We introduce regularities in commutative Banach algebras in such a way that each regularity defines a joint spectrum on the algebra that satisfies the spectral mapping formula.

1. Introduction

Let $B$ be a complex commutative Banach algebra with unit element denoted by $e$. The space of linear continuous functionals on $B$ is denoted by $B'$. We call regularity in $B$ every nontrivial open subset $R \subset B$ which satisfies the following conditions:

$$ab \in R \text{ iff } a \in R, b \in R, \quad (1.1)$$

$$R = R^*, \text{ where } R^* = \{ b \in B \mid \forall \varphi \in B' \varphi(b) = 0 \Rightarrow 0 \in \varphi(R) \}. \quad (1.2)$$

The set $G(B)$ of invertible elements of $B$ is the main example of a regularity. As was proved in [4], the set of elements of $B$ which are not topological zero divisors is also a regularity.

In the present paper, we investigate a construction of joint spectra in $B$ by means of regularities in $B$.

Let $\sigma(a) = \{ \mu \in \mathbb{C} \mid a - \mu e \notin G(B) \}$ be the ordinary spectrum in $B$.

Recall that according to the terminology introduced by Želazko [6], a subspectrum $\tau$ in $B$ is a mapping which associates to every $k$-tuple $(a_1, \ldots, a_k) \in B^k$ a nonempty compact set $\tau(a_1, \ldots, a_k)$ such that

(a) $\tau(a_1, \ldots, a_k) \subset \prod_{i=1}^k \sigma(a_i)$,

(b) $\tau(p(a_1, \ldots, a_k)) = p(\tau(a_1, \ldots, a_k))$ for every polynomial mapping $p = (p_1, \ldots, p_m): \mathbb{C}^k \to \mathbb{C}^m$.

In Theorem 2.1, we prove that an arbitrary subspectrum $\tau$ in $B$ defines a regularity $R_\tau$ by the formula

$$R_\tau = \{ a \in B \mid 0 \notin \tau(a) \}. \quad (1.3)$$
Lemma 2.3 used in the proof of this theorem permits us to obtain an elementary proof of a theorem belonging to Želazko which provides the complete description of all subspectra in $B$.

Let $M(B)$ be the space of multiplicative functionals on $B$ as usually identified with the space of maximal ideals in $B$. $M(B)$ endowed with the Gelfand topology is a compact space. For $a \in B$, $\varphi \in M(B)$, we denote by $\hat{a}(\varphi) = \varphi(a)$ the Gelfand transform of $a$.

Theorem of Želazko [6] states that for every subspectrum $\tau$ in $B$, there is a unique compact subset $K \subset M(B)$ such that

$$\tau(a_1, \ldots, a_k) = \{(\varphi(a_1), \ldots, \varphi(a_k)) \mid \varphi \in K\}, \quad (1.4)$$

for $(a_1, \ldots, a_k) \in B^k$.

Our proof emphasizes the role played by the spectral mapping formula (b) while the original elegant proof in [6] involves more advanced methods.

The principal result of the paper is Theorem 4.1 which states that for an arbitrary regularity $R$ the formula

$$\sigma_R = (a_1, \ldots, a_k) = \{\lambda_1, \ldots, \lambda_k \in \mathbb{C}^k \mid I_B(a_1 - \lambda_1, \ldots, a_k - \lambda_k) \cap R = \emptyset\} \quad (1.5)$$

defines a subspectrum in $B$. By $I_B(a_1 - \lambda_1, \ldots, a_k - \lambda_k)$ the ideal generated in $B$ by the elements $a_1 - \lambda_1, \ldots, a_k - \lambda_k$ is denoted.

It follows that, given an arbitrary subspectrum $\tau$, we can construct the regularity $R_\tau$ and then the subspectrum $\sigma_{R_\tau}$. Both subspectra $\tau$ and $\sigma_{R_\tau}$, according to Želazko theorem, are uniquely determined by compact subsets of $M(B)$, say $K$ and $K_1$, respectively.

We show that

$$K_1 = \tilde{K} = \{\varphi \in M(B) \mid \forall a \in B \varphi(a) = 0 \implies 0 \in \hat{a}(K)\}. \quad (1.6)$$

The idea of describing spectra of single elements in a (noncommutative) Banach algebra by means of regularities appears in [1] by Kordula and Müller (see also [2]). The present paper is concerned with the case of a commutative Banach algebra and characterizes those regularities and corresponding spectra which admit an extension to a subspectrum.

2. Regularity corresponding to a subspectrum

Let $\tau$ be a subspectrum in a commutative unital Banach algebra $B$ and let $R_\tau = \{a \in B \mid 0 \notin \tau(a)\}$.

For the completeness of the paper, we include the elementary proof of the basic fact in the following theorem.

**Theorem 2.1.** $R_\tau$ is a regularity.

**Proof.** By the property (a) of subspectra, we have $\emptyset \neq \tau(a) \subset \sigma(a)$ for an arbitrary $a \in B$. In particular, $\emptyset \neq \tau(0) \subset \sigma(0) = \{0\}$. Hence $\tau(0) = \{0\}$ and $0 \notin R_\tau$.

For $|\mu| > \|a\|$, the element $a - \mu$ is invertible. So $0 \notin \sigma(a - \mu)$ and $0 \notin \tau(a - \mu)$ neither.

The set $R_\tau$ is not empty and not equal to $B$. 

The particular case of the spectral mapping formula (b) is the addition formula

\[ \tau(a + b) = \{ \lambda + \mu \mid (\lambda, \mu) \in \tau(a, b) \}, \tag{2.1} \]
corresponding to the polynomial \( p(x, y) = x + y. \)

On the other hand, by (a), we have

\[ \tau(a, b) \subset \sigma(a) \times \sigma(b) \subset \sigma(a) \times D(0, \|b\|). \tag{2.2} \]

If \( 0 \notin \tau(a) \) and \( \|b\| < \min\{ |\lambda| \mid \lambda \in \tau(a) \}, \) then \( 0 \notin \tau(a + b) \). The set \( R_\tau \) is open.

We apply the spectral mapping formula in the case of \( p(x, y) = xy. \) We obtain

\[ \tau(ab) = \{ \lambda \mu \mid (\lambda, \mu) \in \tau(a, b) \}. \tag{2.3} \]

Immediately, we conclude that \( 0 \notin \tau(ab) \) if and only if \( 0 \notin \tau(a) \) and \( 0 \notin \tau(b) \).

The set \( R_\tau \) has property (1.1).

The proof of property (1.2) is based on the following two lemmas.

**Lemma 2.2.** (1) If \( (\mu_1, \ldots, \mu_k) \in \tau(a_1, \ldots, a_k) \) and \( b_1, \ldots, b_m \in B \), then there exist \( \lambda_1, \ldots, \lambda_m \in \mathbb{C} \) such that

\[ (\mu_1, \ldots, \mu_k, \lambda_1, \ldots, \lambda_m) \in \tau(a_1, \ldots, a_k, b_1, \ldots, b_m). \tag{2.4} \]

(2) If \( (0, \ldots, 0) \in \tau(a_1, \ldots, a_k) \) and \( b_1^i, \ldots, b_k^i \in B, 1 \leq i \leq m, \) then

\[ (0, \ldots, 0) \in \tau\left( \sum_{j=1}^k a_j b_j^1, \ldots, \sum_{j=1}^k a_j b_j^m \right). \tag{2.5} \]

**Proof.** (1) The spectral mapping property (b) applied to the polynomial \( p(x_1, \ldots, x_k, y_1, \ldots, y_m) = (x_1, \ldots, x_k) \) gives us the first formula.

(2) We can find in \( \tau(a_1, \ldots, a_k, b_1^1, \ldots, b_k^1, \ldots, b_1^m, \ldots, b_k^m) \) an element of the form \( (0, \ldots, 0, \lambda_1, \ldots, \lambda_k, \ldots, \lambda_k^m, \ldots, \lambda_k^m) \) using the first part of the lemma. If we apply the spectral mapping property to the polynomial mapping

\[ p(x_1, \ldots, x_k, y_1^1, \ldots, y_k^1, \ldots, y_1^m, \ldots, y_k^m) = \left( \sum_{j=1}^k x_j y_j^1, \ldots, \sum_{j=1}^k x_j y_j^m \right), \tag{2.6} \]

we obtain the desired property. \( \square \)

**Lemma 2.3.** Let \( (0, \ldots, 0) \in \tau(a_1, \ldots, a_k) \) for some \( a_1, \ldots, a_k \in B. \) Then there exists a maximal ideal \( J \in M(B) \) such that \( I_B(a_1, \ldots, a_k) \subset J \) and \( (0, \ldots, 0) \in \tau(b_1, \ldots, b_m) \) for arbitrary \( b_1, \ldots, b_m \in J. \)

**Proof.** If \( b_1, \ldots, b_m \in I_0 = I_B(a_1, \ldots, a_k), \) then \( (0, \ldots, 0) \in \tau(b_1, \ldots, b_m) \) by Lemma 2.2(2).

Denote by \( \mathcal{J} \) the family of all ideals \( I \) in \( B \) which contain \( I_0 \) and have the property that \( (0, \ldots, 0) \in \tau(b_1, \ldots, b_m) \) for arbitrary \( b_1, \ldots, b_m \in I. \) For every linearly ordered subfamily \( I_\alpha, \alpha \in S \) of \( \mathcal{J}, \) the set \( \bigcup_{\alpha \in S} I_\alpha \in \mathcal{J}. \) So by Kuratowski-Zorn lemma, the family \( \mathcal{J} \) contains a maximal element \( J. \) It remains to prove that \( J \in M(B). \) Suppose that \( J \) is not maximal.
There exists $c \in B$ such that $c + \lambda \notin J$ for all $\lambda \in \mathbb{C}$.
However, by Lemma 2.2(1), for arbitrary $c_1, \ldots, c_k \in J$, the set

$$\delta(c_1, \ldots, c_k) = \{ \lambda \in \mathbb{C} \mid 0 \in \tau(c_1, \ldots, c_k, c - \lambda) \} \quad (2.7)$$

is nonempty. It is a compact set as an intersection of the compact set $\tau(c_1, \ldots, c_k, c)$ with a line.

By the spectral mapping property again,

$$\delta(c_1, \ldots, c_k, b_1, \ldots, b_m) \subset \delta(c_1, \ldots, c_k) \cap \delta(b_1, \ldots, b_m). \quad (2.8)$$

The family of compact sets $\delta(c_1, \ldots, c_k)$ has the finite intersection property, so there exists $\lambda_0 \in \mathbb{C}$ which belongs to $\delta(c_1, \ldots, c_k)$ for every $(c_1, \ldots, c_k) \in J^k$.

By Lemma 2.2(2), the ideal generated by $J$ and $c - \lambda_0$ also belongs to $J$, which is a contradiction. Lemma 2.3 is proved. \hfill $\square$

We return to the proof of Theorem 2.1.

Take $a \notin R_F$. In order to prove that $R^F_F = R_F$, we must find a functional $\phi \in B'$ such that $\phi(a) = 0$ and $0 \notin \phi(R_F)$. By definition $0 \in \tau(a)$ and by Lemma 2.2(2), $(0, \ldots, 0) \in \tau(b_1, \ldots, b_m)$ for all $b_1, \ldots, b_m \in I_B(a)$. Lemma 2.3 says that in particular, $a$ belongs to some $J \in M(B)$ that does not intersect $R_F$. $J$ being a maximal ideal, it is equal to the kernel of a linear (multiplicative) functional. The proof follows. \hfill $\square$

Since the way from Lemma 2.3 to Želazko theorem is short, we include the complete proof of this important theorem.

**Theorem 2.4** [6]. For every subspectrum $\tau$ on a commutative algebra $B$, there exists a unique compact set $K \subset M(B)$ such that

$$\tau(a_1, \ldots, a_k) = \{ (\varphi(a_1), \ldots, \varphi(a_k)) \mid \varphi \in K \}. \quad (2.9)$$

**Proof.** We define $K$ as the set of those multiplicative functionals $\varphi$ on $B$ for which

$$(0, \ldots, 0) \in \tau(b_1, \ldots, b_m) \quad \text{for arbitrary } b_1, \ldots, b_m \in \ker \varphi. \quad (2.10)$$

If $(\mu_1, \ldots, \mu_k) \in \tau(a_1, \ldots, a_k)$, then $(0, \ldots, 0) \in \tau(a_1 - \mu_1, \ldots, a_k - \mu_k)$ and by Lemma 2.3, the ideal generated by $a_1 - \mu_1, \ldots, a_k - \mu_k$ is contained in the kernel of a multiplicative functional $\varphi$ such that condition (2.10) is satisfied.

This proves that $K$ is nonempty and

$$\tau(a_1, \ldots, a_k) \subset \{ (\varphi(a_1), \ldots, \varphi(a_k)) \mid \varphi \in K \}. \quad (2.11)$$

Now suppose that $\varphi \in K$ and $a_1, \ldots, a_k \in B$. Obviously, $a_1 - \varphi(a_1), \ldots, a_k - \varphi(a_k) \in \ker \varphi$ and $(0, \ldots, 0) \in \tau(a_1 - \varphi(a_1), \ldots, a_k - \varphi(a_k))$ that implies that $(\varphi(a_1), \ldots, \varphi(a_k)) \in \tau(a_1, \ldots, a_k)$.

It remains to prove that $K$ is compact. Let $\phi \notin K$. There exist $b_1, \ldots, b_m \in \ker \phi$ such that $(\phi(b_1), \ldots, \phi(b_m)) \notin \tau(b_1, \ldots, b_m)$. By the definition of the Gelfand topology and the compactness of $\tau(b_1, \ldots, b_m)$, the property $(\psi(b_1), \ldots, \psi(b_m)) \notin \tau(b_1, \ldots, b_m)$ holds for $\psi$ in some neighborhood of $\phi$. The set $K^c$ is open and $K$ is compact. \hfill $\square$
Let $X$ be a topological Hausdorff space and $\mathcal{F}$ a family of continuous functions on $X$. For an arbitrary set $C \subset X$, we define the $\mathcal{F}$-rationally convex hull of $C$ as follows:

$$\tilde{C} = \{ x \in X \mid \forall f \in \mathcal{F} \ f(x) = 0 \implies 0 \in f(C) \}. \quad (3.1)$$

The term $\mathcal{F}$-rationally convex hull is justified at least when $C$ is compact and $\mathcal{F}$ is a vector space that contains constant functions.

The case is being $x \in \tilde{C}$ if and only if

$$\left| \frac{f}{g} (x) \right| \leq \sup_{y \in C} \left| \frac{f}{g} (y) \right| \quad (3.2)$$

for every $f, g \in \mathcal{F}$ with $0 \notin g(C)$.

A subset $C \subset X$ is $\mathcal{F}$-rationally convex if $\tilde{C} = C$.

The hull $R^\#$ that appears in the definition of a regularity is just the $B'$-rationally convex hull of a set $R \subset B$. Condition (1.2) means that every regularity is $B'$-rationally convex.

We observe some basic properties of regularities.

**Proposition 3.1.** Let $\emptyset \neq R \subset B$.

1. If $R \subset B$ satisfies (1.1), then it contains the set $G(B)$ of all invertible elements in $B$,
2. if $R$ is a regularity, then

$$R^c = \bigcup_{I \in M(B), I \cap R = \emptyset} \{ I \}. \quad (3.3)$$

**Proof.** (1) Let $b \in R$. Then $b = be \in R$. By condition (1.1), $e \in R$. If $a \in G(B)$, then $aa^{-1} = e \in R$ and again by (1.1), we obtain that $a \in R$.

(2) By the definition, $\bigcup_{I \in M(B), I \cap R = \emptyset} \{ I \} \subset R^c$.

Let $a \notin R$. By condition (1.2), there exists $\phi \in B'$ such that $\phi(a) = 0$ and $0 \notin \phi(R)$. In particular, $(\ker \phi) \cap G(B) = \emptyset$. By Gleason-Kahane-Żelazko theorem, $\phi \in M(B)$ (see [5, page 81]) and

$$a \in \ker \phi \subset \bigcup_{I \in M(B), I \cap R = \emptyset} \{ I \}. \quad (3.4)$$

**Proposition 3.2.** A nontrivial open subset $R \subset B$ is a regularity if and only if $G(B) \subset R$ and $R^\# = R$.

**Proof.** We show that the right-hand side condition implies the property (1.1). By condition (1.2) and Gleason-Kahane-Żelazko theorem, $ab \notin R$ if and only if $\varphi(ab) = 0$ for some $\varphi \in M(B)$ with $\ker \varphi \cap R = \emptyset$. This holds if and only if $\varphi(a) = 0$ or $\varphi(b) = 0$. The proof follows.

In general, condition (1.1) does not imply (1.2). The simplest counterexample is the set $Q = B \setminus \{ 0 \}$, where $B$ is an integral domain.

We observe the following hereditary property.
Proposition 3.3. Let $R$ be a regularity in $B$. Let $A$ be a commutative unital Banach algebra and $\phi : A \to B$ a continuous homomorphism of algebras. Then $Q = \phi^{-1}(R)$ is a regularity in $A$.

Proof. The set $Q$ is obviously open in $A$. Moreover,

$$G(A) \subset \phi^{-1}(G(B)) \subset \phi^{-1}(R) = Q. \tag{3.5}$$

By Proposition 3.2, it is sufficient to prove that $Q^c = Q$. To this end, given $a \notin Q$, we must find $\varphi \in A'$ such that $\varphi(a) = 0$ and $\ker \varphi \cap Q = \emptyset$. Since $\phi(a) \notin R$, there exists $\psi \in B'$ such that $\psi(\phi(a)) = 0$ and $\ker \psi \cap R = \emptyset$. Hence, $\varphi = \psi \circ \phi$ has the desired properties. $\square$

We denote by $\hat{B}$ the set of all Gelfand transforms of elements of $B$.

Theorem 3.4. Let $R$ be a regularity in $B$ and let

$$K = \{ \varphi \in M(B) \mid 0 \notin \varphi(R) \} = \{ \varphi \in M(B) \mid \ker \varphi \cap R = \emptyset \}. \tag{3.6}$$

Then $K$ is a nonempty, compact, $\hat{B}$-rationally convex set.

Proof. As we know by Proposition 3.1(2), $R^c$ is a union of a nonempty family of maximal ideals of $B$ which are precisely kernels of each $\varphi \in K$. Hence $K$ is nonempty.

If $\varphi \in K^c$, then $\hat{a}(\varphi) = 0$ for some $a \in R$. If at the same time $\varphi \in \hat{K}$, we obtain $0 \in \hat{a}(K)$. Hence, $\varphi_0(a) = 0$ for some $\varphi_0 \in K$. This contradicts the definition of $K$, and so $\hat{K} \setminus K = \emptyset$.

Take again $\varphi \in K^c$ and $a \in R$ such that $\varphi(a) = 0$. Since $R$ is open, there exists $\delta > 0$ such that $\|a - b\| < \delta$ implies that $b \in R$. The set $V = \{ \psi \in M(B) \mid |\hat{a}(\psi)| < \delta \}$ is a neighborhood of $\varphi$ in $M(B)$. For $\psi \in V$, we have that $a - \psi(a) \in R$ and $\psi(a - \psi(a)) = 0$. It follows that $V \subset K^c$. So $K^c$ is open, $K$ is closed, and hence compact. $\square$

4. Subspectrum associated to a regularity

Let $R$ be a regularity in $B$. For $(a_1, \ldots, a_k) \in B^k$, denote

$$\sigma_R(a_1, \ldots, a_k) = \{ (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k \mid I_B(a_1 - \lambda_1, \ldots, a_k - \lambda_k) \cap R = \emptyset \}. \tag{4.1}$$

Theorem 4.1. For an arbitrary regularity $R$ in a commutative unital Banach algebra, $\sigma_R$ is a subspectrum. If $K = \{ \varphi \in M(B) \mid 0 \notin \varphi(R) \}$, then

$$\sigma_R(a_1, \ldots, a_k) = \{ (\varphi(a_1), \ldots, \varphi(a_k)) \mid \varphi \in K \}. \tag{4.2}$$
Proof. The condition (a) defining subspectrum is obviously satisfied because \(G(B) \subset R\).

We introduce the operator \(T : B \to C(K)\) by the formula

\[
T(a) = \hat{a} |_K.
\]  

(4.3)

The operator \(T\) is a continuous homomorphism of algebras and its image \(A\) is a unital subalgebra of \(C(K)\). If \(a \in R\), then \(T(a)\) nowhere vanishes on \(K\), hence it is invertible in \(C(K)\). Conversely, if \(a \notin R\), then by the property \(R^* = R\) and Gleason-Kahane-Żelazko theorem, there exists \(\varphi \in K\) such that \(\varphi(a) = 0\). So \(\hat{a}\) vanishes at \(\varphi \in K\) and \(T(a)\) is not invertible in \(C(K)\). It follows that \(T(R) = G(C(K)) \cap A\).

Theorem 3.1 in [3] states that the mapping

\[
\tau(f_1, \ldots, f_k) = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k | I_A(f_1 - \lambda_1, \ldots, f_k - \lambda_k) \cap G(C(K)) = \emptyset\}
\]  

(4.4)

is a subspectrum on \(A\). We extend \(T\) on \(A^k\) in a natural way: \(T(a_1, \ldots, a_k) = (T(a_1), \ldots, T(a_k))\).

Notice that

\[
\sigma_R(a_1, \ldots, a_k) = \tau(T(a_1), \ldots, T(a_k)) = \tau(T(a_1, \ldots, a_k)).
\]  

(4.5)

Then for an arbitrary polynomial mapping \(p : \mathbb{C}^k \to \mathbb{C}^m\), we have

\[
p(\sigma_R(a_1, \ldots, a_k)) = p(\tau(T(a_1), \ldots, T(a_k))) = \tau(p(T(a_1), \ldots, T(a_k))) = \tau(T(p(a_1, \ldots, a_k))) = \sigma_R(p(a_1, \ldots, a_k)).
\]  

(4.6)

Thus the spectral mapping formula (b) holds for \(\sigma_R\).

For every \(\varphi \in K\) and \(a_1, \ldots, a_k \in B\), we have

\[
I_B(a_1 - \varphi(a_1), \ldots, a_k - \varphi(a_k)) \subset \ker \varphi.
\]  

(4.7)

The kernel of \(\varphi\) does not intersect \(R\), so \((\varphi(a_1), \ldots, \varphi(a_k)) \in \sigma_R(a_1, \ldots, a_k)\).

Now suppose that \((\mu_1, \ldots, \mu_k) \in \sigma_R(a_1, \ldots, a_k)\), which implies that \((0, \ldots, 0) \in \sigma_R(a_1 - \mu_1, \ldots, a_k - \mu_k)\). By Lemma 2.3, we know that the ideal \(I_B(a_1 - \mu_1, \ldots, a_k - \mu_k)\) is contained in the kernel of some \(\varphi \in M(B)\) and \(0 \notin \sigma_R(b)\) for all \(b \in \ker \varphi\). It follows that \(\varphi \in K\) and \((\mu_1, \ldots, \mu_k) = (\varphi(a_1), \ldots, \varphi(a_k))\). \(\square\)

The set \(K\) is exactly the compact set which describes the subspectrum \(\sigma_R\) in the sense of Želazko theorem (Theorem 2.4).

In Section 3, we have studied the regularity associated with a given subspectrum. According to the definition, the regularity associated with \(\sigma_R\) is the set \(R_1 = \{a \in B \mid 0 \notin \sigma_R(a)\}\). Obviously, \(R \subset R_1\). If \(a \in R_1\), then \(I_B(a) \cap R \neq \emptyset\). There exists \(b \in B\) such that \(ab \in R\). Hence \(a \in R\) by property (1.1). We conclude that \(R_1 = R\).
It is well known that different subspectra can lead to the same set of regular elements. Let $\tau$ be the approximate point spectrum. The corresponding regularity $R_{\tau}$ is the set of all elements of $B$ which are not topological zero divisors while the set $K_{\tau}$ defining $\tau$ via formula (2.9) is the set of maximal ideals which consists of joint topological zero divisors.

The spectrum $\sigma_{R_{\tau}}$ was studied in [4] and it corresponds to $K$ equal to the set of all maximal ideals consisting of topological zero divisors, which in general differs from $K_{\tau}$.

If $K \subset M(B)$ is compact and $\tau$ is the subspectrum defined by formula (2.9), then the regularity $R_{\tau}$ can be described as

$$\{ a \in B \mid 0 \notin \hat{a}(K) \}. \quad (4.8)$$

**Proposition 4.2.** Let $K_1, K_2 \subset M(B)$ and let

$$R_i = \{ a \in B \mid 0 \notin \hat{a}(K_i) \}, \quad (4.9)$$

$i = 1, 2$. Then $R_1 = R_2$ if and only if $\tilde{K}_1 = \tilde{K}_2$.

**Proof.** Suppose that $R_1 = R_2$. It means that for $a \in B$, the Gelfand transform $\hat{a}$ vanishes on $K_1$ if and only if it vanishes on $K_2$. If $\hat{a}(\varphi) = 0$, then $\hat{a}(K_1)$ contains zero if and only if $\hat{a}(K_2)$ does. Hence $\tilde{K}_1 = \tilde{K}_2$.

Now suppose that $\tilde{K}_1 = \tilde{K}_2$ and that $a \notin R_1$. It follows that $\hat{a}(\varphi) = 0$ for some $\varphi \in K_1 \subset \tilde{K}_2$. We obtain $0 \in \hat{a}(K_2)$. So $a \notin R_2$. This shows that $R_1 \subset R_2$, and $R_2 \subset R_1$. Similarly, we can prove the opposite. Then $R_1 = R_2$. □

For a given regularity $R$ in $B$, the subspectrum $\sigma_R$ is the largest subspectrum having $R$ as the corresponding regularity.

**Proposition 4.3.** Let $R$ be a regularity and let $\tau$ be a subspectrum such that $R_{\tau} = R$. Then for every $k$-tuple $(a_1, \ldots, a_k) \in B^k$,

$$\tau(a_1, \ldots, a_k) \subset \sigma_R(a_1, \ldots, a_k). \quad (4.10)$$

**Proof.** If $R$ is a regularity, then according to Theorem 4.1,

$$\sigma_R(a_1, \ldots, a_k) = \{ (\varphi(a_1), \ldots, \varphi(a_k)) \mid \varphi \in K \}, \quad (4.11)$$

where $K = \tilde{K}$ as Theorem 3.4 asserts.

If $\tau$ is a subspectrum of the form

$$\tau(a_1, \ldots, a_k) = \{ (\varphi(a_1), \ldots, \varphi(a_k)) \mid \varphi \in K_1 \} \quad (4.12)$$

and $R_{\tau} = R$, then $\tilde{K}_1 = \tilde{K} = K$ by Proposition 4.2. In particular, $K_1 \subset K$ and

$$\tau(a_1, \ldots, a_k) \subset \sigma_R(a_1, \ldots, a_k). \quad (4.13)$$

□
References


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