ON RIEMANNIAN MANIFOLDS ENDOWED WITH A LOCALLY CONFORMAL COSYMPLECTIC STRUCTURE

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We deal with a locally conformal cosymplectic manifold $M(\phi, \Omega, \xi, \eta, g)$ admitting a conformal contact quasi-torse-forming vector field T . The presymplectic 2-form Ω is a locally conformal cosymplectic 2-form. It is shown that T is a 3-exterior concurrent vector field. Infinitesimal transformations of the Lie algebra of $\wedge M$ are investigated. The Gauss map of the hypersurface M_ξ normal to ξ is conformal and $M_\xi \times M_\xi$ is a Chen submanifold of $M \times M$.

1. Introduction

Locally conformal cosymplectic manifolds have been investigated by Olszak and Rosca [7] (see also [6]).

In the present paper, we consider a $(2m+1)$ -dimensional Riemannian manifold $M(\phi, \Omega, \xi, \eta, g)$ endowed with a locally conformal cosymplectic structure. We assume that M admits a principal vector field (or a conformal contact quasi-torse-forming), that is,

$$\nabla T = sd p + T \wedge \xi = sd p + \eta \otimes T - T^b \otimes \xi, \quad (1.1)$$

with $ds = s\eta$.

First, we prove certain geometrical properties of the vector fields T and ϕT . The existence of T and ϕT is determined by an exterior differential system in involution (in the sense of Cartan [3]).

The principal vector field T is 3-exterior concurrent (see also [8]), it defines a Lie relative contact transformation of the co-Reeb form η , and the Lie differential of T^b with respect to T is conformal to T^b . The vector field ϕT is an infinitesimal transformation of generators T and ξ . The vector fields ξ , T , and ϕT commute and the distribution $D_T = \{T, \phi T, \xi\}$ is involutive. The divergence and the Ricci curvature of T are computed.

Next, we investigate infinitesimal transformations on the Lie algebra of $\wedge M$.

In the last section, we study the hypersurface M_ξ normal to ξ . We prove that M_ξ is Einsteinian, its Gauss map is conformal, and the product submanifold $M_\xi \times M_\xi$ in $M \times M$ is a $\mathcal{U}$ -submanifold in the sense of Chen.

2. Preliminaries

Let (M, g) be an n -dimensional Riemannian manifold endowed with a metric tensor g . Let ΓTM and $\flat : TM \rightarrow T^*M, Z \mapsto Z^\flat$ be the set of sections of the tangent bundle TM and the musical isomorphism defined by g , respectively. Following a standard notation, we set $A^q(M, TM) = \text{Hom}(\Lambda^q TM, TM)$ and notice that the elements of $A^q(M, TM)$ are the vector-valued q -forms ($q \leq n$) (see also [9]). Denote by $d^\nabla : A^q(M, TM) \rightarrow A^{q+1}(M, TM)$ the exterior covariant derivative operator with respect to the Levi-Civita connection ∇ . It should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. If $dp \in A^1(M, TM)$ denotes the soldering form on M , one has $d^\nabla(dp) = 0$.

The cohomology operator d^ω acting on ΛM is defined by $d^\omega \gamma = d\gamma + \omega \wedge \gamma$, where ω is a closed 1-form. If $d^\omega \gamma = 0$, γ is said to be d^ω -closed.

Let R be the curvature operator on M . Then, for any vector field Z on M , the second covariant differential is defined as

$$\nabla^2 Z = d^\nabla(\nabla Z) \in A^2(M, TM) \quad (2.1)$$

and satisfies

$$\nabla^2 Z(V, W) = R(V, W)Z, \quad Z, V, W \in \Gamma TM. \quad (2.2)$$

Let $O = \text{vect}\{e_A \mid A = 1, \dots, n\}$ be an adapted local field of orthonormal frames over M and let $O^* = \text{covect}\{\omega^A\}$ be its associated coframe. With respect to O and O^* , É. Cartan's structure equation can be written, in the indexless manner, as

$$\begin{aligned} \nabla e &= \theta \otimes e \in A^1(M, TM), \\ d\omega &= -\theta \wedge \omega, \\ d\theta &= -\theta \wedge \theta + \Theta. \end{aligned} \quad (2.3)$$

In the above equations, θ , respectively, Θ are the local connection forms in the bundle $\mathbb{O}(M)$, respectively, the curvature forms on M .

3. Locally conformal cosymplectic structure

Let $M(\phi, \Omega, \xi, \eta, g)$ be a $(2m+1)$ -dimensional Riemannian manifold carrying a quintuple of structure tensor fields, where ϕ is an automorphism of the tangent bundle TM , Ω a presymplectic form of rank $2m$, ξ the Reeb vector field, and $\eta = \xi^\flat$ the associated Reeb covector, g the metric tensor.

We assume in the present paper that η is closed and λ is a scalar ($\lambda \in \Lambda^0 M$) such that $d\lambda = \lambda' \eta$, with $\lambda' \in \Lambda^0 M$.

We agree to denominate the manifold M a *locally conformal cosymplectic manifold* if it satisfies

$$\begin{aligned}\phi^2 &= -I + \eta \otimes \xi, & \phi\xi &= 0, & \eta \circ \phi &= 0, & \eta(\xi) &= 1, \\ \nabla\xi &= \lambda(dp - \eta \otimes \xi), \\ d\lambda &= \lambda' \eta, \\ \Omega(Z, Z') &= g(\phi Z, \phi Z'), & \Omega^m \wedge \eta &\neq 0,\end{aligned}\tag{3.1}$$

where $dp \in A^1(M, TM)$ denotes the canonical vector-valued 1-form (or the soldering form [5]) on M . Then Ω is called the fundamental 2-form on M and is expressed by

$$\Omega = \sum_{i=1}^m \omega^i \wedge \omega^{i*}, \quad i^* = i + m.\tag{3.2}$$

By the well-known relations

$$\theta_j^i = \theta_{j*}^{i*}, \quad \theta_j^{i*} = \theta_i^{j*}, \quad i^* = i + m,\tag{3.3}$$

one derives by differentiation of Ω

$$d^{-2\lambda\eta}\Omega = 0 \quad (d\Omega = 2\lambda\eta \wedge \Omega),\tag{3.4}$$

which shows that the presymplectic 2-form Ω is a locally conformal cosymplectic form. Operating on ϕdp by d^∇ , it follows that

$$d^\nabla(\phi dp) = 2\lambda\Omega \otimes \xi + 2\eta \wedge \phi dp.\tag{3.5}$$

On the other hand, we agree to call a vector field T , such that

$$\nabla T = s dp + T \wedge \xi = s dp + \eta \otimes T - T^\flat \otimes \xi,\tag{3.6}$$

a *principal vector field* on M , or a *conformal contact quasi-torse-forming* if

$$ds = s\eta.\tag{3.7}$$

In these conditions, since the q th covariant differential ∇^q of a vector field $Z \in \Gamma TM$ is defined inductively, that is, $\nabla^q Z = d^\nabla(\nabla^{q-1} Z)$, one derives from (3.6)

$$\nabla^4 T = -\lambda^3 \eta \wedge T^\flat \otimes dp.\tag{3.8}$$

As a natural concept of concurrent vector fields and by reference to [8], this proves that T is a 3-exterior concurrent vector field.

Since, as it is known, the divergence of a vector field Z is defined by

$$\operatorname{div} Z = \sum_A g(\nabla_{e_A} Z, e_A),\tag{3.9}$$

one derives from (3.2) and (3.6)

$$\begin{aligned}\operatorname{div} \xi &= 2m\lambda, \\ \operatorname{div} T &= T^0 + (2m+1)s,\end{aligned}\tag{3.10}$$

where $T^0 = \eta(T)$. On the other hand, from (3.6), we derive

$$\begin{aligned}dT^a + T^b \theta_b^a + \lambda T^0 \omega^a &= s\omega^a + T^a \eta, \quad a, b \in \{1, \dots, 2m\}, \\ dT^0 &= -(1+\lambda)T^b + [s + (1+\lambda)T^0]\eta.\end{aligned}\tag{3.11}$$

After some calculations, one gets

$$dT^b = \lambda dT^0 \wedge \eta = \lambda(1+\lambda)\eta \wedge T^b,\tag{3.12}$$

which proves that T^b is an exterior recurrent form [1].

Taking the Lie differential of η with respect of T , one gets

$$\mathcal{L}_T \eta = dT^0,\tag{3.13}$$

and so it turns out that

$$d(\mathcal{L}_T \eta) = 0.\tag{3.14}$$

Following a known terminology, T defines a relative contact transformation of the co-Reeb form η .

Next, we will point out some properties of the vector field ϕT .

By virtue of (3.11), one derives

$$\nabla \phi T = (s - \lambda T^0) \phi dp + \phi T \otimes \eta,\tag{3.15}$$

and so, by (3.6) and (3.2), one gets

$$\begin{aligned}[\phi T, T] &= -\lambda T^0 \phi T, \\ [\phi T, \xi] &= (1-\lambda) \phi T, \\ [T, \xi] &= 0.\end{aligned}\tag{3.16}$$

The above relations prove that ϕT admits an infinitesimal transformation of generators T and ξ . In addition, it is seen that ξ and the principal vector field T commute and that the distribution $D_T = \{T, \phi T, \xi\}$ is involutive.

By Orsted lemma [1], if one takes

$$\mathcal{L}_T T^b = \rho T^b + [T, \xi]^b,\tag{3.17}$$

one gets at once by (3.16)

$$\mathcal{L}_T T^b = \rho T^b,\tag{3.18}$$

which shows that the Lie differential of $T^\flat$ with respect to the principal vector field T is conformal to $T^\flat$.

Moreover, making use of the contact ϕ -Lie derivative operator $(\mathcal{L}_\xi \phi)Z = [\xi, \phi] - \phi[\xi, Z]$, one gets in the case under discussion

$$(\mathcal{L}_\xi \phi)T = (\lambda - 1)\phi T. \quad (3.19)$$

Hence, ξ defines a ϕ -Lie transformation of the principal vector field T .

It is worth to point out that the existence of T and ϕT is determined by an exterior differential system Σ whose characteristic numbers are $r = 3$, $s_0 = 1$, $s_1 = 2$ ($r = s_0 + s_1$). Consequently, the system Σ is in *involution* (in the sense of Cartan [3]) and so T and ϕT depend on 1 arbitrary function of 2 arguments (É. Cartan's test).

Recall Yano's formula for any vector field Z , that is,

$$\operatorname{div}(\nabla_Z Z) - \operatorname{div}(\operatorname{div} Z)Z = \mathcal{R}(Z, Z) - (\operatorname{div} Z)^2 + \sum_{A,B} (\nabla_{e_A} Z, e_B)(\nabla_{e_B} Z, e_A), \quad (3.20)$$

where $\mathcal{R}$ denotes the Ricci tensor.

Then, since one has

$$\begin{aligned} \operatorname{div} T &= T^0 + (2m+1)s, \\ \nabla_T T &= (s + T^0)T - \|T\|^2 \xi, \end{aligned} \quad (3.21)$$

it follows by (3.20) that the Ricci tensor corresponding to T is expressed by

$$\mathcal{R}(T, T) = (s + T^0)(T^0 + (2m+1)s) - 4m^2 - s^2. \quad (3.22)$$

Finally, in the same order of ideas, since one has $i_{\phi T} T^\flat = 0$, then, by the Lie differentiation, one derives $\mathcal{L}_{\phi T} T^\flat = 0$, which shows that ϕT defines a Lie Pfaffian transformation of the dual form of the vector field T .

Besides, by the Ricci identity involving the triple $T, \phi T, \xi$, that is,

$$(\mathcal{L}_\xi g)(T, \phi T) = g(\nabla_\xi T, \phi T) + g(T, \nabla_\xi \phi T), \quad (3.23)$$

one gets $(\mathcal{L}_\xi g)(T, \phi T) = 0$.

Hence, one may say that the Lie structure vanishes.

Thus, we have the following.

THEOREM 3.1. *Let $M(\phi, \Omega, \xi, \eta, g)$ be a $(2m+1)$ -dimensional Riemannian manifold endowed with a locally conformal cosymplectic structure and a principal vector field T defined as a conformal contact quasi-torse-forming and structure scalar λ .*

The following properties hold.

- (i) Ω is a locally conformal cosymplectic 2-form.
- (ii) The principal vector field T is 3-exterior concurrent, that is,

$$\nabla^4 T = -\lambda^3 \eta \wedge T^\flat \otimes dp. \quad (3.24)$$

- (iii) T defines a Lie relative contact transformation of the co-Reeb form η .

- (iv) ϕT is an infinitesimal transformation of generators T and ξ . The vector fields ξ , T , and ϕT commute and the distribution $D_T = \{T, \phi T, \xi\}$ is involutive.
- (v) The Lie differential of $T^\flat$ with respect to T is conformal to $T^\flat$.
- (vi) $\operatorname{div} T = T^0 + (2m+1)s$.
- (vii) The Ricci tensor corresponding to T is expressed by

$$\mathcal{R}(T, T) = (s + T^0)(T^0 + (2m+1)s) - 4m^2 - s^2. \quad (3.25)$$

- (viii) The dual form $T^\flat$ of T is an exterior recurrent form.

4. Conformal symplectic form

We will point out some problems regarding the conformal symplectic form Ω . Taking the Lie differential of Ω with respect to the Reeb vector field ξ , we quickly get

$$d(\mathcal{L}_\xi \Omega) = 2\lambda\Omega. \quad (4.1)$$

Hence, we may say that ξ defines a *conformal Lie derivative* of Ω .

Next, taking the Lie differential of Ω with respect to the vector field ϕT , one gets in two steps

$$\mathcal{L}_{\phi T} \Omega = d(T^0 \eta - T^\flat), \quad (4.2)$$

and, by (3.12), one derives at once

$$d(\mathcal{L}_{\phi T} \Omega) = 0. \quad (4.3)$$

Consequently, from above, we may state that the vector field ϕT defines a relative almost-Pfaffian transformation of the form Ω (see [6]).

In the same order of ideas, one derives after some longer calculations

$$d(\mathcal{L}_T \Omega) = 2\lambda\eta \wedge d(\phi T)^\flat - 2\lambda(1+\lambda)T^\flat \wedge \Omega + [s + (1+s)T^0 + 4\lambda^2 T^0]\eta \wedge \Omega, \quad (4.4)$$

and we may say that the principal vector field T defines a *Lie almost-conformal transformation* of Ω .

Finally, we agree to define the 3-form

$$\psi = T^\flat \wedge \Omega, \quad (4.5)$$

the *principal 3-form* on the manifold M under consideration.

Making use of (3.4) and (3.12), one derives

$$d\psi = \lambda(1+\lambda)\eta \wedge \psi. \quad (4.6)$$

This shows that ψ is a recurrent 3-form. Consequently, since one gets

$$i_{\phi T} T^\flat = 0, \quad i_{\phi T} \Omega = T^0 \eta - T^\flat, \quad (4.7)$$

one derives

$$i_{\phi T}\psi = T^0\eta \wedge T^b, \quad (4.8)$$

and so one obtains

$$\mathcal{L}_{\phi T}\psi = 0. \quad (4.9)$$

Hence, we may say that the Lie derivative defines ϕT as a *Pfaffian transformation* of ψ . Thus, we may state the following theorem.

THEOREM 4.1. *Let $M(\phi, \Omega, \xi, \eta, g)$ be a locally conformal cosymplectic manifold. Then, the following hold.*

- (i) *The Reeb vector field ξ defines a conformal Lie derivative of Ω .*
- (ii) *The vector field ϕT defines a relative almost-Pfaffian transformation of the 2-form Ω .*
- (iii) *The principal vector field T defines a Lie almost-conformal transformation of Ω .*
- (iv) *Let $\psi = T^b \wedge \Omega$ be the principal 3-form on the manifold M . Then ψ is a recurrent 2-form and the Lie derivative defines ϕT as a Pfaffian transformation of ψ .*

5. Hypersurface M_ξ normal to ξ

We denote by M_ξ the hypersurface of M normal to ξ . Since $d\eta = 0$ ($\eta = \xi^b$), one may consider the $2m$ -dimensional manifold M_ξ and the 1-dimensional foliation in the direction of ξ is totally geodesic.

Recall that the Weingarten map

$$A : T_{\bar{p}}(M_\xi) \longrightarrow T_{\bar{p}}(M_\xi), \quad \forall \bar{p} \in M_\xi, \quad (5.1)$$

is a linear and selfadjoint application and Ω_η is symplectic.

Then, if Z^T is any horizontal vector field, one gets by $d\eta = 0$

$$AZ^T = \nabla_{Z^T}\xi = -Z^T, \quad (5.2)$$

and this shows that Z^T is a principal vector field of M_ξ .

Recall that $II = \langle d\bar{p}, d\bar{p} \rangle$ and $III = \langle \nabla\xi, \nabla\xi \rangle$ denote the second and the third fundamental forms associated with the immersion $x : M_\xi \rightarrow M$.

Then, by the expression of $\nabla\xi$, one finds that $II = g^T$ and $III = g^T$, where g^T means the horizontal component of g . Hence, we may conclude that the immersion $x : M_\xi \rightarrow M$ is horizontally umbilical and has $2m$ principal curvatures equal to 1.

The expression of III proves that the Gauss map is conformal and it can also be seen that M_ξ is Einsteinian.

On the other hand, since obviously the mean curvature field ξ is nowhere zero, by reference to [4], it follows that the product submanifold $M_\xi \times M_\xi$ in $M \times M$ is a $\mathcal{U}$ -submanifold (i.e., its allied mean curvature vanishes), or a *Chen* submanifold.

We may summarize the above by the following.

THEOREM 5.1. *Let $M(\phi, \Omega, \xi, \eta, g)$ be a locally conformal cosymplectic manifold and $x: M_\xi \rightarrow M$ the immersion of one hypersurface normal to ξ . Then, the following hold.*

- (i) *The Gauss map associated to the immersion $x: M_\xi \rightarrow M$ is conformal.*
- (ii) *The product submanifold $M_\xi \times M_\xi$ in $M \times M$ is a $\mathcal{U}$ -submanifold.*

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