FUZZY n -NORMED LINEAR SPACE

AL. NARAYANAN AND S. VIJAYABALAJI

Received 24 March 2005 and in revised form 29 September 2005

The primary purpose of this paper is to introduce the notion of fuzzy n -normed linear space as a generalization of n -normed space. Ascending family of α - n -norms corresponding to fuzzy n -norm is introduced. Best approximation sets in α - n -norms are defined. We also provide some results on best approximation sets in α - n -normed space.

1. Introduction

A satisfactory theory of 2-norm and n -norm on a linear space has been introduced and developed by Gähler in [9, 10]. Following Misiak [16], Kim and Cho [13] and Malčeski [15] developed the theory of n -normed space. In [11], Gunawan and Mashadi gave a simple way to derive an $(n-1)$ -norm from the n -norm and realized that any n -normed space is an $(n-1)$ -normed space. Best approximation theory in 2-normed space can be viewed in the papers [3, 4, 5, 9]. Different authors introduced the definitions of fuzzy norms on a linear space. For reference, one may see [2, 6, 7, 8, 12, 14, 17]. Following Cheng and Mordeson [2], Bag and Samanta [1] introduced the concept of fuzzy norm on a linear space.

In the present paper, we introduce the concept of fuzzy n -normed linear space as a generalization of n -normed space by Gunawan and Mashadi [11]. Bag and Samanta [1] introduced α -norms on a linear space corresponding to the fuzzy norm on a linear space. As an analogue of Bag and Samanta [1], we introduce the notion of α - n -norm on a linear space corresponding to the fuzzy n -norm on a linear space. Based on Elumalai et al. [3] and Elumalai and Souruparani [5], we introduce the notion of best approximation sets in α - n -norms and establish some results on it.

2. Preliminaries

For the sake of completeness, we reproduce the following definitions due to Gähler [9], Gunawan and Mashadi [11], Elumalai et al. [3], and Bag and Samanta [1].

Definition 2.1 [9]. Let X be a real vector space of dimension greater than 1 and let $\|\bullet, \bullet\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

- (1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
- (2) $\|x, y\| = \|y, x\|$,
- (3) $\|\alpha x, y\| = |\alpha| \|x, y\|$, where α is real,
- (4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

$\|\bullet, \bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet, \bullet\|)$ is called a linear 2-normed space.

Definition 2.2 [11]. Let $n \in \mathbb{N}$ (natural numbers) and let X be a real vector space of dimension $d \geq n$. (Here we allow d to be infinite.) A real-valued function $\|\bullet, \dots, \bullet\|$ on $\underbrace{X \times \dots \times X}_n$ satisfying the following four properties,

- (1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,
- (2) $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation,
- (3) $\|x_1, x_2, \dots, \alpha x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$ (real),
- (4) $\|x_1, x_2, \dots, x_{n-1}, y + z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$,

is called an n -norm on X and the pair $(X, \|\bullet, \dots, \bullet\|)$ is called an n -normed space.

Definition 2.3 [3]. Let $(X, \|\bullet, \bullet\|)$ be a linear 2-normed space and let G be an arbitrary nonempty subset of X and $x_0 \in X$. Then, for every $x \in X$ and for every $z \in X \setminus G$ which is independent of x and x_0 , $d_z(x, G) \leq \|x - x_0, z\| + d_z(x_0, G)$, where $d_z(x, G) = \inf_{g \in G} \|x - g, z\|$. For each $G \subset X$ and $x_0 \in X$, define $D_z(x_0, G) = \{x \in X : d_z(x, G) = \|x - x_0, z\| + d_z(x_0, G)\}$ for any $z \in X \setminus G$ which is independent of x and x_0 .

Also $P_{G,z}(x) = \{g_0 \in G : \|x - g_0, z\| = d_z(x, G)\}$ and $P_{G,z}^{-1}(x_0) = \{x \in X : \|x - x_0, z\| = d_z(x, G)\}$, where $x_0 \in G$.

Definition 2.4 [1]. Let X be a linear space over F (field of real or complex numbers). A fuzzy subset N of $X \times \mathbb{R}$ ($\mathbb{R}$, set of real numbers) is called a fuzzy norm on X if and only if for all $x, u \in X$ and $c \in F$,

- (N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x, t) = 0$,
- (N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x, t) = 1$ if and only if $x = 0$,
- (N3) for all $t \in \mathbb{R}$ with $t > 0$, $N(cx, t) = N(x, t/|c|)$, if $c \neq 0$,
- (N4) for all $s, t \in \mathbb{R}$, $x, u \in X$, $N(x + u, s + t) \geq \min\{N(x, s), N(u, t)\}$,
- (N5) $N(x, \circ)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t \rightarrow \infty} N(x, t) = 1$.

The pair (X, N) will be referred to as a fuzzy normed linear space.

THEOREM 2.5 [1]. Let (X, N) be a fuzzy normed linear space. Assume further that

- (N6) $N(x, t) > 0$ for all $t > 0$ implies $x = 0$.

Define $\|x\|_\alpha = \inf\{t : N(x, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

Then $\{\|\bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on X (or) α -norms on X corresponding to the fuzzy norm on X .

3. Fuzzy n -normed linear space

By generalizing Definition 2.2, we obtain a satisfactory notion of fuzzy n -normed linear space as follows.

Definition 3.1. Let X be a linear space over a real field F . A fuzzy subset N of $\underbrace{X \times \cdots \times X}_n \times \mathbb{R}$ ($\mathbb{R}$, set of real numbers) is called a fuzzy n -norm on X if and only if

- (N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, \dots, x_n, t) = 0$,
- (N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, \dots, x_n, t) = 1$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,
- (N3) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \dots, x_n$,
- (N4) for all $t \in \mathbb{R}$ with $t > 0$,

$$N(x_1, x_2, \dots, cx_n, t) = N\left(x_1, x_2, \dots, x_n, \frac{t}{|c|}\right), \quad \text{if } c \neq 0, c \in F \text{ (field)}, \quad (3.1)$$

- (N5) for all $s, t \in \mathbb{R}$,

$$N(x_1, x_2, \dots, x_n + x'_n, s + t) \geq \min \{N(x_1, x_2, \dots, x_n, s), N(x_1, x_2, \dots, x'_n, t)\}, \quad (3.2)$$

- (N6) $N(x_1, x_2, \dots, x_n, \circ)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t \rightarrow \infty} N(x_1, x_2, \dots, x_n, t) = 1$.

Then (X, N) is called a fuzzy n -normed linear space or in short f- n -NLS.

Remark 3.2. From (N3), it follows that in an f- n -NLS,

- (N4) for all $t \in \mathbb{R}$ with $t > 0$,

$$N(x_1, x_2, \dots, cx_i, \dots, x_n, t) = N\left(x_1, x_2, \dots, x_i, \dots, x_n, \frac{t}{|c|}\right), \quad \text{if } c \neq 0, \quad (3.3)$$

- (N5) for all $s, t \in \mathbb{R}$,

$$\begin{aligned} &N(x_1, x_2, \dots, x_i + x'_i, \dots, x_n, s + t) \\ &\geq \min \{N(x_1, x_2, \dots, x_i, \dots, x_n, s), N(x_1, x_2, \dots, x'_i, \dots, x_n, t)\}. \end{aligned} \quad (3.4)$$

The following example agrees with our notion of f- n -NLS.

Example 3.3. Let $(X, \|\bullet, \bullet, \dots, \bullet\|)$ be an n -normed space as in Definition 2.2. Define

$$\begin{aligned} &N(x_1, x_2, \dots, x_n, t) \\ &= \begin{cases} \frac{t}{t + \|x_1, x_2, \dots, x_n\|}, & \text{when } t > 0, t \in \mathbb{R}, (x_1, x_2, \dots, x_n) \in \underbrace{X \times \cdots \times X}_n, \\ 0, & \text{when } t \leq 0. \end{cases} \end{aligned} \quad (3.5)$$

Then (X, N) is an f- n -NLS.

Proof. (N1) for all $t \in \mathbb{R}$ with $t \leq 0$, we have by our definition

$$N(x_1, x_2, \dots, x_n, t) = 0. \quad (3.6)$$

(N2) for all $t \in \mathbb{R}$ with $t > 0$, we have $N(x_1, x_2, \dots, x_n, t) = 1$

- (i) if and only if $t/(t + \|x_1, x_2, \dots, x_n\|) = 1$,
- (ii) if and only if $t = t + \|x_1, x_2, \dots, x_n\|$,
- (iii) if and only if $\|x_1, x_2, \dots, x_n\| = 0$,
- (iv) if and only if $x_1, x_2, \dots, x_n$ are linearly dependent.

(N3) for all $t \in \mathbb{R}$ with $t > 0$,

$$\begin{aligned} N(x_1, x_2, \dots, x_n, t) &= \frac{t}{t + \|x_1, x_2, \dots, x_n\|} \\ &= \frac{t}{t + \|x_1, x_2, \dots, x_n, x_{n-1}\|} = N(x_1, x_2, \dots, x_n, x_{n-1}, t) = \dots \end{aligned} \quad (3.7)$$

(N4) For all $t \in \mathbb{R}$ with $t > 0$ and $c \in F$, $c \neq 0$,

$$\begin{aligned} N\left(x_1, x_2, \dots, x_n, \frac{t}{|c|}\right) &= \frac{t/|c|}{(t/|c|) + \|x_1, x_2, \dots, x_n\|} \\ &= \frac{t/|c|}{(t + |c|\|x_1, x_2, \dots, x_n\|)/|c|} \\ &= \frac{t}{t + |c|\|x_1, x_2, \dots, x_n\|} \\ &= \frac{t}{t + \|x_1, x_2, \dots, cx_n\|} = N(x_1, x_2, \dots, cx_n, t). \end{aligned} \quad (3.8)$$

(N5) We have to prove

$$N(x_1, x_2, \dots, x_n + x'_n, s + t) \geq \min \{N(x_1, x_2, \dots, x_n, s), N(x_1, x_2, \dots, x'_n, t)\}. \quad (3.9)$$

If

- (a) $s + t < 0$,
- (b) $s = t = 0$,
- (c) $s + t > 0$; $s > 0, t < 0$; $s < 0, t > 0$, then the above relation is obvious. If
- (d) $s > 0, t > 0, s + t > 0$, then

$$\begin{aligned} N(x_1, x_2, \dots, x_n + x'_n, s + t) &= \frac{s + t}{s + t + \|x_1, x_2, \dots, x_n + x'_n\|} \\ &\geq \frac{s + t}{s + t + \|x_1, x_2, \dots, x_n\| + \|x_1, x_2, \dots, x'_n\|}. \end{aligned} \quad (3.10)$$

If

$$\frac{s}{s + \|x_1, x_2, \dots, x_n\|} \geq \frac{t}{t + \|x_1, x_2, \dots, x'_n\|}, \quad (3.11)$$

then

$$\frac{s}{s + \|x_1, x_2, \dots, x_n\|} - \frac{t}{t + \|x_1, x_2, \dots, x'_n\|} \geq 0, \quad (3.12)$$

which implies

$$s(t + \|x_1, x_2, \dots, x'_n\|) - t(s + \|x_1, x_2, \dots, x_n\|) \geq 0, \quad (3.13)$$

which in turn implies

$$s\|x_1, x_2, \dots, x'_n\| - t\|x_1, x_2, \dots, x_n\| \geq 0. \quad (3.14)$$

So

$$\begin{aligned} & \frac{s+t}{s+t + \|x_1, x_2, \dots, x_n\| + \|x_1, x_2, \dots, x'_n\|} - \frac{t}{t + \|x_1, x_2, \dots, x'_n\|} \\ &= \frac{s\|x_1, x_2, \dots, x'_n\| - t\|x_1, x_2, \dots, x_n\|}{(s+t + \|x_1, x_2, \dots, x_n\| + \|x_1, x_2, \dots, x'_n\|)(t + \|x_1, x_2, \dots, x'_n\|)}. \end{aligned} \quad (3.15)$$

By (3.14),

$$\frac{s+t}{s+t + \|x_1, x_2, \dots, x_n\| + \|x_1, x_2, \dots, x'_n\|} - \frac{t}{t + \|x_1, x_2, \dots, x'_n\|} \geq 0, \quad (3.16)$$

which implies

$$\frac{s+t}{s+t + \|x_1, x_2, \dots, x_n\| + \|x_1, x_2, \dots, x'_n\|} \geq \frac{t}{t + \|x_1, x_2, \dots, x'_n\|}. \quad (3.17)$$

Similarly, if

$$\frac{t}{t + \|x_1, x_2, \dots, x'_n\|} \geq \frac{s}{s + \|x_1, x_2, \dots, x_n\|}, \quad (3.18)$$

then

$$\frac{s+t}{s+t + \|x_1, x_2, \dots, x_n\| + \|x_1, x_2, \dots, x'_n\|} \geq \frac{s}{s + \|x_1, x_2, \dots, x_n\|}. \quad (3.19)$$

Thus,

$$N(x_1, x_2, \dots, x_n + x'_n, s+t) \geq \min \{N(x_1, x_2, \dots, x_n, s), N(x_1, x_2, \dots, x'_n, t)\}. \quad (3.20)$$

(N6) For all $t_1, t_2 \in \mathbb{R}$, if $t_1 < t_2 \leq 0$, then, by our definition,

$$N(x_1, x_2, \dots, x_n, t_1) = N(x_1, x_2, \dots, x_n, t_2) = 0. \quad (3.21)$$

Suppose $t_2 > t_1 > 0$, then

$$\begin{aligned} & \frac{t_2}{t_2 + \|x_1, x_2, \dots, x_n\|} - \frac{t_1}{t_1 + \|x_1, x_2, \dots, x_n\|} \\ &= \frac{\|x_1, x_2, \dots, x_n\| (t_2 - t_1)}{(t_2 + \|x_1, x_2, \dots, x_n\|)(t_1 + \|x_1, x_2, \dots, x_n\|)} \geq 0, \end{aligned} \quad (3.22)$$

for all $(x_1, x_2, \dots, x_n) \in \underbrace{X \times \dots \times X}_n$, implies

$$\frac{t_2}{t_2 + \|x_1, x_2, \dots, x_n\|} \geq \frac{t_1}{t_1 + \|x_1, x_2, \dots, x_n\|}, \quad (3.23)$$

which in turn implies $N(x_1, x_2, \dots, x_n, t_2) \geq N(x_1, x_2, \dots, x_n, t_1)$.

Thus $N(x_1, x_2, \dots, x_n, t)$ is a nondecreasing function.

Also,

$$\begin{aligned} \lim_{t \rightarrow \infty} N(x_1, x_2, \dots, x_n, t) &= \lim_{t \rightarrow \infty} \frac{t}{t + \|x_1, x_2, \dots, x_n\|} \\ &= \lim_{t \rightarrow \infty} \frac{t}{t(1 + (1/t)\|x_1, x_2, \dots, x_n\|)} = 1. \end{aligned} \quad (3.24)$$

Thus (X, N) is an f - n -NLS. □

As a consequence of Theorem 2.5, we introduce an interesting notion of ascending family of α - n -norms corresponding to the fuzzy n -norm in the following theorem.

THEOREM 3.4. *Let (X, N) be an f - n -NLS. Assume the condition that*

(N7) $N(x_1, x_2, \dots, x_n, t) > 0$ for all $t > 0$ implies $x_1, x_2, \dots, x_n$ are linearly dependent.

Define $\|x_1, x_2, \dots, x_n\|_\alpha = \inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

Then $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of n -norms on X . These n -norms are called α - n -norms on X corresponding to the fuzzy n -norm on X .

Proof. (1) $\|x_1, x_2, \dots, x_n\|_\alpha = 0$. This

- (i) implies $\inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\} = 0$,
- (ii) implies, for all $t \in \mathbb{R}$, $t > 0$, $N(x_1, x_2, \dots, x_n, t) \geq \alpha > 0$, $\alpha \in (0, 1)$,
- (iii) implies, by (N7), $x_1, x_2, \dots, x_n$ are linearly dependent.

Conversely assume that $x_1, x_2, \dots, x_n$ are linearly dependent. This

- (i) implies, by (N2), $N(x_1, x_2, \dots, x_n, t) = 1$ for all $t > 0$,
- (ii) implies, for all $\alpha \in (0, 1)$, $\inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\} = 0$,
- (iii) implies $\|x_1, x_2, \dots, x_n\|_\alpha = 0$.

(2) As $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation, it follows that $\|x_1, x_2, \dots, x_n\|_\alpha$ is invariant under any permutation.

(3) If $c \neq 0$, then

$$\begin{aligned} \|x_1, x_2, \dots, cx_n\|_\alpha &= \inf \{s : N(x_1, x_2, \dots, cx_n, s) \geq \alpha\} \\ &= \inf \left\{ s : N\left(x_1, x_2, \dots, x_n, \frac{s}{|c|}\right) \geq \alpha \right\}. \end{aligned} \quad (3.25)$$

Let $t = s/|c|$, then

$$\begin{aligned} \|x_1, x_2, \dots, cx_n\|_\alpha &= \inf \{ |c|t : N(x_1, x_2, \dots, x_n, t) \geq \alpha \} \\ &= |c| \inf \{ t : N(x_1, x_2, \dots, x_n, t) \geq \alpha \} \\ &= |c| \|x_1, x_2, \dots, x_n\|_\alpha. \end{aligned} \quad (3.26)$$

If $c = 0$, then

$$\begin{aligned} \|x_1, x_2, \dots, cx_n\|_\alpha &= \|x_1, x_2, \dots, 0\|_\alpha \\ &= 0 = 0 \|x_1, x_2, \dots, x_n\|_\alpha \\ &= |c| \|x_1, x_2, \dots, x_n\|_\alpha, \quad \forall c \in F \text{ (field)}. \end{aligned} \quad (3.27)$$

(4)

$$\begin{aligned} &\|x_1, x_2, \dots, x_n\|_\alpha + \|x_1, x_2, \dots, x'_n\|_\alpha \\ &= \inf \{ t : N(x_1, x_2, \dots, x_n, t) \geq \alpha \} + \inf \{ s : N(x_1, x_2, \dots, x'_n, s) \geq \alpha \} \\ &= \inf \{ t + s : N(x_1, x_2, \dots, x_n, t) \geq \alpha, N(x_1, x_2, \dots, x'_n, s) \geq \alpha \} \\ &\geq \inf \{ t + s : N(x_1, x_2, \dots, x_n + x'_n, t + s) \geq \alpha \} \\ &\geq \inf \{ r : N(x_1, x_2, \dots, x_n + x'_n, r) \geq \alpha \}, \quad r = t + s \\ &= \|x_1, x_2, \dots, x_n + x'_n\|_\alpha. \end{aligned} \quad (3.28)$$

Therefore, $\|x_1, x_2, \dots, x_n + x'_n\|_\alpha \leq \|x_1, x_2, \dots, x_n\|_\alpha + \|x_1, x_2, \dots, x'_n\|_\alpha$.

Thus $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an α - n -norm on X .

Let $0 < \alpha_1 < \alpha_2$. Then

$$\begin{aligned} \|x_1, x_2, \dots, x_n\|_{\alpha_1} &= \inf \{ t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_1 \}, \\ \|x_1, x_2, \dots, x_n\|_{\alpha_2} &= \inf \{ t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_2 \}. \end{aligned} \quad (3.29)$$

As $\alpha_1 < \alpha_2$,

$$\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_2\} \subset \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_1\} \quad (3.30)$$

implies

$$\inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_2\} \geq \inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_1\} \quad (3.31)$$

which implies

$$\|x_1, x_2, \dots, x_n\|_{\alpha_2} \geq \|x_1, x_2, \dots, x_n\|_{\alpha_1}. \quad (3.32)$$

Hence, $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of α - n -norms on X corresponding to the fuzzy n -norm on X . $\square$

4. Best approximation sets in α - n -normed space

Inspired by this α - n -norm on X , we introduce the notion of two subsets of X , namely, $D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $P_{G, x_2, x_3, \dots, x_n}(x)$.

Definition 4.1. Let $(X, \|\bullet, \bullet, \dots, \bullet\|_\alpha)$ be an α - n -normed space corresponding to the fuzzy n -norm N on X . Let G be an arbitrary nonempty subset of X and $x_0 \in X$. Then for every $x \in X$ and for every $x_2, x_3, \dots, x_n \in X \setminus G$ which is independent of x and x_0 ,

$$d_{x_2, x_3, \dots, x_n}(x, G) \leq \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \quad (4.1)$$

where

$$d_{x_2, x_3, \dots, x_n}(x, G) = \inf_{g \in G} \|x - g, x_2, x_3, \dots, x_n\|_\alpha. \quad (4.2)$$

For each $G \subset X$ and $x_0 \in X$, we define

$$\begin{aligned} D_{x_2, x_3, \dots, x_n}(x_0, G) \\ = \{x \in X : d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)\} \end{aligned} \quad (4.3)$$

for any $x_2, x_3, \dots, x_n \in X \setminus G$ which is independent of x and x_0 .

We denote

$$\begin{aligned} P_{G, x_2, x_3, \dots, x_n}(x) &= \{g_0 \in G : \|x - g_0, x_2, x_3, \dots, x_n\|_\alpha = d_{x_2, x_3, \dots, x_n}(x, G)\}, \\ P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0) &= \{x \in X : \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha = d_{x_2, x_3, \dots, x_n}(x, G)\}, \end{aligned} \quad (4.4)$$

where $x_0 \in G$.

We give the following examples in the α -2-normed linear space and α - n -normed linear space for the sets $D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $P_{G, x_2, x_3, \dots, x_n}(x)$. It is easy to find the set $P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0)$.

Example 4.2. Let $X = \mathbb{R}^3$ be a linear space over $\mathbb{R}$.

Define $\|\bullet, \bullet\| : X \times X \rightarrow \mathbb{R}$ by

$$\begin{aligned} \|x_1, x_2\|_1 &= \max\{|a_1 b_2 - a_2 b_1|, |b_1 c_2 - b_2 c_1|, |a_1 c_2 - a_2 c_1|\}, \\ \|x_1, x_2\|_2 &= \frac{1}{2} \{\max\{|a_1 b_2 - a_2 b_1|, |b_1 c_2 - b_2 c_1|, |a_1 c_2 - a_2 c_1|\}\}, \end{aligned} \quad (4.5)$$

where $x_i = (a_i, b_i, c_i) \in \mathbb{R}^3$, $i = 1, 2$. Then $(X, \|\bullet, \bullet\|_1)$ and $(X, \|\bullet, \bullet\|_2)$ are 2-normed linear spaces.

Define $N : X \times X \times \mathbb{R} \rightarrow [0, 1]$ by

$$N(x_1, x_2, t) = \begin{cases} 1, & \text{if } t > \|x_1, x_2\|_1, \\ 0.5, & \text{if } \|x_1, x_2\|_2 < t \leq \|x_1, x_2\|_1, \\ 0, & \text{if } t \leq \|x_1, x_2\|_2. \end{cases} \quad (4.6)$$

Then (X, N) is a fuzzy 2-normed linear space. Define $\|x_1, x_2\|_\alpha = \inf\{t : N(x_1, x_2, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

The α -2-norms are given by

$$\begin{aligned} \|x_1, x_2\|_\alpha &= \|x_1, x_2\|_1, & \text{when } 1 > \alpha > 0.5, \\ &= \|x_1, x_2\|_2, & \text{when } 0 < \alpha \leq 0.5. \end{aligned} \quad (4.7)$$

Let $G = \{(a, 0, 0) : a \in \mathbb{R}\}$ be a subset of X .

Choose $x_0 = (0, 1, 1)$ and $x_2 \in K = \{(0, 0, k) : k \in \mathbb{R} \setminus \{0\}\}$.

Then

$$\begin{aligned} D_{x_2}(x_0, G) &= \{x = (0, b, 0), b \in \mathbb{R}^+ \setminus \{0\} : d_{x_2}(x, G) = \|x - x_0, x_2\|_\alpha + d_{x_2}(x_0, G)\}, \\ P_{G, x_2}(x) &= \{g' = (a, 0, 0) : -1 \leq a \leq 1\}. \end{aligned} \quad (4.8)$$

Example 4.3. Let $X = \mathbb{R}^{n+1}$ be a linear space over $\mathbb{R}$.

Define $\|\bullet, \bullet, \dots, \bullet\| : \underbrace{X \times \dots \times X}_n \rightarrow \mathbb{R}$ by

$$\begin{aligned} \|x_1, x_2, \dots, x_n\|_1 &= \max\{\Delta_1, \Delta_2, \dots, \Delta_n\}, \\ \|x_1, x_2, \dots, x_n\|_2 &= \frac{1}{2} \{\max\{\Delta_1, \Delta_2, \dots, \Delta_n\}\}, \end{aligned} \quad (4.9)$$

where

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{n(n+1)} \end{vmatrix}, \\ \Delta_2 &= \begin{vmatrix} a_{13} & \cdots & a_{1(n+1)} & a_{11} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n3} & \cdots & a_{n(n+1)} & a_{n1} \end{vmatrix}, \\ &\vdots \\ \Delta_n &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{aligned} \quad (4.10)$$

and $x_i = (a_{i1}, a_{i2}, \dots, a_{i(n+1)}) \in \mathbb{R}^{n+1}$, $i = 1, 2, \dots, n$.

Then $(X, \|\bullet, \bullet, \dots, \bullet\|_1)$ and $(X, \|\bullet, \dots, \bullet, \bullet\|_2)$ are n -normed linear spaces. Define $N : \underbrace{X \times \dots \times X}_n \times \mathbb{R} \rightarrow [0, 1]$ by

$$N(x_1, x_2, \dots, x_n, t) = \begin{cases} 1, & \text{if } t > \|x_1, x_2, \dots, x_n\|_1, \\ 0.5, & \text{if } \|x_1, x_2, \dots, x_n\|_2 < t \leq \|x_1, x_2, \dots, x_n\|_1, \\ 0, & \text{if } t \leq \|x_1, x_2, \dots, x_n\|_2. \end{cases} \quad (4.11)$$

Then (X, N) is a fuzzy n -normed linear space. Define $\|x_1, x_2, \dots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

The α - n -norms are given by

$$\begin{aligned} \|x_1, x_2, \dots, x_n\|_\alpha &= \|x_1, x_2, \dots, x_n\|_1, \quad \text{when } 1 > \alpha > 0.5, \\ &= \|x_1, x_2, \dots, x_n\|_2, \quad \text{when } 0 < \alpha \leq 0.5. \end{aligned} \quad (4.12)$$

Let $G = \{(a, 0, 0, \dots, n \text{ times } 0) : a \in \mathbb{R}\}$ be a subset of X .

Choose $x_0 = (0, 1, 1, \dots, n \text{ times } 1)$ and

$$x_2, x_3, \dots, x_n \in K = \left\{ \left(0, 0, k_3^{(i)}, \dots, k_{n+1}^{(i)} \right) : k_3^{(i)}, \dots, k_{n+1}^{(i)} \in \mathbb{R} \setminus \{0\} \right\}. \quad (4.13)$$

That is,

$$\begin{aligned} x_2 &= \left(0, 0, k_3^{(2)}, \dots, k_{n+1}^{(2)} \right), \\ x_3 &= \left(0, 0, k_3^{(3)}, \dots, k_{n+1}^{(3)} \right), \\ &\vdots \\ x_n &= \left(0, 0, k_3^{(n)}, \dots, k_{n+1}^{(n)} \right). \end{aligned} \quad (4.14)$$

Then

$$\begin{aligned} D_{x_2, x_3, \dots, x_n}(x_0, G) &= \{x = (0, b, 0, \dots, (n-1) \text{ times } 0), b \in \mathbb{R}^+ \setminus \{0\} : \\ &\quad d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)\}, \end{aligned} \quad (4.15)$$

where $d_{x_2, x_3, \dots, x_n}(x, G) = \max\{|b|\Delta, |a|\Delta\}$,

$$\Delta = \begin{vmatrix} k_3^{(2)} & k_4^{(2)} & \dots & k_{n+1}^{(2)} \\ k_3^{(3)} & k_4^{(3)} & \dots & k_{n+1}^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ k_3^{(n)} & k_4^{(n)} & \dots & k_{n+1}^{(n)} \end{vmatrix}, \quad (4.16)$$

$$\|x - x_0, x_2, \dots, x_n\|_\alpha = |b - 1|\Delta, \quad d_{x_2, x_3, \dots, x_n}(x_0, G) = \max\{\Delta, |a|\Delta\}$$

and also $P_{G, x_2, x_3, \dots, x_n}(x) = \{g' = (a, 0, \dots, n \text{ times } 0) : -1 \leq a \leq 1\}$.

By routine calculation the following theorems are validate from the Examples 4.2 and 4.3.

THEOREM 4.4. For $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $y \in D_{x_2, x_3, \dots, x_n}(x, G)$,

- (i) $\|y - x_0, x_2, x_3, \dots, x_n\|_\alpha = \|y - x, x_2, x_3, \dots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha$,
- (ii) $y - x + x_0 \in D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Proof. (i) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $y \in D_{x_2, x_3, \dots, x_n}(x, G)$.

Then by (4.3) we have

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \\ d_{x_2, x_3, \dots, x_n}(y, G) &= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x, G). \end{aligned} \quad (4.17)$$

Consider

$$\begin{aligned} &\|y - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= \|y - x_0 - x + x, x_2, x_3, \dots, x_n\|_\alpha \\ &= \|(y - x) + (x - x_0), x_2, x_3, \dots, x_n\|_\alpha \\ &\leq \|y - x, x_2, x_3, \dots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= (d_{x_2, x_3, \dots, x_n}(y, G) - d_{x_2, x_3, \dots, x_n}(x, G)) \\ &\quad + (d_{x_2, x_3, \dots, x_n}(x, G) - d_{x_2, x_3, \dots, x_n}(x_0, G)) \\ &= d_{x_2, x_3, \dots, x_n}(y, G) - d_{x_2, x_3, \dots, x_n}(x_0, G) \\ &\leq \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha. \end{aligned} \quad (4.18)$$

Therefore,

$$\|y - x_0, x_2, x_3, \dots, x_n\|_\alpha = \|y - x, x_2, x_3, \dots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha. \quad (4.19)$$

(ii) By (4.2), we have

$$\begin{aligned} &d_{x_2, x_3, \dots, x_n}(y - x + x_0, G) \\ &\geq d_{x_2, x_3, \dots, x_n}(y, G) - \|y - (y - x + x_0), x_2, x_3, \dots, x_n\|_\alpha \\ &= d_{x_2, x_3, \dots, x_n}(y, G) - \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= (\|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x, G)) - \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + (\|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)) \\ &\quad - \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\ &= \|(y - x + x_0) - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G). \end{aligned} \quad (4.20)$$

Again by (4.2), it follows that

$$d_{x_2, x_3, \dots, x_n}(y - x + x_0, G) = \|(y - x + x_0) - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \quad (4.21)$$

which implies $y - x + x_0 \in D_{x_2, x_3, \dots, x_n}(x_0, G)$. $\square$

THEOREM 4.5. Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$. Then

- (i) $[x_0, x] = \{\lambda x_0 + (1 - \lambda)x : 0 \leq \lambda \leq 1\} \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$,
- (ii) $D_{x_2, x_3, \dots, x_n}(x, G) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Proof. (i) Let $y = \lambda x_0 + (1 - \lambda)x$ such that $0 \leq \lambda \leq 1$. Then

$$\begin{aligned}
 d_{x_2, x_3, \dots, x_n}(y, G) &\geq d_{x_2, x_3, \dots, x_n}(x, G) - \|x - y, x_2, x_3, \dots, x_n\|_\alpha \\
 &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) - \|x - y, x_2, x_3, \dots, x_n\|_\alpha \\
 &= \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G).
 \end{aligned} \tag{4.22}$$

By (4.2), we have

$$d_{x_2, x_3, \dots, x_n}(y, G) = \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \tag{4.23}$$

which implies $y \in D_{x_2, x_3, \dots, x_n}(x_0, G)$.

(ii) Let $y \in D_{x_2, x_3, \dots, x_n}(x, G)$. Then, by (4.3) and Theorem 4.4(i),

$$\begin{aligned}
 d_{x_2, x_3, \dots, x_n}(y, G) &= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x, G) \\
 &= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + (\|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)) \\
 &= \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G),
 \end{aligned} \tag{4.24}$$

which implies $y \in D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Therefore, $D_{x_2, x_3, \dots, x_n}(x, G) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$. □

THEOREM 4.6. Let $x_0, y_0 \in X$ and $\lambda \neq 0$. Then

- (i) $D_{x_2, x_3, \dots, x_n}(x_0, G) + y_0 = D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0)$,
- (ii) $D_{x_2, x_3, \dots, x_n}(x_0, \lambda G) = \lambda D_{x_2, x_3, \dots, x_n}(x_0/\lambda, G)$.

Proof. (i) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$. Then

$$\begin{aligned}
 d_{x_2, x_3, \dots, x_n}(x + y_0, G + y_0) &= d_{x_2, x_3, \dots, x_n}(x, G) \\
 &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\
 &= \|x + y_0 - (x_0 + y_0), x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G).
 \end{aligned} \tag{4.25}$$

Therefore, $x + y_0 \in D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0)$.

Conversely, let $y \in D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0)$. Then

$$\begin{aligned}
 d_{x_2, x_3, \dots, x_n}(y - y_0, G) &= d_{x_2, x_3, \dots, x_n}(y, G + y_0) \\
 &= \|y - y_0 - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0) \\
 &= \|(y - y_0) - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G).
 \end{aligned} \tag{4.26}$$

Therefore, $y - y_0 \in D_{x_2, x_3, \dots, x_n}(x_0, G)$, and so

$$D_{x_2, x_3, \dots, x_n}(x_0, G) + y_0 = D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0). \tag{4.27}$$

(ii) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, \lambda G)$. Then

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}\left(\frac{x}{\lambda}, G\right) &= \frac{1}{|\lambda|} d_{x_2, x_3, \dots, x_n}(x, \lambda G) \\ &= \frac{1}{|\lambda|} (\|x - x_0, x_2, x_3, \dots, x_n\|_{\alpha} + d_{x_2, x_3, \dots, x_n}(x_0, \lambda G)) \\ &= \left\| \frac{x}{\lambda} - \frac{x_0}{\lambda}, x_2, x_3, \dots, x_n \right\|_{\alpha} + d_{x_2, x_3, \dots, x_n}\left(\frac{x_0}{\lambda}, G\right). \end{aligned} \quad (4.28)$$

Therefore, $x/\lambda \in D_{x_2, x_3, \dots, x_n}(x_0/\lambda, G)$.

Conversely, let $x \in D_{x_2, x_3, \dots, x_n}(x_0/\lambda, G)$. Then

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(\lambda x, \lambda G) &= |\lambda| d_{x_2, x_3, \dots, x_n}(x, G) \\ &= |\lambda| \left(\left\| x - \frac{x_0}{\lambda}, x_2, x_3, \dots, x_n \right\|_{\alpha} + d_{x_2, x_3, \dots, x_n}\left(\frac{x_0}{\lambda}, G\right) \right) \\ &= \|\lambda x - x_0, x_2, x_3, \dots, x_n\|_{\alpha} + d_{x_2, x_3, \dots, x_n}(x_0, \lambda G). \end{aligned} \quad (4.29)$$

Therefore, $\lambda x \in D_{x_2, x_3, \dots, x_n}(x_0, \lambda G)$.

Thus, $D_{x_2, x_3, \dots, x_n}(x_0, \lambda G) = \lambda D_{x_2, x_3, \dots, x_n}(x_0/\lambda, G)$. $\square$

THEOREM 4.7. Let $G \subset G_1$ and $x_0 \in X$, where G_1 is a subset of X such that

$$d_{x_2, x_3, \dots, x_n}(x_0, G) = d_{x_2, x_3, \dots, x_n}(x_0, G_1). \quad (4.30)$$

Then $D_{x_2, x_3, \dots, x_n}(x_0, G_1) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Proof. Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G_1)$. Then, by (4.30), we have

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &\geq d_{x_2, x_3, \dots, x_n}(x, G_1) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_{\alpha} + d_{x_2, x_3, \dots, x_n}(x_0, G_1) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_{\alpha} + d_{x_2, x_3, \dots, x_n}(x_0, G). \end{aligned} \quad (4.31)$$

By (4.2), it follows that

$$d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_{\alpha} + d_{x_2, x_3, \dots, x_n}(x_0, G), \quad (4.32)$$

which implies $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Hence, $D_{x_2, x_3, \dots, x_n}(x_0, G_1) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$. $\square$

THEOREM 4.8. (i) $P_{G, x_2, x_3, \dots, x_n}(x_0) \subset P_{G, x_2, x_3, \dots, x_n}(x)$ for every $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$,

(ii) $D_{x_2, x_3, \dots, x_n}(x_0, G) = P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0)$ for every $x_0 \in \bar{G}$.

Proof. (i) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $g \in P_{G, x_2, x_3, \dots, x_n}(x_0)$.

Now,

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &= \|x - x_0, x_2, x_3, \dots, x_n\|_{\alpha} + d_{x_2, x_3, \dots, x_n}(x_0, G) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_{\alpha} + \|x_0 - g_0, x_2, x_3, \dots, x_n\|_{\alpha}. \end{aligned} \quad (4.33)$$

By Theorem 4.4(i), we have

$$d_{x_2, x_3, \dots, x_n}(x, G) = \|x - g_0, x_2, x_3, \dots, x_n\|_\alpha, \quad (4.34)$$

which implies $g_0 \in P_{G, x_2, x_3, \dots, x_n}(x)$, which in turn implies $P_{G, x_2, x_3, \dots, x_n}(x_0) \subset P_{G, x_2, x_3, \dots, x_n}(x)$.

(ii) Let $x_0 \in \overline{G}$ and $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$. Then

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha, \quad \text{where } x_0 \in \overline{G}, \end{aligned} \quad (4.35)$$

which implies $x \in P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0)$. So

$$D_{x_2, x_3, \dots, x_n}(x_0, G) \subset P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0). \quad (4.36)$$

Conversely, let $x \in P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0)$.

Then $x_0 \in P_{G, x_2, x_3, \dots, x_n}(x)$.

Since $x_0 \in \overline{G}$, $d_{x_2, x_3, \dots, x_n}(x_0, G) = 0$.

Hence, we have

$$d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \quad (4.37)$$

which implies $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$, which in turn implies

$$P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0) \subset D_{x_2, x_3, \dots, x_n}(x_0, G). \quad (4.38)$$

From (4.36) and (4.38), we have

$$D_{x_2, x_3, \dots, x_n}(x_0, G) = P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0). \quad (4.39)$$

□

References

- [1] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), no. 3, 687–705.
- [2] S. C. Cheng and J. N. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), no. 5, 429–436.
- [3] S. Elumalai, Y. J. Cho, and S. S. Kim, *Best approximation sets in linear 2-normed spaces*, Commun. Korean Math. Soc. **12** (1997), no. 3, 619–629.
- [4] S. Elumalai and R. Ravi, *Farthest points on suns*, Math. Today **9** (1991), 13–18.
- [5] S. Elumalai and M. Souruparani, *A characterization of best approximation and the operators in linear 2-normed spaces*, Bull. Calcutta Math. Soc. **92** (2000), no. 4, 235–248.
- [6] C. Felbin, *Finite-dimensional fuzzy normed linear space*, Fuzzy Sets and Systems **48** (1992), no. 2, 239–248.
- [7] ———, *The completion of a fuzzy normed linear space*, J. Math. Anal. Appl. **174** (1993), no. 2, 428–440.
- [8] ———, *Finite dimensional fuzzy normed linear space. II*, J. Anal. **7** (1999), 117–131.
- [9] S. Gähler, *Lineare 2-normierte Räume*, Math. Nachr. **28** (1964), 1–43.

- [10] ———, *Untersuchungen über verallgemeinerte m -metrische Räume, I*, Math. Nachr. **40** (1969), 165–189.
- [11] H. Gunawan and M. Mashadi, *On n -normed spaces*, Int. J. Math. Math. Sci. **27** (2001), no. 10, 631–639.
- [12] A. K. Katsaras, *Fuzzy topological vector spaces. II*, Fuzzy Sets and Systems **12** (1984), no. 2, 143–154.
- [13] S. S. Kim and Y. J. Cho, *Strict convexity in linear n -normed spaces*, Demonstratio Math. **29** (1996), no. 4, 739–744.
- [14] S. V. Krishna and K. K. M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Systems **63** (1994), no. 2, 207–217.
- [15] R. Malčeski, *Strong n -convex n -normed spaces*, Mat. Bilten **21** (1997), no. 47, 81–102.
- [16] A. Misiak, *n -inner product spaces*, Math. Nachr. **140** (1989), 299–319.
- [17] G. S. Rhie, B. M. Choi, and D. S. Kim, *On the completeness of fuzzy normed linear spaces*, Math. Japon. **45** (1997), no. 1, 33–37.

AL. Narayanan: Department of Mathematics, Annamalai University, Annamalai Nagar-608002, Tamilnadu, India

E-mail address: alagappan_cdm@rediffmail.com

S. Vijayabalaji: Department of Mathematics, Annamalai University, Annamalai Nagar-608002, Tamilnadu, India

E-mail address: balaji_nandini@rediffmail.com