \right]^{(m)}
\]

\[
= \sum_{i=0}^{n} \sum_{k=0}^{m} \binom{m}{k} P_{ni}^{(k)}(x) \frac{r!}{(r-m+k)!} (x-x_i)^{r-m+k}\tag{4.8}
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} \frac{r!}{(r-m+k)!} \sum_{i=0}^{n} P_{ni}^{(k)}(x)(x-x_i)^{r-m+k}.\]
We have
\[ A^{(m)}(x) = 0, \quad \text{for } r \geq m + 1, \] (4.9)
by the above assumption. Thus,
\[ A^{(m+1)}(x) = 0. \] (4.10)

On the other hand, we have
\begin{align*}
A^{(m+1)}(x) &= \frac{d}{dx} A^{(m)}(x) \\
&= \sum_{k=0}^{m} \binom{m}{k} \frac{r!}{(r-m+k)!} \sum_{i=0}^{n} P^{(k+1)}_{ni}(x)(x-x_i)^{r-m+k} \\
&\quad + \sum_{k=0}^{m} \binom{m}{k} \frac{r!}{(r-m+k-1)!} \sum_{i=0}^{n} P^{(k)}_{ni}(x)(x-x_i)^{r-m+k-1} \\
&= 0.
\end{align*}
(4.11)

We now rewrite the above relation in the form
\begin{align*}
\sum_{i=0}^{n} P^{(m+1)}_{ni}(x)(x-x_i)^{r} + \sum_{k=0}^{m} \binom{m}{k} \frac{r!}{(r-m+k)!} \sum_{i=0}^{n} P^{(k+1)}_{ni}(x)(x-x_i)^{r-m+k} \\
&\quad + \sum_{k=0}^{m} \binom{m}{k} \frac{r!}{(r-m+k-1)!} \sum_{i=0}^{n} P^{(k)}_{ni}(x)(x-x_i)^{r-m+k-1} = 0.
\end{align*}
(4.12)

For \( r - m + k - 1 \geq k + 1 \), that is, \( r \geq m + 2 \), we have
\[ \sum_{i=0}^{n} P^{(k)}_{ni}(x)(x-x_i)^{r-m+k-1} = 0 \] (4.13)
by the above assumption. We also have
\[ \sum_{i=0}^{n} P^{(k+1)}_{ni}(x)(x-x_i)^{r-m+k} = 0, \] (4.14)
if \( r - m + k \geq k + 2 \), that is, \( r \geq m + 2 \). Thus (4.7) holds. This completes the proof. \( \square \)

**Theorem 4.2.** Let \( f \in C^{r+1}(a,b) \) and let \( P_r(t) \) be an arbitrary polynomial of degree \( \leq r \) and let \( 0 \leq k \leq n, 1 \leq m \leq k \). Then
\[ f^{(m)}(x) = L^{(m)}_{n}(x) + E_{k,r}(x), \] (4.15)
where
\[ E_{k,r}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} P^{(m)}_{ni}(x) \int_{x_i}^{x} [f^{(k+1)}(t) - P_r(t)](t-x_i)^k dt. \] (4.16)
Inequalities in polynomial interpolation

Proof. We define

\[ v_i(x) = \int_{x_i}^{x} \left[ f^{(k+1)}(t) - P_r(t) \right] (t - x_i)^k \, dt \]
\[ = \int_{x_i}^{x} g(t) (t - x_i)^k \, dt, \quad (4.17) \]

where, obviously, \( g(t) = f^{(k+1)}(t) - P_r(t) \). We denote

\[ R_{k,r}(x) = f(x) - L_n(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x)v_i(x), \quad (4.18) \]

see Theorem 2.2. Then we have

\[ R_{k,r}^{(m)}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} \left[ p_{ni}(x)v_i(x) \right]^{(m)} \]
\[ = \frac{(-1)^k}{k!} \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{m}{j} p_{ni}^{(j)}(x)v_i^{(m-j)}(x) \]
\[ = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}^{(m)}(x)v_i(x) + \frac{(-1)^k}{k!} \sum_{i=0}^{n} \sum_{j=0}^{m-1} \binom{m}{j} p_{ni}^{(j)}(x)v_i^{(m-j)}(x). \quad (4.19) \]

We introduce the notation

\[ B(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} \sum_{j=0}^{m-1} \binom{m}{j} p_{ni}^{(j)}(x)v_i^{(m-j)}(x) \quad (4.20) \]

such that

\[ R_{k,r}^{(m)}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}^{(m)}(x)v_i(x) + B(x). \quad (4.21) \]

We now rewrite \( B(x) \) in the form

\[ B(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} \sum_{j=0}^{m-2} \binom{m}{j} p_{ni}^{(j)}(x)v_i^{(m-j)}(x) + \frac{(-1)^k}{k!} m \sum_{i=0}^{n} p_{ni}^{(m-1)}(x)v_i'(x). \quad (4.22) \]

We have

\[ v_i'(x) = g(x)(x - x_i)^k \quad (4.23) \]

such that

\[ \sum_{i=0}^{n} p_{ni}^{(m-1)}(x)v_i'(x) = g(x) \sum_{i=0}^{n} p_{ni}^{(m-1)}(x)(x - x_i)^k = 0, \quad (4.24) \]

for \( k \geq m \)—see Lemma 4.1.
We also have

\[ v_{i}^{(m-j)}(x) = \sum_{l=0}^{m-j-1} \binom{m-j-1}{l} g^{(l)}(x) \frac{k!}{(k-m+j+l+1)!} (x-x_{i})^{k-m+j+l+1}, \]  

(4.25)

for \( m \geq j + 2 \) such that

\[ \sum_{i=0}^{n} \sum_{j=0}^{m-2} \binom{m}{j} p_{ni}^{(j)}(x) v_{i}^{(m-j)}(x) = \sum_{j=0}^{m-2} \binom{m}{j} \sum_{l=0}^{m-j-1} \binom{m-j-1}{l} \frac{k!}{(k-m+j+l+1)!} \]

\[ \times \sum_{i=0}^{n} p_{ni}^{(j)}(x)(x-x_{i})^{k-m+j+l+1} = 0, \]

(4.26)

if \( k-m+j+l+1 \geq j+1 \), that is, \( k \geq m \), since \( l \geq 0 \)—see also Lemma 4.1. Hence, \( B(x) = 0 \) in all cases. Now from (4.21) it follows that

\[ R_{k,r}^{(m)}(x) = \frac{(-1)^{k}}{k!} \sum_{i=0}^{n} p_{ni}^{(m)}(x) v_{i}(x) = \frac{(-1)^{k}}{k!} \sum_{i=0}^{n} p_{ni}^{(m)}(x) \int_{x_{i}}^{x} [f^{(k+1)}(t) - P_{r}(t)](t-x_{i})^{k} dt. \]  

(4.27)

On the other hand, we have

\[ [f(x) - L_{n}(x)]^{(m)} = f^{(m)}(x) - L_{n}^{(m)}(x). \]  

(4.28)

This completes the proof. \( \square \)

**Theorem 4.3.** Under the assumptions of Theorem 4.2,

\[ |f^{(m)}(x) - L_{n}^{(m)}(x)| \leq \frac{E_{r}(f^{(k+1)})}{(k+1)!} \sum_{i=0}^{n} |p_{ni}^{(m)}(x)| |x-x_{i}|^{k+1}, \]  

(4.29)

where \( E_{r}(\cdot) \) is defined by (3.6).

**Proof.** Let \( P_{r}(t) = P_{r}^{*}(t) \), where \( P_{r}^{*}(t) \) is defined by (3.6) for the function \( g(t) = f^{(k+1)}(t) \). We have

\[ |R_{k,r}^{(m)}(x)| = \left| \frac{(-1)^{k}}{k!} \sum_{i=0}^{n} p_{ni}^{(m)}(x) \int_{x_{i}}^{x} [f^{(k+1)}(t) - P_{r}^{*}(t)](t-x_{i})^{k} dt \right| \]

\[ \leq \frac{||f^{(k+1)}(t) - P_{r}^{*}(t)||}{(k+1)!} \sum_{i=0}^{n} |p_{ni}^{(m)}(x)| |x-x_{i}|^{k+1} \]  

(4.30)

\[ = \frac{E_{r}(f^{(k+1)})}{(k+1)!} \sum_{i=0}^{n} |p_{ni}^{(m)}(x)| |x-x_{i}|^{k+1}, \]
If we choose \( Pr \) since
\[
\left| \int_{x_i}^{x} (t - x_i)^k \, dt \right| = \frac{|x - x_i|^{k+1}}{k + 1}. \tag{4.31}
\]

Theorem 4.4. Under the assumptions of Theorem 3.3 and Lemma 4.1,
\[
|f^{(m)}(x) - L_n^{(m)}(x)| \leq \frac{\Gamma_{k+1} - y_{k+1}}{2(k+1)!} \sum_{i=0}^{n} |p_{ni}^{(m)}(x)| |x - x_i|^{k+1},
\]
\[
|f^{(m)}(x) - L_n^{(m)}(x)| \leq \frac{1}{k!} \sum_{i=0}^{n} (S_{ki} - y_{k+1}) |p_{ni}^{(m)}(x)| |x - x_i|^{k+1}, \tag{4.32}
\]
\[
|f^{(m)}(x) - L_n^{(m)}(x)| \leq \frac{1}{k!} \sum_{i=0}^{n} (\Gamma_{k+1} - S_{ki}) |p_{ni}^{(m)}(x)| |x - x_i|^{k+1}.
\]

Proof. We choose \( P_r(t) = \Gamma_{k+1} + y_{k+1}/2 \) in Theorem 4.2. Then we get
\[
|f^{(m)}(x) - L_n^{(m)}(x)| \leq \frac{1}{k!} \sum_{i=0}^{n} |p_{ni}^{(m)}(x)| \left| \int_{x_i}^{x} f^{(k+1)}(t) - \frac{\Gamma_{k+1} + y_{k+1}}{2} (t - x_i) \, dt \right|
\]
\[
\leq \frac{\Gamma_{k+1} - y_{k+1}}{2(k+1)!} \sum_{i=0}^{n} |p_{ni}^{(m)}(x)| \left| \int_{x_i}^{x} (t - x_i) \, dt \right| \tag{4.33}
\]
\[
= \frac{\Gamma_{k+1} - y_{k+1}}{2(k+1)!} \sum_{i=0}^{n} |p_{ni}^{(m)}(x)| |x - x_i|^{k+1}.
\]

If we choose \( P_r(t) = y_{k+1} \) in Theorem 4.2, then we get
\[
|f^{(m)}(x) - L_n^{(m)}(x)| \leq \frac{1}{k!} \sum_{i=0}^{n} |p_{ni}^{(m)}(x)| \left| \int_{x_i}^{x} f^{(k+1)}(t) - y_{k+1} (t - x_i) \, dt \right|
\]
\[
\leq \frac{1}{k!} \sum_{i=0}^{n} (S_{ki} - y_{k+1}) |p_{ni}^{(m)}(x)| |x - x_i|^{k+1}, \tag{4.34}
\]

since \( \int_{x_i}^{x} |f^{(k+1)}(t) - y_{k+1}| \, dt = |f^{(k)}(x) - f^{(k)}(x_i) - y_{k+1}(x - x_i)| \).

In a similar way we prove that the third inequality holds. \( \square \)

References


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