INTERACTION OF SHOCK WAVES IN GAS DYNAMICS: UNIFORM IN TIME ASYMPTOTICS

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We construct a uniform in time asymptotics describing the interaction of two isothermal shock waves with opposite directions of motion. We show that any smooth regularization of the problem implies the realization of the stable scenario of interaction.

1. Introduction

We consider the gas dynamics system in the isothermal case

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \quad x \in \mathbb{R}^1,\ t > 0, \qquad \frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}\left(\rho u^2 + c_0^2 \rho\right) = 0, \tag{1.1}$$

together with the initial data in the form of two shock waves with opposite directions of motion

$$\begin{aligned} \rho|_{t=0} &= \rho_0 + e_1 H\left(-x + x_1^0\right) + e_2 H\left(x - x_2^0\right), \\ u|_{t=0} &= u_1 H\left(-x + x_1^0\right) + u_2 H\left(x - x_2^0\right). \end{aligned} \tag{1.2}$$

Here, $H(x)$ is the Heaviside function, $H(x) = 1$ for $x > 0$, and $H(x) = 0$ for $x < 0$, $e_i = \rho_i - \rho_0 > 0$ are amplitudes of jumps, and $\rho_i, u_i, c_0 > 0$ are constants. For definiteness, we assume that $\rho_1 \geqslant \rho_2$ and $x_1^0 < x_2^0$. The initial shock waves are assumed to be stable, so that

$$\begin{aligned} u_1 &= c_0\left(\sqrt{\frac{\rho_1}{\rho_0}} - \sqrt{\frac{\rho_0}{\rho_1}}\right) > 0, \\ u_2 &= -c_0\left(\sqrt{\frac{\rho_2}{\rho_0}} - \sqrt{\frac{\rho_0}{\rho_2}}\right) < 0. \end{aligned} \tag{1.3}$$

The solution of problem (1.1), (1.2) seems nowadays to be well known. Indeed, the standard procedure of "step-by-step" consideration before and after the interaction time instant $t = t^*$ shows that the solution is described by the two noninteracting shock waves

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for $t < t^*$, namely,

$$\begin{aligned} \rho &= \rho_0 + e_1 H\big(-x+\varphi_{10}(t)\big) + e_2 H\big(x-\varphi_{20}(t)\big), \\ u &= u_1 H\big(-x+\varphi_{10}(t)\big) + u_2 H\big(x-\varphi_{20}(t)\big), \end{aligned} \tag{1.4}$$

where $\varphi_{i0} = \varphi_{i0_t} t + x_i^0$ are the phases of the shocks,

$$\varphi_{10_t} = u_1 + c_0\sqrt{\frac{\rho_0}{\rho_1}} = c_0\sqrt{\frac{\rho_1}{\rho_0}}, \qquad \varphi_{20_t} = u_2 - c_0\sqrt{\frac{\rho_0}{\rho_2}} = -c_0\sqrt{\frac{\rho_2}{\rho_0}}, \tag{1.5}$$

and $\varphi_{10}(t^*) = \varphi_{20}(t^*) \overset{\text{def}}{=} x^*$ is the point of intersection of the paths $x = \varphi_{i0}(t)$, $i = 1,2$.

Next, at the time t^*, the initial conditions (1.2) are replaced by the shock wave with the amplitudes $\rho_1 - \rho_2$ and $u_1 - u_2$ of the jumps of ρ and u, which are concentrated at the point $x = x^*$. Solving this Riemann problem, we obtain that the solution for $t > t^*$ is again represented by two noninteracting shock waves with uniquely defined new amplitudes and new paths of propagation (see, e.g., [2, 9]). Let us call this behavior of the solution the "stable scenario."

However, the uniqueness of weak solutions for hyperbolic systems of conservation laws has been proved (with additional conditions) only for the case of sufficiently small amplitudes of shocks (see [1, 2, 8]). Apart from the above mentioned solution, the Riemann problem admits a family of artificial solutions. Therefore, the described construction cannot be treated as a well-posed one for the case of arbitrary amplitudes of shocks.

It is clear that the weak point of this scheme is the consideration of shock waves as noninteracting ones for time close to t^*. Moreover, this conflicts with the physical sense of the problem since the actual gas dynamics includes viscosity phenomena. Therefore, it is necessary to smooth the solution for time close to t^*, and to consider the process of interaction in detail.

Whitham [10] was the first to solve a similar problem for the inviscid Burgers-Hopf equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{1.6}$$

with the quadratic nonlinearity $f(u) = u^2$. Passing to the Burgers regularization and using the Hopf-Cole transformation, G. Whitham found the exact solution for the initial data similar to (1.2) and, as a result, established that the regularization implies the choice of a stable scenario of interaction. However, this procedure works uniquely for the quadratic nonlinearity. A progress in this problem has been achieved only recently by Danilov and Shelkovich for (1.6) with convex nonlinearities (see [5]; see also [3, 4]). Since it is impossible to find exact solutions in the general case, they constructed an asymptotic solution in the framework of the "weak asymptotic method" [3, 4, 5, 6, 7]. The main point here is the treatment of the solution $u_\varepsilon(x,t)$ of the regularized problem as a $\mathscr{C}^\infty(0,T;\mathscr{C}^\infty(\mathbb{R}^1))$ mapping for $\varepsilon = \text{const} > 0$ and a $\mathscr{C}(0,T;\mathscr{D}'(\mathbb{R}^1))$ mapping uniformly in $\varepsilon \in [0,1]$, where ε is a parameter of regularization. Respectively, a family $u_\varepsilon(t,x)$ is called

an *asymptotic* mod $\mathbb{O}_{\mathcal{D}'}(\varepsilon)$ *solution* of (1.6) if the relation

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_\varepsilon \psi\, dx - \int_{-\infty}^{\infty} f(u_\varepsilon)\frac{\partial \psi}{\partial x}\, dx = \mathbb{O}(\varepsilon) \tag{1.7}$$

holds for any test function $\psi = \psi(x)$. The main advantage of this approach is the possibility to describe the interaction of nonlinear waves by an ordinary differential equation. Let us note that this method allows also to describe soliton interactions for nonintegrable problems [4, 7].

Our aim is a generalization of the weak asymptotic method for hyperbolic systems of conservation laws. Using system (1.1) as a simple but meaningful example, we show that this tool easily allows to construct an asymptotic solution. At the same time, we obtain in this way a scattering-type problem for a dynamical system (instead of an equation in the scalar case). Analysis of this problem requires the use of the specifics of the original problem. However, this can be done, and we obtain a uniform in time description of the interaction of two shock waves in the case of opposite directions of motion.

2. Construction of the asymptotic solution

Following the ideas sketched above, we arrive at what follows.

Definition 2.1. Sequences $\rho_\varepsilon(t,x)$ and $u_\varepsilon(t,x)$ are called a weak asymptotic mod $\mathbb{O}_{\mathcal{D}'}(\varepsilon)$ solution of system (1.1) if $\rho_\varepsilon(t,x)$ and $u_\varepsilon(t,x)$ belong to $\mathcal{C}^\infty([0,T]\times\mathbb{R}^1)$ for $\varepsilon = \text{const} > 0$ and to $\mathcal{C}(0,T;\mathcal{D}'(\mathbb{R}^1))$ uniformly in $\varepsilon \in [0,\text{const}]$, and if the relations

$$\begin{gathered}\frac{d}{dt}\int_{-\infty}^{\infty} \rho_\varepsilon \psi_1\, dx - \int_{-\infty}^{\infty} \rho_\varepsilon u_\varepsilon \frac{\partial \psi_1}{\partial x}\, dx = \mathbb{O}(\varepsilon),\\ \frac{d}{dt}\int_{-\infty}^{\infty} \rho_\varepsilon u_\varepsilon \psi_2\, dx - \int_{-\infty}^{\infty} (\rho_\varepsilon u_\varepsilon^2 + c_0^2\rho_\varepsilon)\frac{\partial \psi_2}{\partial x}\, dx = \mathbb{O}(\varepsilon)\end{gathered} \tag{2.1}$$

hold for any test functions, $\psi_i = \psi_i(x) \in \mathcal{D}(\mathbb{R}^1)$.

It is necessary to note that the parabolic regularization of (1.1) with $\mathbb{O}(\varepsilon)$ viscosity terms implies $\mathbb{O}(\varepsilon)$ corrections in relations (2.1).

To present the asymptotic solution, let us denote $\omega = \omega(\eta) \in \mathcal{C}^\infty(\mathbb{R}^1)$ an auxiliary function such that

$$\lim_{\eta\to-\infty} \omega = 0, \qquad \lim_{\eta\to\infty} \omega = 1. \tag{2.2}$$

For simplicity, we assume that ω tends to its limiting values at an exponential rate. Moreover, we assume that

$$\omega'_\eta > 0, \qquad \omega(\eta) + \omega(-\eta) = 1. \tag{2.3}$$

Obviously, this implies that $\omega(\eta) - 1/2$ is an odd function and $\omega((x-\phi)/\varepsilon) \to H(x-\phi)$ as $\varepsilon \to 0$.

Now, let us write the weak asymptotic solution for the problem (1.1), (1.2) in the following form:

$$\begin{aligned}\rho_\varepsilon &= \rho_0 + e_1\omega\left(\frac{-x+\phi_1}{\varepsilon}\right) + e_2\omega\left(\frac{x-\phi_2}{\varepsilon}\right) + r\omega\left(\frac{-x+\phi_1}{\varepsilon}\right)\omega\left(\frac{x-\phi_1}{\varepsilon}\right),\\ u_\varepsilon &= u_1\omega\left(\frac{-x+\phi_1}{\varepsilon}\right) + u_2\omega\left(\frac{x-\phi_2}{\varepsilon}\right) + v\omega\left(\frac{-x+\phi_1}{\varepsilon}\right)\omega\left(\frac{x-\phi_1}{\varepsilon}\right).\end{aligned} \tag{2.4}$$

Here, the phases $\phi_i = \phi_i(t,\tau)$ are defined by

$$\phi_i = \varphi_{i0}(t) + \psi_0(t)\varphi_{i1}(\tau), \quad i = 1,2, \tag{2.5}$$

where

$$\psi_0(t) = \varphi_{20}(t) - \varphi_{10}(t), \qquad \tau = \frac{\psi_0(t)}{\varepsilon}, \tag{2.6}$$

and the phases of noninteracting shock waves, φ_{i0}, and ρ_0, e_i, u_i are the same as in (1.4), (1.5).

Thus, τ plays the role of a "fast" time and for time t before (after) the interaction $\psi_0(t) > 0$ and $\tau \to +\infty$ ($\psi_0(t) < 0$ and $\tau \to -\infty$) as $\varepsilon \to 0$.

The functions $\varphi_{i1} = \varphi_{i1}(\tau)$, $r = r(\tau)$, and $v = v(\tau)$ are assumed to be smooth and such that

$$\varphi_{i1} \longrightarrow 0, \quad r \longrightarrow 0, \quad v \longrightarrow 0 \quad \text{as } \tau \longrightarrow +\infty, \tag{2.7}$$

$$\varphi_{i1} \longrightarrow \overline{\varphi}_{i1}, \quad r \longrightarrow \overline{r}, \quad v \longrightarrow \overline{v} \quad \text{as } \tau \longrightarrow -\infty, \tag{2.8}$$

where $\overline{\varphi}_{i1}, \overline{r}$, and $\overline{v}$ are some constants.

The first assumption (2.7) implies that the anzatz (2.4) describes the two noninteracting waves (1.4) before the interaction. In order to describe the behavior of the anzatz after the instant time of interaction, we have to analyze the product $\omega((-x+\phi_1)/\varepsilon)\omega((x-\phi_2)/\varepsilon)$.

Lemma 2.2. *Under the assumptions mentioned above, the following relations hold:*

$$\omega^k\left(\frac{x-\phi_i}{\varepsilon}\right) = H(x-\phi_i) + \varepsilon d_k\delta(x-\phi_i) + \mathbb{O}_{\mathcal{D}'}(\varepsilon^2), \tag{2.9}$$

$$\begin{aligned}\omega^k\left(\frac{-x+\phi_1}{\varepsilon}\right)\omega^\ell\left(\frac{x-\phi_2}{\varepsilon}\right) &= b_{k\ell}\{H(-x+\phi_1) - H(-x+\phi_2)\}\\ &\quad - \varepsilon\{c_{k\ell}\delta(x-\phi_1) + c_{\ell k}\delta(x-\phi_2)\} + \mathbb{O}_{\mathcal{D}'}(\varepsilon^2),\end{aligned} \tag{2.10}$$

where $k,\ell \geqslant 1$, d_k are some constants, $d_1 = 0$ and

$$\begin{aligned}b_{k\ell} &= k\int_{-\infty}^{\infty} \omega^{k-1}(\eta)\omega'_\eta(\eta)\omega^\ell(-\sigma-\eta)d\eta,\\ c_{k\ell} &= k\int_{-\infty}^{\infty} \eta\,\omega^{k-1}(\eta)\omega'_\eta(\eta)\omega^\ell(-\sigma-\eta)d\eta.\end{aligned} \tag{2.11}$$

Here and in what follows

$$\sigma = \frac{\phi_2 - \phi_1}{\varepsilon}, \qquad \omega'_\eta(\eta) = \frac{d\omega(\eta)}{d\eta}. \tag{2.12}$$

Proof. Relation (2.9) is almost obvious. Let us only note that the equality $d_1 = 0$ is a direct consequence of the equality in (2.3). Furthermore, considering the left-hand side of relation (2.10) in the weak sense, we obtain the following:

$$\begin{aligned}
I &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \omega^k\left(\frac{-x+\phi_1}{\varepsilon}\right)\omega^\ell\left(\frac{x-\phi_2}{\varepsilon}\right)\psi(x)dx \\
&= \int_{-\infty}^{\infty} \omega^k\left(\frac{-x+\phi_1}{\varepsilon}\right)\omega^\ell\left(\frac{x-\phi_2}{\varepsilon}\right)\frac{d}{dx}\int_{-\infty}^{x}\psi(x')dx'\,dx \\
&= -\int_{-\infty}^{\infty}\psi_0(x)\left\{\omega^\ell\left(\frac{x-\phi_2}{\varepsilon}\right)\frac{\partial}{\partial x}\omega^k\left(\frac{-x+\phi_1}{\varepsilon}\right) + \omega^k\left(\frac{-x+\phi_1}{\varepsilon}\right)\frac{\partial}{\partial x}\omega^\ell\left(\frac{x-\phi_2}{\varepsilon}\right)\right\}dx \\
&= k\int_{-\infty}^{\infty}\omega^{k-1}(\eta)\omega'_\eta(\eta)\omega^\ell(-\sigma-\eta)\psi_0(\phi_1-\varepsilon\eta)d\eta \\
&\quad - \ell\int_{-\infty}^{\infty}\omega^{\ell-1}(\eta)\omega'_\eta(\eta)\omega^k(-\sigma-\eta)\psi_0(\phi_2+\varepsilon\eta)d\eta,
\end{aligned} \tag{2.13}$$

where $\psi_0(x) = \int_{-\infty}^{x}\psi(x')dx'$, and $\psi \in \mathcal{D}(\mathbb{R}^1)$, and we took into account the exponential rate of vanishing of the product $\omega(\eta)\omega(-\eta)$ as $\eta \to \pm\infty$. Now, applying the Taylor expansion and using notation (2.10), we can rewrite the right-hand side in (2.13) in the following form:

$$I = b_{k\ell}\psi_0(\phi_1) - b_{\ell k}\psi_0(\phi_2) - \varepsilon c_{k\ell}\psi(\phi_1) - \varepsilon c_{\ell k}\psi(\phi_2) + \mathcal{O}(\varepsilon^2). \tag{2.14}$$

A detailed analysis of the integrals in (2.11) implies the following statement.

Lemma 2.3. *The convolutions $b_{k\ell}$ and $c_{k\ell}$ exist and have the following properties:*

$$\begin{gathered}
b_{k\ell}(\sigma) = b_{\ell k}(\sigma) > 0, \qquad \frac{\sigma}{2}\frac{db_{kk}}{d\sigma} + \frac{dc_{kk}}{d\sigma} = 0, \\
\sigma\frac{db_{12}}{d\sigma} + \frac{d}{d\sigma}(c_{12}+c_{21}) = 0,
\end{gathered} \tag{2.15}$$

$$\begin{gathered}
\lim_{\sigma\to+\infty} b_{k\ell}(\sigma) = \lim_{\sigma\to+\infty} c_{k\ell}(\sigma) = 0, \quad \lim_{\sigma\to-\infty} b_{k\ell} = 1, \quad \textit{for any } k,\ell \geqslant 1, \\
\lim_{\sigma\to-\infty} c_{1\ell} = 0, \quad \lim_{\sigma\to-\infty} c_{2\ell} = c_2 \stackrel{\text{def}}{=} \int_{-\infty}^{\infty}\eta\left(\omega^2(\eta)\right)'_\eta d\eta, \quad \textit{for } \ell \geqslant 1.
\end{gathered} \tag{2.16}$$

Moreover,

$$\begin{gathered}
\text{for } \ell \geqslant 1,\ k > 1, \quad \lim_{\sigma\to+\infty} \frac{b_{\ell k}}{b_{11}^{\ell}} = 0, \quad \lim_{\sigma\to+\infty} \frac{c_{11}}{b_{11}} = \alpha_1\sigma, \\
\text{for } i = 1,2, \quad \lim_{\sigma\to+\infty} \frac{c_{i2}}{b_{11}^{i}} = \beta_i, \quad \lim_{\sigma\to+\infty} \frac{c_{21}}{b_{11}} = 0, \\
\text{for } j = 1,2, \quad \lim_{\sigma\to+\infty} \sigma^{j-1}\left(\frac{db_{1j}}{d\sigma}\Big/ b_{11}\right) = \gamma_j, \quad \lim_{\sigma\to+\infty} \left(\frac{dc_{12}}{d\sigma}\Big/ b_{11}\right) = c,
\end{gathered} \tag{2.17}$$

where α_1, β_i, γ_j, and c are constants.

Using the first equation in (2.15) and the obvious relation $\psi_0(\phi_i) = \int_{-\infty}^{\infty} H(-x+\phi_i)\psi(x)dx$, we see that (2.14) implies the desired relation (2.10). This completes the proof of Lemma 2.2. □

Applying the statement of Lemma 2.2, we obtain that the weak asymptotic of the anzatz (2.4) has the following form:

$$\begin{aligned}
\rho_\varepsilon = {} & \rho_0 - rb_{11} + (e_1 + rb_{11})H(-x+\phi_1) + (e_2 + rb_{11})H(x-\phi_2) \\
& - \varepsilon r c_{11}(\delta(x-\phi_1) + \delta(x-\phi_2)) + \mathbb{O}_{\mathcal{D}'}(\varepsilon^2),
\end{aligned} \tag{2.18}$$

$$u_\varepsilon = -vb_{11} + (u_1 + vb_{11})H(-x+\phi_1) + (u_2 + vb_{11})H(x-\phi_2) + \mathbb{O}_{\mathcal{D}'}(\varepsilon^2). \tag{2.19}$$

Therefore, for time after the interaction, we obtain two shock waves (or one wave if $\lim_{\tau\to-\infty}(\phi_1 - \phi_2) = 0$) with new amplitudes and new trajectories of motion. It is clear also that assumptions (2.8) are critical ones. Indeed, the breakdown of (2.8) implies the realization of a scenario of shock waves interaction which is qualitatively different from the stable scenario.

Now, let us find equations for the functions ϕ_i, r, and v. To this end, we should calculate weak asymptotic expansions for the expressions in the integrals (2.1). Applying the statement of Lemmas 2.2 and 2.3 and using the notation

$$V = vb_{11}, \qquad R = rb_{11}, \qquad B_{ik} = \frac{b_{ik}}{b_{11}^{i}}, \qquad C_{ik} = \frac{c_{ik}}{b_{11}^{i}}, \tag{2.20}$$

after simple calculations, we obtain the following.

Lemma 2.4. *Under the assumptions (2.2) and (2.3), the following relations hold:*

$$\begin{aligned}
\rho_\varepsilon u_\varepsilon = {} & \rho_1 u_1 H(-x+\phi_1) + \rho_2 u_2 H(x+\phi_2) \\
& + G_1\{H(-x+\phi_1) - H(-x+\phi_2)\} \\
& + \varepsilon G_2 \delta(x-\phi_1) + \varepsilon G_3 \delta(x-\phi_2) + \mathbb{O}_{\mathcal{D}'}(\varepsilon^2),
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\rho_\varepsilon u_\varepsilon^2 = {} & \rho_1 u_1^2 H(-x+\phi_1) + \rho_2 u_2^2 H(x+\phi_2) \\
& + G_4\{H(-x+\phi_1) - H(-x+\phi_2)\} + \mathbb{O}_{\mathcal{D}'}(\varepsilon),
\end{aligned} \tag{2.22}$$

where

$$
\begin{aligned}
&G_i = g_{i0} + g_{i1}V + g_{i2}R + B_{22}RV, \quad i = 1,2,3,\\
&G_4 = g_{40} + g_{41}V + g_{42}R + g_{43}V^2 + g_{44}RV + B_{33}RV^2,\\
&g_{10} = (e_1u_2 + e_2u_1)b_{11}, \qquad g_{20} = g_{30} = (e_1u_2 + e_2u_1)c_{11},\\
&g_{11} = \rho_0 + (e_1 + e_2)B_{12}, \qquad g_{12} = (u_1 + u_2)B_{12},\\
&g_{21} = \rho_0C_{11} + e_1C_{21} + e_2C_{12}, \qquad g_{22} = u_1C_{21} + u_2C_{12},\\
&g_{31} = \rho_0C_{11} + e_2C_{21} + e_1C_{12}, \qquad g_{32} = u_2C_{21} + u_1C_{12},\\
&g_{40} = 2\rho_0u_1u_2b_{11} + [(e_1u_2 + e_2u_1)(u_1 + u_2) + (e_1 + e_1)u_1u_2]b_{12},\\
&g_{41} = 2[\rho_0(u_1 + u_2) + (e_1u_1 + e_2u_2)B_{13} + (e_1u_2 + e_2u_1)b_{11}B_{22}],\\
&g_{42} = (u_1^2 + u_2^2)B_{13} + 2u_1u_2b_{11}B_{22}, \qquad g_{43} = \rho_0B_{22} + (e_1 + e_2)B_{23},\\
&g_{44} = 2(u_1 + u_2)B_{23}.
\end{aligned}
\tag{2.23}
$$

Now, we should calculate the time derivatives. Since

$$
\frac{d\tau(t)}{dt} = \frac{\psi_{0_t}}{\varepsilon}, \quad \psi_{0_t} = \varphi_{20_t} - \varphi_{10_t}, \tag{2.24}
$$

in order to obtain the precision $\mathbb{O}(\varepsilon)$ in the right-hand side of relations (2.1), we have to take into account the terms of order $\mathbb{O}_{\mathscr{D}'}(\varepsilon)$ in (2.18) and (2.21). At the same time, the phase derivatives do not include $\mathbb{O}(1/\varepsilon)$ terms since

$$
\frac{d\phi_i}{dt} = \varphi_{i0_t} + \psi_{0_t}\varphi_{i1} + \frac{\psi_0}{\varepsilon}\psi_{0_t}\varphi_{i1}' = \varphi_{i0_t} + \psi_{0_t}(\tau\varphi_{i1})'. \tag{2.25}
$$

Here and in the sequel, the apostrophe denotes derivative with respect to τ.

Next, using formulas (2.18), (2.25), and notation (2.20), we find that

$$
\begin{aligned}
\frac{\partial\rho_\varepsilon}{\partial t} = {}& (e_1 + R)\frac{d\phi_1}{dt}\delta(x - \phi_1) - (e_2 + R)\frac{d\phi_2}{dt}\delta(x - \phi_2)\\
&+ \frac{\psi_{0t}}{\varepsilon}R'\{H(-x + \phi_1) - H(-x + \phi_2)\}\\
&- \psi_{0t}(RC_{11})'\{\delta(x - \phi_1) - \delta(x - \phi_2)\} + \mathbb{O}_{\mathscr{D}'}(\varepsilon).
\end{aligned}
\tag{2.26}
$$

Now we need to use the following almost obvious statement.

Lemma 2.5. *Let $S(\tau)$ be a function from the Schwartz space, and let a function $\phi_k(\tau) \in \mathscr{C}^\infty$ have the representation*

$$
\phi_k(\tau) = x^* + \varepsilon\chi_k(\tau), \quad \chi_k(0) = 0, \tag{2.27}
$$

where $x^* = \text{const}$ *and* χ_k *is a slowly increasing function. Then*

$$S(\tau)H(-x+\phi_k(\tau)) = S(\tau)H(-x+x^*) + \varepsilon S(\tau)\chi_k(\tau)\delta(x-x^*) + \mathbb{O}_{\mathcal{D}'}(\varepsilon^2). \tag{2.28}$$

Moreover,

$$S(\tau)\delta(x-x^*) = S(\tau)\delta(x-\phi_k(\tau)) + \mathbb{O}_{\mathcal{D}'}(\varepsilon). \tag{2.29}$$

Denoting x^*, t^* the point and time instant of interaction of the paths $x = \varphi_{i0}(t)$, $i = 1,2$, we obtain the equality

$$\tau = \frac{\psi_0(t)}{\varepsilon} = \psi_{0_t}\frac{t-t^*}{\varepsilon}. \tag{2.30}$$

Thus, for functions of the form (2.5), we have

$$\phi_k = x^* + \varepsilon\tau\left(\frac{\varphi_{k0_t}}{\psi_{0_t}} + \varphi_{k1}(\tau)\right) \overset{\text{def}}{=} x^* + \varepsilon\chi_k(\tau). \tag{2.31}$$

The assumptions for r and the properties of the convolutions $b_{k\ell}$, $c_{k\ell}$ imply exponential rate of vanishing of the functions R' and $(RC_{11})'$. This and the statement of Lemma 2.5 imply the following relation:

$$\begin{aligned}\frac{1}{\varepsilon}R'\{H(-x+\phi_1) - H(-x+\phi_2)\} &= R'(\chi_1-\chi_2)\delta(x-x^*) + \mathbb{O}_{\mathcal{D}'}(\varepsilon)\\ &= -\frac{\sigma}{2}R'\{\delta(x-\phi_1)+\delta(x-\phi_2)\} + \mathbb{O}_{\mathcal{D}'}(\varepsilon).\end{aligned} \tag{2.32}$$

Therefore,

$$\frac{\partial\rho_\varepsilon}{\partial t} = \left\{(e_1+R)\frac{d\phi_1}{dt} - \psi_{0_t}L_0\right\}\delta(x-\phi_1) - \left\{(e_2+R)\frac{d\phi_2}{dt} + \psi_{0_t}L_0\right\}\delta(x-\phi_2) + \mathbb{O}_{\mathcal{D}'}(\varepsilon), \tag{2.33}$$

where

$$L_0 = \frac{\sigma}{2}R' + (RC_{11})'. \tag{2.34}$$

It remains to use equality (2.15). Introduce the following notation:

$$\begin{gathered} k_{ij} = \frac{\sigma}{2}b_{ij} + c_{ij}, \qquad \dot{B}_{11} = \frac{1}{b_{11}}\frac{db_{11}}{d\sigma},\\ K_{1i} = \frac{k_{1i}}{b_{11}}, \quad i = 1,2, \qquad K_{21} = \frac{k_{21}}{b_{11}}, \qquad K_{22} = \frac{k_{22}}{b_{11}^2}. \end{gathered} \tag{2.35}$$

Then

$$L_0 = K_{11}(R' - R\dot{B}_{11}\sigma'). \tag{2.36}$$

Preparing similar calculations for the time derivative of $\rho_\varepsilon u_\varepsilon$, we obtain the formula

$$\begin{aligned}\frac{\partial \rho_\varepsilon u_\varepsilon}{\partial t} &= \left\{(\rho_1 u_1 + G_1)\frac{d\phi_1}{dt} - \psi_{0_t} L_1\right\}\delta(x-\phi_1) \\ &\quad - \left\{(\rho_2 u_2 + G_1)\frac{d\phi_2}{dt} + \psi_{0_t} L_2\right\}\delta(x-\phi_2) + \mathbb{O}_{\mathcal{D}'}(\varepsilon),\end{aligned} \tag{2.37}$$

where

$$\begin{aligned}
&L_i = \ell_{i1} + \ell_{i2}\sigma', \quad \ell_{i1} = M_i V' + N_i R', \quad i = 1,2, \\
&M_1 = \rho_0 K_{11} + e_2 K_{12} + e_1 K_{21} + R K_{22}, \qquad N_1 = u_2 K_{12} + u_1 K_{21} + V K_{22}, \\
&M_2 = \rho_0 K_{11} + e_1 K_{12} + e_2 K_{21} + R K_{22}, \qquad N_2 = u_1 K_{12} + u_2 K_{21} + V K_{22}, \\
&\ell_{12} = \{(e_2 - e_1)D_{12} - M_1 \dot{B}_{11}\}V + \{(u_2 - u_1)D_{12} - N_1 \dot{B}_{11}\}R, \\
&\ell_{22} = -\{(e_2 - e_1)D_{12} + M_2 \dot{B}_{11}\}V - \{(u_2 - u_1)D_{12} + N_2 \dot{B}_{11}\}R, \\
&D_{12} = \frac{1}{b_{11}}\left\{\frac{\sigma}{2}\frac{db_{12}}{d\sigma} + \frac{dc_{12}}{d\sigma}\right\}.
\end{aligned} \tag{2.38}$$

Next, using formulas (2.18), (2.21), and (2.22), it is easy to calculate the derivatives

$$\begin{aligned}
\frac{\partial \rho_\varepsilon}{\partial x} &= -(e_1 + R)\delta(x-\phi_1) + (e_2 + R)\delta(x-\phi_2) + \mathbb{O}_{\mathcal{D}'}(\varepsilon), \\
\frac{\partial \rho_\varepsilon u_\varepsilon}{\partial x} &= -(\rho_1 u_1 + G_1)\delta(x-\phi_1) + (\rho_2 u_2 + G_1)\delta(x-\phi_2) + \mathbb{O}_{\mathcal{D}'}(\varepsilon), \\
\frac{\partial \rho_\varepsilon u_\varepsilon^2}{\partial x} &= -(\rho_1 u_1^2 + G_4)\delta(x-\phi_1) + (\rho_2 u_2^2 + G_4)\delta(x-\phi_2) + \mathbb{O}_{\mathcal{D}'}(\varepsilon).
\end{aligned} \tag{2.39}$$

Substituting expressions (2.32), (2.37), and (2.39) into relations (2.1), collecting coefficients of $\delta(x-\phi_1)$ and $\delta(x-\phi_2)$ and setting them equal to zero, we obtain

$$(e_1 + R)\frac{d\phi_1}{dt} = \psi_{0_t} L_0 + G_1 + \rho_1 u_1, \tag{2.40}$$

$$(e_2 + R)\frac{d\phi_2}{dt} = -\psi_{0_t} L_0 + G_1 + \rho_2 u_2, \tag{2.41}$$

$$(\rho_1 u_1 + G_1)\frac{d\phi_1}{dt} = \psi_{0_t} L_1 + G_4 + \rho_1 u_1^2 + c_0^2(e_1 + R), \tag{2.42}$$

$$(\rho_2 u_2 + G_1)\frac{d\phi_2}{dt} = -\psi_{0_t} L_2 + G_4 + \rho_2 u_2^2 + c_0^2(e_2 + R). \tag{2.43}$$

Theorem 2.6. *Let there exist a smooth solution R, V, and ϕ_i, $i = 1,2$, of the system (2.40)–(2.43) such that relations (2.7) and (2.8) hold. Then the weak asymptotic solution (2.4) describes the stable scenario of the shock waves interaction uniformly in time.*

Proof. To prove this statement, it is enough to consider system (2.40)–(2.43) for $\tau \to \pm\infty$. Let $\tau \to +\infty$. Assumption (2.7) and the vanishing of the convolutions imply the relations

$$G_i, L_i, R, V \longrightarrow 0, \quad \frac{d\phi_i}{dt} \longrightarrow \varphi_{i0_t}, \quad i = 1,2, \text{ as } \tau \longrightarrow +\infty. \tag{2.44}$$

Thus, the system (2.40)–(2.43) transforms into the following:

$$e_i\varphi_{i0_t} = \rho_i u_i, \quad \rho_i u_i \varphi_{i0_t} = \rho_i u_i^2 + c_0^2 \rho_i, \quad i = 1,2. \tag{2.45}$$

Obviously, equalities (2.45) are the Rankine-Hugoniot conditions for the shock waves with amplitudes (e_i, u_i), $i = 1,2$, which propagate over the unperturbed gas with the state $(\rho_0, u_0 = 0)$.

Now, let us consider system (2.40)–(2.43) for $\tau \to -\infty$, that is, for times after the interaction.

Assumptions (2.8) and stabilization of the convolutions imply the relations

$$\phi_i \longrightarrow \overline{\phi}_i(t), \quad R \longrightarrow \overline{r}, \quad V \longrightarrow \overline{v} \quad \text{as } \tau \longrightarrow -\infty. \tag{2.46}$$

Using the explicit formulas (2.23), it is easy to establish that

$$G_1 \longrightarrow \rho^* u^* - \rho_1 u_1 - \rho_2 u_2, \qquad G_4 \longrightarrow \rho^* u^{*2} - \rho_1 u_1^2 - \rho_2 u_2^2, \tag{2.47}$$

where

$$\rho^* = \rho_0 + e_1 + e_2 + \overline{r}, \qquad u^* = u_1 + u_2 + \overline{v}. \tag{2.48}$$

Therefore, the system (2.40)–(2.43) reduces to

$$\begin{aligned}
(\rho^* - \rho_2)\overline{\phi}_{1_t} &= \rho^* u^* - \rho_2 u_2, \\
(\rho^* u^* - \rho_2 u_2)\overline{\phi}_{1_t} &= \rho^* u^{*2} - \rho_2 u_2^2 + c_0^2(\rho^* - \rho_2), \\
(\rho^* - \rho_1)\overline{\phi}_{2_t} &= \rho^* u^* - \rho_1 u_1, \\
(\rho^* u^* - \rho_1 u_1)\overline{\phi}_{2_t} &= \rho^* u^{*2} - \rho_1 u_1^2 + c_0^2(\rho^* - \rho_1).
\end{aligned} \tag{2.49}$$

Thus, we obtain the Rankine-Hugoniot conditions for two shock waves which propagate over the backgrounds (ρ_2, u_2) and (ρ_1, u_1), respectively. Obviously, equalities (2.49) imply the standard formulas for the limiting velocities of the front motions

$$\overline{\phi}_{1_t} = u^* + c_0\sqrt{\frac{\rho_2}{\rho^*}}, \qquad \overline{\phi}_{2_t} = u^* - c_0\sqrt{\frac{\rho_1}{\rho^*}}. \tag{2.50}$$

Moreover, solving (2.49) for ρ^*, u^*, we find the expressions

$$\rho^* = \frac{\rho_1\rho_2}{\rho_0}, \qquad u^* = u_1 + u_2, \tag{2.51}$$

which coincide with the well-known solution of the Riemann problem in the situation under consideration. Thus, we can treat the system (2.40)–(2.43) as a generalization of the Rankine-Hugoniot conditions for two interacting shocks with opposite directions of motion. □

3. Investigation of the dynamical system

First of all, let us pass from (2.40)–(2.43) to a system of three autonomous equations. To do this, let us solve (2.40), (2.41) with respect to ϕ_{1_t}, ϕ_{2_t} and subtract one from the other. Since

$$\frac{d\phi_2}{dt} - \frac{d\phi_1}{dt} = \psi_{0_t}\frac{d}{d\tau}\frac{\psi_2 - \phi_1}{\varepsilon} = \psi_{0_t}\frac{d\sigma}{d\tau}, \tag{3.1}$$

we obtain the equality

$$\frac{d\sigma}{d\tau} = \frac{1}{\psi_{0_t}}\left\{\frac{G_1 + \rho_2 u_2}{e_2 + R} - \frac{G_1 + \rho_1 u_1}{e_1 + R}\right\} - L_0\left\{\frac{1}{e_2 + R} + \frac{1}{e_1 + R}\right\}. \tag{3.2}$$

Next, let us note that (2.40)–(2.43) imply the following compatibility conditions:

$$\frac{(-1)^{k+1}\psi_{0_t}L_0 + G_1 + \rho_k u_k}{e_k + R} = \frac{(-1)^{k+1}\psi_{0_t}L_0 + G_4 + \rho_k u_k^2 + c_0^2(e_k + R)}{\rho_k u_k + G_1}, \tag{3.3}$$

for $k = 1,2$. Now, solving (3.2), (3.3) with respect to the derivatives $\sigma_\tau{}'$, $R_\tau{}'$, $V_\tau{}'$, we can rewrite these equations in the standard form

$$\psi_{0_t}\frac{dU}{d\tau} = F(U), \quad U = (\sigma, R, V). \tag{3.4}$$

Obviously, assumptions (2.7) imply the following scattering-type conditions:

$$\frac{\sigma}{\tau} \longrightarrow 1, \quad R \longrightarrow 0, \quad V \longrightarrow 0 \quad \text{as } \tau \longrightarrow \infty. \tag{3.5}$$

Our aim is to establish the existence of a global solution for the problem (3.4), (3.5), and to discover the behavior of σ, R, and V for $\tau \to -\infty$. However, the explicit formulas for the right-hand side F are rather unwieldy. In order to avoid too complicated algebraic calculations, we restrict ourselves to the special case $\rho_1 = \rho_2$. It is easy to establish that this choice implies the equality $u_1 = -u_2$, and moreover, $V \equiv 0$. Thus, in the special case, we pass from the system (3.4) to the following system of two equations:

$$\begin{aligned} \psi_{0_t}\frac{d\sigma}{d\tau} &= 4u_1\frac{\alpha_2 + q_1 K_{11} R}{\alpha_3 + \alpha_4 R} \overset{\text{def}}{=} F_1, \\ \psi_{0_t}\frac{dR}{d\tau} &= 2u_1\frac{e_1 q_0 + (q_0 - \alpha_0)R - \alpha_1 R^2}{\alpha_3 + \alpha_4 R} \overset{\text{def}}{=} F_2, \end{aligned} \tag{3.6}$$

where $\psi_{0_t} = -2u_1\rho_1/e_1$, and

$$\begin{aligned}
&q_0 = \rho_0 b_{11} + e_1 b_{12}, \qquad q_1 = \frac{\rho_0\rho_1}{2e_1^2} + B_{13} - b_{11}B_{22},\\
&\alpha_0 = \frac{\rho_1^2}{2e_1} + e_1 q_1 + 2\rho_1 D_{12} + \dot{B}_{11}\left(\rho_1 N + 2K_{11}\left(q_0 - \frac{\rho_1^2}{2e_1}\right)\right),\\
&\alpha_1 = q_1(1 - 2K_{11}\dot{B}_{11}), \qquad \alpha_2 = K_{11}\left(\frac{\rho_1^2}{2e_1} - q_0\right) - \frac{\rho_1 N}{2},\\
&\alpha_3 = e_1 N - \rho_1 K_{11}, \quad \alpha_4 = N + 4K_{11}D_{12}, \quad N = K_{21} - K_{12}.
\end{aligned} \tag{3.7}$$

There are four curves that specify the behavior of the system (3.6) trajectories. Let us denote

$$\gamma_1^{\pm} = \{R = R_1^{\pm}(\sigma),\ \sigma \in \mathbb{R}^1\}, \quad R_1^{\pm} = \frac{1}{2\alpha_1}\left(q_0 - \alpha_0 \pm \sqrt{(\alpha_0 - q_0)^2 + 4\alpha_1 e_1 q_0}\right), \tag{3.8}$$

the isoclines of F_1,

$$\gamma_2 = \{R = R_2(\sigma),\ \sigma \in \mathbb{R}^1\}, \quad R_2 = -\frac{\alpha_2}{q_1 K_{11}}, \tag{3.9}$$

the isocline of F_2, and

$$\gamma_s = \{R = R_3(\sigma),\ \sigma \in \mathbb{R}^1\}, \qquad R_s = -\frac{\alpha_3}{\alpha_4}, \tag{3.10}$$

the curve of singularities.

The statement of Lemma 2.3 implies the following relations:

$$\begin{gathered}
N \longrightarrow c_2, \qquad K_{11} \longrightarrow \frac{\sigma}{2}, \qquad D_{12}, \dot{B}_{11} \longrightarrow 0, \qquad q_0 \longrightarrow \rho_1, \qquad q_1, \alpha_1 \longrightarrow \frac{\rho_0\rho_1}{2e_1^2},\\
\alpha_0 \longrightarrow \frac{\rho_1(\rho_0 + \rho_1)}{2e_1}, \qquad \alpha_2 \longrightarrow -\frac{\rho_1(\sigma(\rho_1 - 2\rho_0) + 2c_2 e_1)}{4e_1},\\
\alpha_3 = -\frac{\sigma\rho_1}{2} + e_1 c_2, \quad \alpha_4 \longrightarrow c_2 \quad \text{as } \sigma \longrightarrow -\infty.
\end{gathered} \tag{3.11}$$

Therefore, for $\sigma \to -\infty$,

$$R_1^{+} \longrightarrow \frac{e_1^2}{\rho_0}, \qquad R_1^{-} \longrightarrow -2e_1, \qquad R_2 \longrightarrow -\frac{e_1}{\rho_0}(\rho_1 - 2e_1), \qquad R_s \longrightarrow \sigma\frac{\rho_1}{2c_2}. \tag{3.12}$$

To consider the behavior of the curves γ for $\sigma \to +\infty$, we should make more precise estimates of the convolutions. Indeed, with the accuracy of Lemma 2.3, we obtain the relations $\alpha_1 \to 0$, $\alpha_4 \to 0$ as $\sigma \to +\infty$ and loose the signs of the curves. The simplest way to overcome this difficulty is to note that the limiting Rankine-Hugoniot conditions (2.45) and (2.49) do not depend on the choice of the regularization ω. Thus we can use a specific

regularization, for instance, $\omega(\eta) = (1+\tanh\eta)/2$. For such a choice, we find

$$\begin{gathered} N \longrightarrow \frac{1}{2}, \qquad K_{11} \longrightarrow -\frac{\sigma}{2(2\sigma-1)}, \qquad D_{12} \longrightarrow \frac{\sigma-1}{2\sigma-1}, \\ \dot{B}_{11} \longrightarrow -4\frac{\sigma-1}{2\sigma-1}, \qquad q_0 \longrightarrow 0, \qquad q_1 \longrightarrow \frac{\rho_0\rho_1}{2e_1^2}, \\ \alpha_0 \longrightarrow e_1 q_1, \qquad \alpha_1 \longrightarrow \frac{q_1}{4\sigma^2}, \qquad \alpha_2 \longrightarrow -\frac{\rho_1(\rho_1+2e_1)}{8e_1}, \\ \alpha_3 \longrightarrow \frac{1}{4}(\rho_1+2e_1), \qquad \alpha_4 \longrightarrow \frac{1}{8\sigma^2} \quad \text{as } \sigma \longrightarrow \infty. \end{gathered} \tag{3.13}$$

Thus,

$$R_1^+ \longrightarrow 0, \qquad R_1^- \longrightarrow -4\sigma^2 e_1, \qquad R_2 \longrightarrow -\frac{e_1}{\rho_0}(\rho_1+2e_1), \qquad R_s \longrightarrow -2\sigma^2(\rho_1+2e_1). \tag{3.14}$$

Numerical simulations show that the curves $\gamma_1^{\pm}$, γ_2, and γ_s do not intersect for finite σ. This implies that the system (3.6) does not have any critical points. At the same time, the parts of the curve γ_1^+ for $\sigma \to \pm\infty$ play the role of a saddle point and an attractive node, respectively. Indeed, let $\sigma \to -\infty$ and $R = e_1^2/\rho_0 + r$. Then, formulas (3.11) imply the following linearization of system (3.6):

$$\frac{d\sigma}{d\tau} = \frac{\rho_0}{\rho_1}\left(1+\frac{r}{e_1}\right), \quad \frac{dr}{d\tau} = -\frac{\rho_0+\rho_1}{\rho_1}\frac{r}{\sigma} \quad \text{for } \sigma \ll -1. \tag{3.15}$$

The solution $r = r(\sigma)$ has the form

$$e_1 \ln\left(|r||\sigma|^{\beta}\right) + r = \text{const}, \quad \beta = \frac{(\rho_0+\rho_1)}{\rho_0} > 0. \tag{3.16}$$

So, in the leading term

$$\sigma = \frac{\rho_0}{\rho_1}\tau, \qquad |r| = |c\sigma|^{-\beta}, \quad c = \text{const}, \tag{3.17}$$

and it is clear that the line $R = e_1^2/\rho_0$, $\sigma = \rho_0\tau/\rho_1$ for $\tau \to -\infty$ is similar to an attractive node.

Let $\sigma \to \infty$. Formulas (3.13) imply the following linearization of system (3.6):

$$\frac{d\sigma}{d\tau} = \left(1+\frac{\beta_1}{e_1}R\right), \quad \frac{dR}{d\tau} = 2\beta_1 R, \quad \beta_1 = \frac{\rho_0}{\rho_1+2e_1} > 0. \tag{3.18}$$

So, the trajectories go out a neighborhood of the line $\sigma = \tau$, $R = 0$ with an exponential velocity, whereas $\sigma = \tau$ and $R = 0$ satisfy system (3.6) for $\tau \to +\infty$.

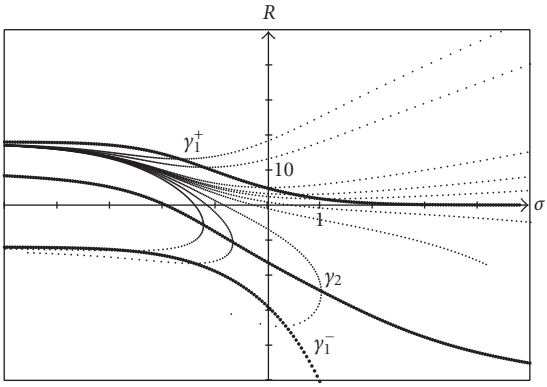

Figure 3.1. The phase portrait for system (3.6).

Therefore, taking into account the signs of the right-hand sides F_1 and F_2, we obtain the phase portrait shown in Figure 3.1.

Now, it is evident the existence of a separatrix which goes from $\overline{\sigma}_- = \rho_0\tau/\rho_1, \overline{R}_- = e_1^2/\rho_0$ (for $\tau \to -\infty$) to $\overline{\sigma}_+ = \tau, \overline{R}_+ = 0$ (for $\tau \to \infty$) lying under the isocline γ_1^+. This implies the following statement.

Theorem 3.1. *There exists the separatrix for system (3.6) which coincides with the "points" $\overline{\sigma}_-, \overline{R}_-$ and $\overline{\sigma}_+, \overline{R}_+$. This separatrix can be specified by the scattering-type conditions*

$$\frac{\sigma}{\tau} \longrightarrow 1, \quad R \longrightarrow 0 \quad \textit{for } \tau \longrightarrow +\infty. \tag{3.19}$$

It seems that a similar statement is true in the general case of arbitrary ρ_1 and ρ_2. In any case, taking into account the limiting values of the convolutions, after cumbersome calculations, it is possible to find the limiting values of the functions

$$\begin{gathered} \overline{\sigma}_- = \tau\frac{\rho_0}{\sqrt{\rho_1\rho_2}}, \quad \overline{R}_- = \frac{e_1e_2}{\rho_0}, \quad \overline{V}_+ = 0 \quad \text{as } \tau \longrightarrow -\infty, \\ \overline{\sigma}_+ = \tau, \quad \overline{R}_+ = 0, \quad \overline{V}_- = 0 \quad \text{as } \tau \longrightarrow +\infty. \end{gathered} \tag{3.20}$$

4. Calculations of the phase corrections

After solving problem (3.4), (3.5), we can find the phase corrections φ_{i1}. Using again the formulas (2.25), we can rewrite (2.40), (2.41) in the following form:

$$\psi_{0_t}\frac{d}{d\tau}(\tau\varphi_{i1}) = \frac{G_1 + \rho_i u_i + (-1)^{i+1}\psi_{0_t}L_0}{e_i + R} - \varphi_{i0_t} \stackrel{\text{def}}{=} f_i(U), \tag{4.1}$$

where $i = 1,2$, and $U = (\sigma, R, V)$.

Now, we readily derive the desired formulas as follows:

$$\varphi_{i1}(\tau) = \frac{1}{\psi_{0_t}}\frac{1}{\tau}\int_0^\tau f_i(U)\,d\tau', \quad i = 1,2. \tag{4.2}$$

Smoothness of U implies the boundedness of f_i at the point $\tau = 0$. Thus, φ_{i1} are bounded at this point. Next, since

$$G_1, L_0, R \longrightarrow 0 \quad \text{as } \tau \longrightarrow +\infty, \qquad \varphi_{i0_t} = \frac{\rho_i u_i}{e_i}, \quad i = 1,2, \tag{4.3}$$

the functions f_i vanish sufficiently rapidly as $\tau \to \infty$. This guarantees the convergence of the integral in the right-hand sides of (4.2) as $\tau \to \infty$. Hence,

$$\varphi_{i1}(\tau) \longrightarrow 0 \quad \text{as } \tau \longrightarrow +\infty, \tag{4.4}$$

which confirms the first *a priori* assumption in (2.7). Furthermore,

$$f_i(\tau) = \frac{\rho^* u^* - \rho_{\bar{i}} u_{\bar{i}}}{\rho^* - \rho_{\bar{i}}} - \varphi_{i0_t} + \mathcal{O}\left(\tau^2 e^{\gamma\tau}\right) \quad \text{as } \tau \longrightarrow +\infty, \tag{4.5}$$

where we use the notation (2.48) and $\overline{r} = \overline{R}_-$, $\overline{v} = \overline{V}_-$ since $B_{11} \to 1$, $\bar{i} = 2$ for $i = 1$, and $\bar{i} = 1$ for $i = 2$, and γ is a number defined by the choice of the regularization ω.

Thus, the integral diverges. By using L'Hospital rule, it is easy to find the limiting value of φ_{i1} as follows:

$$\overline{\varphi}_{i1} = \frac{1}{\psi_{0_t}} \left\{ \frac{\rho^* u^* - \rho_{\bar{i}} u_{\bar{i}}}{\rho^* - \rho_{\bar{i}}} - \varphi_{i0_t} \right\}, \quad i = 1,2. \tag{4.6}$$

This satisfies the first *a priori* assumption in (2.13). Moreover, formulas (4.6) allow to calculate the limiting phases $\overline{\phi}_i = \lim_{\tau \to -\infty} \phi_i$. Indeed, using the Taylor expansion at the time instant $t = t^*$ and taking into account equality (2.30), we derive

$$\begin{aligned} \overline{\phi}_i &\stackrel{\text{def}}{=} \varphi_{i0} + \psi_0(t)\overline{\varphi}_{i1} = x^* + \left(\varphi_{i0_t} + \psi_{0_t}\overline{\varphi}_{i1}\right)(t - t^*) \\ &= x^* + \frac{\rho^* u^* - \rho_{\bar{i}} u_{\bar{i}}}{\rho^* - \rho_{\bar{i}}}(t - t^*), \quad i = 1,2. \end{aligned} \tag{4.7}$$

Obviously, these phases satisfy the first Rankine-Hugoniot conditions (2.49).

5. Conclusion

We considered the gas dynamics system as the most important example of hyperbolic systems of conservation laws. It seems obvious that the above described method of construction of uniform in time asymptotics can be applied to general strictly hyperbolic systems of conservation laws. Of course, the direct application of the method is possible only when interaction of shock waves results in the appearance of new shock waves. It is clear also that for systems of n equations, $n > 2$, the anzatz has to include all the possible stable shock waves. In such a case, sufficiently simple algebraic calculations allow to obtain a result similar to Theorem 2.6. However, the investigation of the corresponding dynamical system is not so easy in the general case. This and the appearance of centered rarefaction, contact discontinuities, and vacuum state are the issues for future investigation.

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