Suppose that $C$ is a nonempty closed convex subset of a real uniformly convex Banach space $X$. Let $T : C \to C$ be an asymptotically quasi-nonexpansive mapping. In this paper, we introduce the three-step iterative scheme for such map with error members. Moreover, we prove that if $T$ is uniformly $L$-Lipschitzian and completely continuous, then the iterative scheme converges strongly to some fixed point of $T$.

1. Introduction

Let $C$ be a subset of normed space $X$, and let $T$ be a self-mapping on $C$. $T$ is said to be nonexpansive provided that $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. $T$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} k_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|$ for all $x, y \in C$ and $n \geq 1$. $T$ is said to be an asymptotically quasi-nonexpansive map if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} k_n = 0$ such that $\|T^n x - p\| \leq (1 + k_n)\|x - p\|$ for all $x \in C$ and $p \in F(T)$, and $n \geq 1$ ($F(T)$ denotes the set of fixed points of $T$, that is, $F(T) = \{x \in C : Tx = x\}$).

From the above definitions, if $F(T) \neq \emptyset$, then asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive mapping.

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk in 1972 [2]. In 2001, Noor [5, 6] introduced the three-step iterative scheme and he studied the approximate solutions of variational inclusions (inequalities) in Hilbert spaces. The three-step iterative approximation problems were studied extensively by Noor [5, 6], Glowinski and Le Tallec [1], and Haubruge et al. [3].

Recently, Xu and Noor [8] introduced the three-step iterative scheme for asymptotically nonexpansive mappings and they proved the following strong convergence theorem in Banach spaces.

**Theorem 1.1** (see [8, Theorem 2.1]). Let $X$ be a real uniformly convex Banach space, let $C$ be a nonempty closed, bounded convex subset of $X$. Let $T$ be a completely continuous and asymptotically nonexpansive self-mapping with sequence $\{k_n\}$ satisfying $k_n \geq 0$ and
\[ \sum_{n=1}^{\infty} k_n < \infty. \] Let \( \{\alpha_n\}, \{\beta_n\}, \) and \( \{\gamma_n\} \) be real sequences in \([0,1]\) satisfying

(i) \( 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1, \)

(ii) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1. \)

For a given \( x_0 \in D, \) define

\[
\begin{align*}
    z_n &= \gamma_n T^n x_n + (1 - \gamma_n) x_n, \\
    y_n &= \beta_n T^n z_n + (1 - \beta_n) x_n, \\
    x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n.
\end{align*}
\] (1.1)

Then \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) converge strongly to a fixed point of \( T. \)

In this paper, we will extend the iterative scheme (1.1) to the iterative scheme of asymptotically quasi-nonexpansive mappings with error members. Moreover, we will prove the strong convergence of iterative scheme to a fixed point of \( T \) (\( C \) need not to be a bounded set), requiring \( T \) to be uniformly \( L \)-Lipschitzian and completely continuous. The results presented in this paper generalize and extend the corresponding main results of Xu and Noor [8].

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

**Definition 2.1** (see [2]). A Banach space \( X \) is said to be uniformly convex if the modulus of convexity of \( X \)

\[
\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| = \epsilon \right\} > 0 \] (2.1)

for all \( 0 < \epsilon \leq 2 \) (i.e., \( \delta_X(\epsilon) \) is a function \((0,2] \to (0,1))\).

**Definition 2.2.** A mapping \( T : C \to C \) is called uniformly \( L \)-Lipschitzian if there exists a constant \( L > 0 \) such that for all \( x, y \in C, \)

\[
\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall \, n \geq 1. \] (2.2)

In what follows, we will make use of the following lemmas.

**Lemma 2.3** (see [4]). Let the nonnegative number sequences \( \{a_n\}, \{b_n\}, \) and \( \{d_n\} \) satisfy that

\[
a_{n+1} \leq (1 + b_n) a_n + d_n, \quad \forall \, n = 1, 2, \ldots, \sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} d_n < \infty. \] (2.3)

Then,

(1) \( \lim_{n \to \infty} a_n \) exists;

(2) if \( \liminf_{n \to \infty} a_n = 0, \) then \( \lim_{n \to \infty} a_n = 0. \)
Lemma 2.4 ([7], J. Schu’s Lemma). Let $X$ be a real uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, $x_n, y_n \in X$, $\limsup_{n \to \infty} \|x_n\| \leq a$, $\limsup_{n \to \infty} \|y_n\| \leq a$, and $\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = a$, $a \geq 0$. Then, $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

3. Main results

In this section, we prove our main theorem. First of all, we will need the following lemmas.

Lemma 3.1. Let $X$ be a real uniformly convex Banach space, $C$ a nonempty closed convex subset of $X$. Let $T$ be an asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n \geq 1}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_0 \in C$ and

$$
\begin{align*}
    z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n, \\
    y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n, \\
    x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n,
\end{align*}
$$

where $\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\beta_n\}, \{\beta'_n\}, \{\beta''_n\}, \{\gamma_n\}, \{\gamma'_n\}$, and $\{\gamma''_n\}$ are real sequences in $[0,1]$ and $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are three bounded sequences in $C$ such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

If $p \in F(T)$, then $\lim_{n \to \infty} \|x_n - p\|$ exists.

Proof. Let $p \in F(T)$. Since $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are bounded sequences in $C$, put

$$
M = \sup_{n \geq 1} \|u_n - p\| \vee \sup_{n \geq 1} \|v_n - p\| \vee \sup_{n \geq 1} \|w_n - p\|. \tag{3.2}
$$

Then $M$ is a finite number. So for each $n \geq 1$, we note that

$$
\begin{align*}
    \|x_{n+1} - p\| &= \|\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\
    &\leq \alpha_n \|T^n y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \tag{3.3} \\
    &\leq \alpha_n (1 + k_n) \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\|, \\
    \|y_n - p\| &= \|\alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\
    &\leq \alpha'_n \|T^n z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \tag{3.4} \\
    &\leq \alpha'_n (1 + k_n) \|z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\|, \\
    \|z_n - p\| &\leq \alpha''_n (1 + k_n) \|x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|u_n - p\|. \tag{3.5}
\end{align*}
$$
Substituting (3.5) into (3.4),

\[
\|y_n - p\| \leq \alpha_n' \beta'' (1 + k_n)^2 \|x_n - p\| + \alpha_n' \gamma'' (1 + k_n) \|u_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|v_n - p\|
\]

\[
\leq (1 - \beta_n' - \gamma_n') \alpha'' (1 + k_n)^2 \|x_n - p\| + \beta_n' \|x_n - p\| + (1 - \beta_n' - \gamma_n') \beta'' (1 + k_n)^2 \|x_n - p\| + m_n
\]

\[
= \beta_n' (1 + k_n)^2 \|x_n - p\| + (1 - \beta_n') \alpha'' (1 + k_n)^2 \|x_n - p\| + m_n
\]

\[
\leq \beta_n (1 + k_n)^2 \|x_n - p\| + (1 - \beta_n') (1 + k_n)^2 \|x_n - p\| + m_n
\]

\[
= (1 + k_n)^2 \|x_n - p\| + m_n,
\]

where \(m_n = \gamma_n' (1 + k_n)M + \gamma_n' M\). Substituting (3.6) into (3.3) again, we have

\[
\|x_{n+1} - p\| \leq \alpha_n (1 + k_n) \|(1 + k_n)^2 \|x_n - p\| + m_n\| \|x_n - p\| + \gamma_n \|w_n - p\|
\]

\[
= \alpha_n (1 + k_n)^3 \|x_n - p\| + \alpha_n (1 + k_n) m_n + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\|
\]

\[
\leq (\alpha_n + \beta_n) (1 + k_n)^3 \|x_n - p\| + (1 + k_n) m_n + \gamma_n \|w_n - p\|
\]

\[
\leq (1 + k_n)^3 \|x_n - p\| + (1 + k_n) m_n + \gamma_n \|w_n - p\|
\]

\[
= (1 + k_n)^3 \|x_n - p\| + (1 + k_n) m_n + \gamma_n M
\]

\[
= (1 + d_n) \|x_n - p\| + b_n,
\]

where \(d_n = 3k_n + 3k_n^2 + k_n^3\) and \(b_n = (1 + k_n) m_n + \gamma_n M\). Since \(\sum_{n=1}^{\infty} d_n < \infty\) and \(\sum_{n=1}^{\infty} b_n < \infty\), by Lemma 2.3, we have that \(\lim_{n \to \infty} \|x_n - p\|\) exists. This completes the proof. \(\square\)

**Lemma 3.2.** Let \(X\) be a real uniformly convex Banach space, \(C\) a nonempty closed convex subset of \(X\). Let \(T\) be an asymptotically quasi-nonexpansive mapping with sequence \(\{k_n\}_{n=1}^{\infty}\) such that \(\sum_{n=1}^{\infty} k_n < \infty\) and \(F(T) \neq \emptyset\). Let \(x_0 \in C\) and for each \(n \geq 0\),

\[
z_n = \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n,
\]

\[
y_n = \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n,
\]

\[
x_{n+1} = \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n,
\]

where \(\{u_n\}, \{v_n\}, \) and \(\{w_n\}\) are three bounded sequences in \(C\) and \(\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\beta_n\}, \{\beta'_n\}, \{\beta''_n\}, \{\gamma_n\}, \{\gamma'_n\}, \) and \(\{\gamma''_n\}\) are real sequences in \([0,1]\) which satisfy the same assumptions as Lemma 3.1 and the additional assumption that \(0 \leq \alpha < \alpha_n, \beta_n, \alpha'_n, \beta'_n \leq \beta < 1\) for some \(\alpha, \beta\) in \((0,1)\). Then \(\lim_{n \to \infty} \|T^n y_n - x_n\| = 0 = \lim_{n \to \infty} \|T^n z_n - x_n\|\).
Proof. For any $p \in F(T)$, it follows from Lemma 3.1, that $\lim_{n \to \infty} \|x_n - p\|$ exists. Let $\lim_{n \to \infty} \|x_n - p\| = a$ for some $a \geq 0$. From (3.6), we have
\[
\|y_n - p\| \leq (1 + k_n)^2 \|x_n - p\| + m_n. \tag{3.9}
\]
Taking $\limsup_{n \to \infty}$ in both sides, we obtain
\[
\limsup_{n \to \infty} \|y_n - p\| \leq \limsup_{n \to \infty} \|x_n - p\| = \lim_{n \to \infty} \|x_n - p\| = a. \tag{3.10}
\]
Note that
\[
\limsup_{n \to \infty} \|T^n y_n - p\| \leq \limsup_{n \to \infty} (1 + k_n) \|y_n - p\| = \limsup_{n \to \infty} \|y_n - p\| \leq a,
\]
a = \lim_{n \to \infty} \|x_{n+1} - p\| = \lim_{n \to \infty} \|a_n T^n y_n + \beta_n x_n + y_n w_n - p\|
\[
= \lim_{n \to \infty} \|a_n \left[ T^n y_n - p + \frac{y_n}{2 \alpha_n} (w_n - p) \right] + \beta_n \left[ x_n - p + \frac{y_n}{2 \beta_n} (w_n - p) \right] + (1 - \alpha_n) \left[ x_n - p + \frac{y_n}{2 \beta_n} (w_n - p) \right] \|. \tag{3.11}
\]
By J. Schu’s Lemma 2.4, we have
\[
\lim_{n \to \infty} \|T^n y_n - x_n + \left( \frac{y_n}{2 \alpha_n} - \frac{y_n}{2 \beta_n} \right) (w_n - p) \| = 0. \tag{3.12}
\]
Since $\lim_{n \to \infty} \|(y_n/2 \alpha_n - y_n/2 \beta_n)(w_n - p)\| = 0$, it follows that
\[
\lim_{n \to \infty} \|T^n y_n - x_n\| = 0. \tag{3.13}
\]
Finally, we will prove that $\lim_{n \to \infty} \|T^n z_n - x_n\| = 0$. To this end, we note that for each $n \geq 1$,
\[
\|x_n - p\| \leq \|T^n y_n - x_n\| + \|T^n y_n - p\| \leq \|T^n y_n - x_n\| + (1 + k_n) \|y_n - p\|. \tag{3.14}
\]
Since $\lim_{n \to \infty} \|T^n y_n - x_n\| = 0 = \lim_{n \to \infty} k_n$, we obtain that
\[
a = \lim_{n \to \infty} \|x_n - p\| \leq \liminf_{n \to \infty} \|y_n - p\|. \tag{3.15}
\]
It follows that
\[
a \leq \liminf_{n \to \infty} \|y_n - p\| \leq \limsup_{n \to \infty} \|y_n - p\| \leq a. \tag{3.16}
\]
This implies that
\[
\lim_{n \to \infty} \|y_n - p\| = a. \tag{3.17}
\]
On the other hand, we note that

\[
\|z_n - p\| = \|\alpha'' T^nx_n + \beta'' x_n + \gamma'' u_n - p\|
\leq \alpha'' (1 + k_n) \|x_n - p\| + \beta'' \|x_n - p\| + \gamma'' \|u_n - p\|
\leq \alpha'' (1 + k_n) \|x_n - p\| + (1 - \alpha'') (1 + k_n) \|x_n - p\| + \gamma'' \|u_n - p\|
\leq (1 + k_n) \|x_n - p\| + \gamma'' \|u_n - p\|.
\]  

(3.18)

By boundedness of the sequence \(\{u_n\}\) and \(\lim_{n \to \infty} k_n = 0 = \lim_{n \to \infty} \gamma''\), we have

\[
\limsup_{n \to \infty} \|z_n - p\| \leq \limsup_{n \to \infty} \|x_n - p\| = a,
\]

and so

\[
\limsup_{n \to \infty} \|T^nx_n - p\| \leq \limsup_{n \to \infty} (1 + k_n) \|z_n - p\| \leq a,
\]

\[
a = \lim_{n \to \infty} \|y_n - p\| = \lim_{n \to \infty} \|\alpha'' T^nx_n + \beta'' x_n + \gamma' v_n - p\|
= \lim_{n \to \infty} \left\| \alpha'' \left[ T^nx_n - p + \frac{\gamma'}{2\alpha''} (v_n - p) \right] + \beta'' \left[ x_n - p + \frac{\gamma'}{2\beta''} (v_n - p) \right] \right\|
\leq \lim_{n \to \infty} \left\| \alpha'' \left[ T^nx_n - p + \frac{\gamma'}{2\alpha''} (v_n - p) \right] + (1 - \alpha'') \left[ x_n - p + \frac{\gamma'}{2\beta''} (v_n - p) \right] \right\|.
\]  

(3.20)

By J. Schu’s Lemma 2.4, we have

\[
\lim_{n \to \infty} \left\| T^nx_n - x_n + \left( \frac{\gamma'}{2\alpha''} - \frac{\gamma'}{2\beta''} \right) (v_n - p) \right\| = 0.
\]  

(3.21)

Since \(\lim_{n \to \infty} \|\gamma'/(2\alpha'' - \gamma'/2\beta'') (v_n - p)\| = 0\), it follows that

\[
\lim_{n \to \infty} \|T^nx_n - x_n\| = 0.
\]  

(3.22)

This completes the proof.

\[\square\]

**Theorem 3.3.** Let \(X\) be a real uniformly convex Banach space, \(C\) a nonempty closed convex subset of \(X\). Let \(T\) be uniformly \(L\)-Lipschitzian, completely continuous, and an asymptotically quasi-nonexpansive mapping with sequence \(\{k_n\}_{n \geq 1}\) such that \(\sum_{n=1}^{\infty} k_n < \infty\) and \(F(T) \neq \emptyset\). Let \(x_0 \in C\) and for each \(n \geq 0\),

\[
z_n = \alpha'' T^nx_n + \beta'' x_n + \gamma'' u_n, \\
y_n = \alpha'' T^nx_n + \beta'' x_n + \gamma' v_n, \\
x_{n+1} = \alpha T^ny_n + \beta x_n + \gamma w_n,
\]

(3.23)

where \(\{u_n\}, \{v_n\}, \text{ and } \{w_n\}\) are three bounded sequences in \(C\) and \(\{\alpha_n\}, \{\alpha'_n\}, \{\alpha''_n\}, \{\beta_n\}, \{\beta'_n\}, \{\gamma_n\}, \{\gamma'_n\}, \text{ and } \{\gamma''_n\}\) are real sequences in \([0,1]\) which satisfy the same assumptions as Lemma 3.1 and the additional assumption that \(0 \leq \alpha < \alpha_n, \beta_n, \alpha''_n, \beta'_n \leq \beta < 1\) for some \(\alpha, \beta\) in \((0,1)\). Then \(\{x_n\}, \{y_n\}, \text{ and } \{z_n\}\) converge strongly to a fixed point of \(T\).
Proof. It follows from Lemma 3.2 that

$$\lim_{n \to \infty} \| T^n y_n - x_n \| = 0 = \lim_{n \to \infty} \| T^n z_n - x_n \|$$

(3.24)

and this implies that

$$\| x_{n+1} - x_n \| = \alpha_n \| T^n y_n - x_n \| + y_n \| w_n - x_n \| \to 0 \text{ as } n \to \infty.$$  

(3.25)

We note that

$$\| T^n x_n - x_n \| \leq \| T^n x_n - T^n y_n \| + \| T^n y_n - x_n \| \leq L \| x_n - y_n \| + \| T^n y_n - x_n \| \to 0 \text{ as } n \to \infty,$$

(3.26)

$$\| x_n - T x_n \| \leq \| x_{n+1} - x_n \| + \| x_{n+1} - T^n x_{n+1} \| + \| T^n x_{n+1} - x_n \| \leq \| x_{n+1} - x_n \| + (1 + k_{n+1}) \| x_{n+1} - x_n \| + L \| T^n x_n - x_n \|.$$

(3.27)

It follows from (3.25), (3.26), and the above inequality that

$$\lim_{n \to \infty} \| x_n - T x_n \| = 0.$$  

(3.28)

By Lemma 3.1, \{x_n\} is bounded. It follows from our assumption that T is completely continuous and that there exists a subsequence \{Tx_{n_k}\} of \{Tx_n\} such that \( \lim_{k \to \infty} \| Tx_{n_k} - p \| = 0 \) as \( k \to \infty \). Moreover, by (3.28), we have \( \| Tx_{n_k} - x_{n_k} \| \to 0 \) which implies that \( x_{n_k} \to p \) as \( k \to \infty \). By (3.28) again, we have

$$\| p - T p \| = \lim_{k \to \infty} \| x_{n_k} - T x_{n_k} \| = 0.$$  

(3.29)

This shows that \( p \in F(T) \). Furthermore, since \( \lim_{n \to \infty} \| x_n - p \| \) exists, we have \( \lim_{n \to \infty} \| x_n - p \| = 0 \), that is, \{x_n\} converges to some fixed point of T. It follows that

$$\| y_n - x_n \| \leq \alpha' \| T^n z_n - x_n \| + y' \| v_n - x_n \| \to 0,$$

$$\| z_n - x_n \| \leq \alpha'' \| T^n x_n - x_n \| + y'' \| u_n - x_n \| \to 0.$$  

(3.30)

Therefore, \( \lim_{n \to \infty} y_n = p = \lim_{n \to \infty} z_n \). This completes the proof. \( \square \)

References


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