GENERALIZATIONS OF HOPFIAN AND CO-HOPFIAN MODULES

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Let R be a ring and M a left R -module. M which satisfies DCC on essential submodules is GCH, and M which satisfies ACC on small submodules is WH. If $M[X]$ is GCH $R[X]$ -module, then M is GCH R -module. Examples show that a GCH module need not be co-Hopfian and a WH module need not be Hopfian.

1. Introduction and preliminaries

In this paper, all rings are associative with identity and all modules are unital left modules unless otherwise specified.

Let R be a ring and M a module. $N \leq M$ will mean N is a submodule of M . A submodule E of M is called *essential* in M (notation $E \leq_e M$) if $E \cap A \neq 0$ for any nonzero submodule A of M . Dually, a submodule S of M is called *small* in M (notation $S \ll M$) if $M \neq S + T$ for any proper submodule T of M . M is said to be Hopfian (co-Hopfian) in case any surjective (injective) homomorphism is automatically an isomorphism. M is called generalized Hopfian (GH) if any of its surjective endomorphisms has a small kernel. M is called weakly co-Hopfian if any injective endomorphism of M is essential. In this paper, we introduce the concepts of GCH modules and WH modules. It is shown that (1) a module M which satisfies DCC on essential submodules is GCH and a module M which satisfies ACC on small submodules is WH; (2) if $M[X]$ is GCH $R[X]$ -module, then M is GCH R -module. Examples show that a GCH module need not be co-Hopfian and a WH module need not be Hopfian. The notions which are not explained here will be found in [4].

LEMMA 1.1 (see [6, 17.3(2)]). *Let K , L , and M be modules. Then two monomorphisms $f : K \rightarrow L$, $g : L \rightarrow M$ are essential if and only if gf is essential.*

LEMMA 1.2 (see [6, 19.3(1)]). *Let K , L , and M be modules. Then two epimorphisms $f : K \rightarrow L$, $g : L \rightarrow M$ are small if and only if gf is small.*

LEMMA 1.3 (see [2, Proposition 5.20]). *Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \leq_e M_1 \oplus M_2$ if and only if $K_1 \leq_e M_1$ and $K_2 \leq_e M_2$.*

2. Modules whose essential injective endomorphisms are isomorphic

Let M be a module. M is said to be a generalized co-Hopfian module (GCH module) if any essential injective endomorphism of M is isomorphic.

PROPOSITION 2.1. *Suppose that M satisfies the condition that N is GCH for every proper essential submodule. Then M itself is GCH.*

Proof. Suppose on the contrary that M is not GCH, then there exists an essential injective homomorphism $g : M \rightarrow M$ which is not an isomorphism. Let $N = \text{Im} g$. Then $N \neq M$ and g induces an isomorphism $\bar{g} : M \rightarrow N$. Then $\bar{g}|_N : N \rightarrow N$ is an essential injective morphism which is not an isomorphism. This is a contradiction since N is a GCH module. $\square$

PROPOSITION 2.2. *Let M be a GCH module and K be a direct summand of M . Then K is GCH.*

Proof. Let $M = K \oplus K'$ and let $f : K \rightarrow K$ be an essential injection of K . f induces an injective endomorphism of M , that is, $f \oplus 1_{K'} : M \rightarrow M$ with $(f \oplus 1_{K'})(k + k') = f(k) + k'$. Since $\text{Im}(f \oplus 1_{K'}) = \text{Im} f \oplus K' \leq_e K \oplus K' = M$ by Lemma 1.3, $f \oplus 1_{K'}$ is essential. Since M is GCH, $f \oplus 1_{K'}$ is isomorphic, and hence f is an isomorphism, as required. $\square$

PROPOSITION 2.3. *Let $M = M_1 \oplus M_2$ and let M_1, M_2 be invariant submodules under any injection of M . Then M is GCH if and only if M_1, M_2 are GCH.*

Proof. “If” part is clear by Proposition 2.2.

“Only if” part. Let $f : M \rightarrow M$ be an essential injection. Then $f|_{M_i} : M_i \rightarrow M_i$ is essential injection. By assumption, $f|_{M_i}$ is isomorphic. Thus, $f(M) = f(M_1 + M_2) = f(M_1) + f(M_2) = M_1 + M_2 = M$, and hence f is surjective, as desired. $\square$

It is easy to know that a module M is co-Hopfian if and only if M is both a weakly co-Hopfian module and a GCH module. In [2], Haghany and Vedadi proved that if DCC holds on nonessential submodules of M , then M is weakly co-Hopfian. We also know that Artinian modules are co-Hopfian modules. Thus it is natural that we consider a module with DCC on essential submodules.

THEOREM 2.4. *Let M be a module with DCC on essential submodules. Then M is GCH.*

Proof. Let $f : M \rightarrow M$ be an essential injection of M . Then, $\cdots \leq \text{Im} f^2 \leq \text{Im} f$ is a descending chain on essential submodules of M by Lemma 1.1. Since M satisfies DCC on essential submodules, there exists n such that $\text{Im} f^n = \text{Im} f^{n+1}$. For any $m \in M$, since $f^n(m) \in \text{Im} f^{n+1}$, there exists $m_1 \in M$ such that $f^n(m) = f^{n+1}(m_1)$. Since f is injective, $m = f(m_1)$. Thus f is surjective, as required. $\square$

LEMMA 2.5. *A quasicontinuous module M is continuous if and only if it is a GCH module.*

A continuous module is a quasicontinuous module, but a quasicontinuous module need not be a continuous module. Thus the following result gives a sufficient condition such that a quasicontinuous module is a continuous module.

COROLLARY 2.6. *Let M be a module with DCC on essential submodules. Then M is quasicontinuous if and only if M is continuous.*

Proof. It follows from Theorem 2.4 and Lemma 2.5. $\square$

PROPOSITION 2.7. *Let P be a property of modules preserved under isomorphism. If a module M has the property P and satisfies DCC on essential submodules with property P , then M is GCH.*

Proof. Suppose that M is not GCH. Then there exists a proper essential submodule N_1 of M with $N_1 \simeq M$. Thus N_1 is not GCH and enjoys P . We have a proper essential submodule N_2 of N_1 with $N_2 \simeq N_1$. Clearly, N_2 is not GCH and satisfies P . Repeating, we obtain a strictly descending chain $N_1 > N_2 > \cdots$ of proper essential submodules each with property P , a contradiction. $\square$

COROLLARY 2.8. *If M has DCC on its non-GCH submodules, then M is GCH.*

Proof. Suppose not, and let P be the property of being non-GCH. Applying Proposition 2.7, we arrive at a contradiction. Thus M must be GCH. $\square$

Example 2.9. A semisimple module M is weakly co-Hopfian if and only if any homogeneous component of M is finitely generated (see [2, Corollary 1.12]). Thus a semisimple module need not be a weakly co-Hopfian module, and hence it is not a co-Hopfian module. However, any semisimple module is GCH.

Let M be a module. The elements of $M[X]$ are formal sums of the form $a_0 + a_1X + \cdots + a_kX^k$ with k an integer greater than or equal to 0 and $a_i \in M$. We denote this sum by $\sum_{i=0}^k a_iX^i$ (a_0X^0 is to be understood as the element $a_0 \in M$). Addition is defined by adding the corresponding coefficients. The $R[X]$ -module structure is given by

$$\left(\sum_{i=0}^k \lambda_i X^i\right) \cdot \left(\sum_{j=0}^z a_j X^j\right) = \sum_{\mu=0}^{k+z} c_\mu X^\mu, \quad (2.1)$$

where $c_\mu = \sum_{i+j=\mu} \lambda_i a_j$, for any $\lambda_i \in R, a_j \in M$.

LEMMA 2.10 (see [5, Lemma 1.7]). *Let $N \leq M$. Then N is essential in M as an R -module if and only if $N[X]$ is essential in $M[X]$ as an $R[X]$ -module.*

THEOREM 2.11. *Let M be a module. If $M[X]$ is GCH $R[X]$ -module, then M is GCH R -module.*

Proof. Let $f : M \rightarrow M$ be an essential injective endomorphism of M . Then $f[X] : M[X] \rightarrow M[X]$ with $f[X](\sum m_i X^i) = \sum f(m_i X^i)$ is an injective endomorphism of $M[X]$. Next, we will show that $\text{Im} f[X] \leq_e M[X]$. It is easy to verify that $\text{Im} f[X] = (\text{Im} f)[X]$. Since $\text{Im} f \leq_e M$, $\text{Im} f[X] \leq_e M[X]$ by Lemma 2.10. Thus $f[X]$ is isomorphic by assumption. The surjectivity of f follows by that of $f[X]$. This completes the proof. $\square$

3. Modules whose small surjective endomorphisms are isomorphic

Let M be a module. M is said to be a weakly Hopfian module (WH module) if any small surjection of M is isomorphic.

THEOREM 3.1. *Let M be a module. If M satisfies the condition that M/N is WH for every small submodule, $0 \neq N \leq M$. Then M itself is WH.*

Proof. Suppose on the contrary that M is not WH. Then there exists a small surjection f of M which is not an isomorphism. Let $N = \text{Ker } f$. Then $0 \neq N$ and f induces an isomorphism $\tilde{f} : M/N \rightarrow M$. If $\eta : M \rightarrow M/N$ denotes the canonical quotient morphism, then $\eta\tilde{f} : M/N \rightarrow M/N$ is a small surjection which is not an isomorphism. This is a contradiction. $\square$

PROPOSITION 3.2. *Let M be a torsion-free module. Then M is WH.*

Proof. Let $f : M \rightarrow M$ be any small surjection. Let $0 \neq x \in M \setminus \text{Ker } f$, then $-x \in M \setminus \text{Ker } f$. For any $r \in R$, $f(xr) = f(x)r$. Since $x \notin \text{Ker } f$, $f(x) \neq 0$. By assumption, $f(x)r \neq 0$ and hence $xr \in M \setminus \text{Ker } f$. Thus, $(M \setminus \text{Ker } f) \cup \{0\}$ is a submodule of M . Since $(M \setminus \text{Ker } f) \cup \{0\} + \text{Ker } f = M$ and f is a small surjection, $(M \setminus \text{Ker } f) \cup \{0\} = M$ and so $\text{Ker } f = 0$, as required. $\square$

PROPOSITION 3.3. *Let M be WH and K is a direct summand of M . Then K is WH.*

Proof. Since K is a direct summand of M , there exists $K' \leq M$ such that $M = K \oplus K'$. Let $f : K \rightarrow K$ be a surjection with $\text{Ker } f \ll K$, then f induces a small surjection $f \oplus 1_{K'} : M \rightarrow M$ with $(f \oplus 1_{K'})(k + k') = f(k) + k'$ by Lemma 1.2. Since M is WH, $f \oplus 1_{K'}$ is isomorphic and hence f is an isomorphism, as desired. $\square$

PROPOSITION 3.4. *Let $M = M_1 \oplus M_2$ and let M_1, M_2 be invariant submodules under any surjection of M . Then M is WH if and only if M_1, M_2 are WH.*

Proof. “If” part is clear by Proposition 3.3.

“Only if” part. Let $f : M \rightarrow M$ be a small epimorphism, then $f|_{M_i} : M_i \rightarrow M_i$ is a small surjection. By assumption, $f|_{M_i}$ is isomorphic. Let $f(m_1 + m_2) = 0$, then $f(m_1) + f(m_2) = 0$ and so $m_1 = m_2 = 0$. Thus f is injective. This completes the proof. $\square$

We know that Noetherian modules are Hopfian modules. It is also easy to know that a module M is Hopfian if and only if M is both a generalized Hopfian module and a WH module. In [1], Ghorbani and Haghany proved that if ACC holds on nonsmall submodules of M , then M is generalized Hopfian. Thus it is natural that we consider the following result.

THEOREM 3.5. *Let M be a module with ACC on small submodules. Then M is WH.*

Proof. Let $f : M \rightarrow M$ be a small surjection of M . Then $\text{Ker } f \leq \text{Ker } f^2 \leq \text{Ker } f^3 \leq \dots$ is an ascending chain on small submodule of M by Lemma 1.2. Since M satisfies ACC on small submodules, there exists a positive number n such that $\text{Ker } f^n = \text{Ker } f^{n+1}$. Next, we will prove that $\text{Ker } f = 0$. Let $x \in \text{Ker } f$, then $f(x) = 0$. Since f is surjective, there exists $y_1 \in M$ such that $f(y_1) = x$. Since f is surjective, there exists $y_2 \in M$ such that $f(y_2) = y_1$. Repeating the process, we obtain $y_{n-1} \in M$ with $f(y_n) = y_{n-1}$. Thus $f(y_1) = f^2(y_2) = \dots = f^n(y_n) = x$. Since $x \in \text{Ker } f$, $f(x) = f(f^n(y_n)) = 0$, that is, $f^{n+1}(y_n) = 0$. So $y_n \in \text{Ker } f^{n+1} = \text{Ker } f^n$. Consequently, $f^n(y_n) = 0$ and hence $x = 0$, as required. $\square$

LEMMA 3.6. *A quasidiscrete module M is discrete if and only if M is a WH module.*

Proof. The necessity of the condition is obvious. Conversely, let $f : M \rightarrow N$ be an epimorphism, with kernel K , onto the direct summand N of M . As M is quasidiscrete, there is

a decomposition $M = A \oplus B$ with $A \leq K$ and $B \cap K \ll B$. Now, $N \simeq M/K = K + B/K \simeq B/(B \cap K)$. Theorem 4.24 in [4] yields $N \simeq B$, let $g : N \rightarrow B$ be an isomorphism. Then,

$$M = A \oplus B \longrightarrow A \oplus N \longrightarrow A \oplus B = M \quad (3.1)$$

is an epimorphism with the small kernel $B \cap K$, where $1 \oplus f|_B : M \rightarrow A \oplus N$ and $1 \oplus g : A \oplus N \rightarrow A \oplus B = M$. By assumption, it is an isomorphism, that is, $B \cap K = 0$. Then $K = A$ is a direct summand of M and f splits. This completes the proof. $\square$

A discrete module is a quasidiscrete module, but a quasidiscrete module need not be a discrete module. Thus the following result gives a sufficient condition such that a quasidiscrete module is a discrete module.

COROLLARY 3.7. *Let M be a module with ACC on small submodules. Then M is quasidiscrete if and only if M is discrete.*

Proof. It follows from Theorem 3.5 and Lemma 3.6. $\square$

PROPOSITION 3.8. *Let P be a property of modules preserved under isomorphism. If a module M has the property P and satisfies ACC on nonzero small submodules N such that M/N has the property P , then M is WH.*

Proof. Suppose that M is not WH. Then there exists a submodule N_1 with $N_1 \neq 0$ and $M/N_1 \simeq M$. Thus M/N_1 is not WH but satisfies P . Hence, there exists a submodule $N_2 \leq N_1$ with $N_1 \neq N_2$ and $M/N_2 \simeq M/N_1$. So we get $0 < N_1 < N_2$ with $M/N_i \simeq M$ for $i = 1, 2$. Repeating the process yields a chain of submodules of the type that contradicts our hypothesis. $\square$

COROLLARY 3.9. *If M is GCH and satisfies ACC on small submodules N such that M/N is GCH, then M is WH.*

Proof. It follows by Proposition 3.8. $\square$

Remark 3.10. Following Proposition 2.7, if a module M is WH and has DCC on essential WH submodules, then M is GCH.

COROLLARY 3.11. *If M has ACC on its nonzero small submodules N such that M/N is not WH, then M is WH.*

Proof. Let P be the property of being WH, and suppose that M is not WH. By Proposition 3.8, M must be WH. This contradiction proves that M is WH. $\square$

Example 3.12. Let M be a torsion-free Abelian group of finite rank such that $\text{End}(M)$ is a principal ideal domain and let $B = \oplus_n M$, where n is an integer. Then the kernel of endomorphism of B is a direct summand which, again, is a direct sum of copies of M . It is clear to see that M is WH, but it need not be Hopfian.

Example 3.13. If R is a semisimple Artinian ring, then a module M is Hopfian if and only if M has finite length (see [3]). Note that a vector space V over a field F is Hopfian if and only if it is finite dimensional. Thus an infinite-dimensional vector space is WH, but it is not Hopfian.

Let R be a ring and $\mathcal{C}$ a class of R -modules, we will say that $\mathcal{C}$ is *socle fine* whenever for any $M, N \in \mathcal{C}$, we have $\text{Soc}(M) \simeq \text{Soc}(N)$ if and only if $M \simeq N$.

THEOREM 3.14. *For any ring R , the following statements are equivalent.*

- (1) R is semisimple.
- (2) The class of all GCH modules is socle fine.
- (3) The class of all WH modules is socle fine.

Proof. It is easy to prove that quasiprojective (quasiinjective) and semisimple modules are GCH(WH) modules. If R is semisimple, then the class of all R -modules is socle fine and hence (1) implies (2) and (3).

(2) $\Rightarrow$ (1) Since R and $\text{Soc}(R)$ are quasiprojective and $\text{Soc}(R) = \text{Soc}(\text{Soc}(R))$, we have $R \simeq \text{Soc}(R)$ by (2). Thus R is semisimple.

(3) $\Rightarrow$ (1) Since $\text{Soc}(E(R)) = \text{Soc}(R) = \text{Soc}(\text{Soc}(R))$ and both of $E(R)$ and $\text{Soc}(R)$ are quasiinjective, we have $E(R) \simeq \text{Soc}(R)$ is semisimple, and so is R . $\square$

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