ON WEAK CONVERGENCE OF ITERATES IN QUANTUM L_p -SPACES ($p \geq 1$)

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Equivalent conditions are obtained for weak convergence of iterates of positive contractions in the L_1 -spaces for general von Neumann algebra and general JBW algebras, as well as for Segal-Dixmier L_p -spaces ($1 \leq p < \infty$) affiliated to semifinite von Neumann algebras and semifinite JBW algebras without direct summands of type I_2 .

1. Introduction and preliminaries

This paper is devoted to a presentation of some results concerning ergodic-type properties of weak convergence of iterates of operators acting in L_1 -space for general von Neumann algebras and JBW algebras, as well as Segal-Dixmier L_p -spaces ($1 \leq p < \infty$) of operators affiliated with semifinite von Neumann algebras and semifinite JBW algebras.

The first results in the field of noncommutative ergodic theory were obtained independently by Sinaĭ and Ansĕlevič [21] and Lance [15]. Developments of the subject are reflected in the monographs of Jajte [13] and Krengel [14] (see also [8, 9, 10, 18]).

We will use facts and the terminology from the general theory of von Neumann algebras (see [5, 7, 17, 19, 22]), the general theory of Jordan and real operator algebras (see [2, 3, 11, 16]), and the theory of noncommutative integration (see [20, 23, 24]).

Let M be a von Neumann algebra, acting on a separable Hilbert space H , M_* is a predual space of M , which always exists according to the Sakai theorem [19]. It is well known that M_* could be identified with L_1 -space for M .

Spaces L_1 and L_2 of the operators affiliated with the semifinite von Neumann algebra M with semifinite faithful trace τ were introduced by Segal (see [20]). This result was extended to L_p -space of operators affiliated with von Neumann algebras M , τ , and integrated with p th power by Dixmier (see [6]). For an alternative exposition of building L_p based on Grothendieck's idea of using rearrangements of functions, see also [24]. The theory of L_p -spaces was extended further to the von Neumann algebras with faithful normal weight ρ . However, these spaces lack some of the properties, for example, in general, these spaces do not intersect.

Recall some standard terminology (see [8, 9, 10, 14]).

Definition 1.1. A linear mapping T from M_* in itself is called a *contraction* if its norm is not greater than one.

Definition 1.2. A contraction T is said to be *positive* if

$$TM_{*+} \subset M_{*+}. \quad (1.1)$$

We will consider the two topologies on the space M_* : the *weak topology*, or the $\sigma(M_*, M)$ topology, and the *strong topology* of the M_* -space norm convergence.

Definition 1.3. A matrix $(a_{n,i})$, $i, n = 1, 2, \dots$, of real numbers is called *uniformly regular* if

$$\sup_n \sum_{i=1}^{\infty} |a_{n,i}| \leq C < \infty; \quad \lim_{n \rightarrow \infty} \sup_i |a_{n,i}| = 0; \quad \lim_{n \rightarrow \infty} \sum_i a_{n,i} = 1. \quad (1.2)$$

2. Main result: the case of quantum L_1 -spaces

2.1. The case of noncommutative L_1 -spaces. The following theorem is valid.

THEOREM 2.1. *The following conditions for a positive contraction T in the predual space of a complex von Neumann algebras M are equivalent.*

- (i) *The sequence $\{T^i\}_{i=1,2,\dots}$ converges weakly.*
- (ii) *For each strictly increasing sequence of natural numbers $\{k_i\}_{i=1,2,\dots}$,*

$$n^{-1} \sum_{i < n} T^{k_i} \quad (2.1)$$

converges strongly.

- (iii) *For any uniformly regular matrix $(a_{n,i})$, the sequence $\{A_n(T)\}_{n=1,2,\dots}$,*

$$A_n(T) = \sum_i a_{n,i} T^i, \quad (2.2)$$

converges strongly.

Proof of Theorem 2.1. We first prove the following lemma.

LEMMA 2.2. *Let there exist a uniformly regular matrix $(a_{n,i})$ such that for each strictly increasing sequence $\{k_i\}_{i=1,2,\dots}$ of natural numbers,*

$$B_n = \sum_i a_{n,i} T^{k_i} \quad (2.3)$$

converges strongly. Then the sequence $\{T^i\}_{i=1,2,\dots}$ converges weakly.

Proof. Let $(a_{n,i})$ be a matrix with the aforementioned properties. Then the limit B_n is not dependant upon the choice of the sequence $\{k_i\}_{i=1,2,\dots}$. In fact, let $\{k_i\}_{i=1,2,\dots}$ and $\{l_i\}_{i=1,2,\dots}$ be the sequences for which the limits B_n are different. This means that for some $x \in M_*$,

$$\sum_i a_{n,i} T^{k_i} x \rightarrow x_1, \quad \sum_i a_{n,i} T^{l_i} x \rightarrow x_2, \quad (2.4)$$

for $n \rightarrow \infty$. For a matrix $(a_{n,i})$, we build increasing sequences $\{i_j\}_{j=1,2,\dots}$ and $\{n_j\}_{j=1,2,\dots}$, such that

$$\lim_{j \rightarrow \infty} \left(\sum_{i < i_{j-1}} |a_{n_j,i}| + \sum_{i > i_j} |a_{n_j,i}| \right) = 0. \quad (2.5)$$

Let

$$m_i = k_i \quad \text{for } i \in [i_{2j-1}, i_{2j}), \quad m_i = l_i \quad \text{for } i \in [i_{2j}, i_{2j+1}), \quad j = 1, 2, \dots \quad (2.6)$$

Then

$$\lim_j \left\| \sum_i a_{n_{2j+1},i} T^{m_i} x - x_1 \right\| = 0; \quad \lim_j \left\| \sum_i a_{n_{2j},i} T^{m_i} x - x_2 \right\| = 0, \quad (2.7)$$

which contradict (2.3), and therefore $x_1 = x_2$. Let now $y \in M$ such that

$$(T^n x - x_1, y) \rightarrow 0, \quad (2.8)$$

when $n \rightarrow \infty$. We choose a subsequence $\{k_i\}$ such that

$$(T^{k_i} x - x_1, y) \rightarrow \gamma \neq 0, \quad (2.9)$$

where γ is a real number. Then, from the uniform regularity of the matrix $(a_{n,i})$, it follows that

$$\lim_n \left(\sum_i a_{n,i} T^{k_i} x - x_1, y \right) = \gamma, \quad (2.10)$$

which contradicts the choice of the matrix $(a_{n,i})$. □

The implication (iii) $\Rightarrow$ (ii) is trivial because the matrix $(a_{n,i})$,

$$a_{n,i} = \frac{1}{n} \sum_{i < n} \delta_{j,k_i}, \quad (2.11)$$

is uniformly regular. Applying Lemma 2.2 to the matrix

$$a_{n,i} = \frac{1}{n}, \quad (2.12)$$

$i \leq n$ and $a_{n,i} = 0$ for $i > n$, we get the implication (ii) $\Rightarrow$ (i).

To prove the implication (i) $\Rightarrow$ (iii), we would need the following lemma.

LEMMA 2.3. *Let Q be a contraction in the Hilbert space H . Then the weak convergence of $Q^n x$ in H , where $x \in H$, implies the strong convergence of*

$$\sum_i a_{n,i} Q^i x \quad (2.13)$$

for any uniformly regular matrix $(a_{n,i})$.

Proof. If the weak limit $Q^n x$ exists and is equal to x_1 , then

$$Qx_1 = Q\left(\lim_{n \rightarrow \infty} Q^n x\right) = x_1, \quad (2.14)$$

where the limit is considered in the weak topology, that is, x_1 is Q -invariant. Replacing x on $x - x_1$ (if necessary), we may suppose that $Q^n x$ converges weakly to $\mathbf{0}$, and hence

$$(Q^n x, x) \longrightarrow 0. \quad (2.15)$$

We are going to show that

$$\sum_n a_{i,n} Q^n x \xrightarrow{\|\cdot\|} \mathbf{0}, \quad (2.16)$$

where $(a_{i,n})$ is uniformly regular matrix. One can see that

$$\left\| \sum_i a_{N,i} Q^i x \right\|^2 \leq \sum_i \sum_j a_{N,i} a_{N,j} (Q^i x, Q^j x) \leq \sum_i \sum_j |a_{N,i} a_{N,j} (Q^i x, Q^j x)|. \quad (2.17)$$

We fix $\varepsilon > 0$. Because Q is a contraction, the limit $\|Q^n x\|$ does exist. Now, we can find $K > 0$, such that for $k > K$ and $j \geq 0$,

$$\|Q^k x\| - \|Q^{k+j} x\| \leq \varepsilon^2, \quad |(Q^k x, x)| \leq \varepsilon. \quad (2.18)$$

Then,

$$\begin{aligned} & |(Q^k x, x) - (Q^{k+j} x, Q^j x)| \\ &= |(Q^k x, x) - (Q^{*j} Q^{k+j} x, x)| \\ &\leq \|Q^k x - Q^{*j} Q^{k+j} x\| \cdot \|x\| = (\|Q^k x - Q^{*j} Q^{k+j} x\|^2)^{1/2} \cdot \|x\| \\ &= (\|Q^k x\|^2 - 2\|Q^{k+j} x\|^2 + \|Q^{*j} Q^{k+j} x\|^2)^{1/2} \cdot \|x\| \\ &\leq (\|Q^k x\|^2 - \|Q^{k+j} x\|^2) \cdot \|x\| \leq \varepsilon \cdot \|x\|, \end{aligned} \quad (2.19)$$

and therefore

$$|(Q^{k+j} x, Q^j x)| \leq \varepsilon \cdot (1 + \|x\|) \quad (2.20)$$

for all $k > K$ and $j \geq 0$, or for $|i - j| \geq k$, the inequality

$$|(Q^i x, Q^j x)| \leq \varepsilon \cdot (1 + \|x\|) \quad (2.21)$$

is valid. We will fix $\eta > 0$, and let N be a natural number such that

$$\max_i |a_{n,i}| < \eta, \quad (2.22)$$

for $n \geq N$. Then the expression (1) for $n \geq N$ could be estimated in the following way:

$$\begin{aligned}
 & \sum_i \sum_j |a_{N,i} a_{N,j} (Q^i x, Q^j x)| \\
 &= \sum_{|i-j| \leq k} |a_{n,i} a_{n,j} (Q^i x, Q^j x)| + \sum_{|i-j| > k} |a_{n,i} a_{n,j} (Q^i x, Q^j x)| \\
 &\leq \sum_i |a_{n,i}| \cdot \eta \cdot \|x\|^2 \cdot (2k-1) + \sum_i \sum_j |a_{n,i} a_{n,j}| \cdot \varepsilon \cdot (1 + \|x\|) \\
 &\leq C \cdot \eta \cdot \|x\|^2 \cdot (2k-1) + C^2 \cdot \varepsilon \cdot (1 + \|x\|).
 \end{aligned} \tag{2.23}$$

From the arbitrariness of the values of ε and η , it follows that the strong convergence is present and the lemma is proven. $\square$

We prove the implication (i) $\Rightarrow$ (iii). Let $x \in M_{*+}$ and the sequence $\{T^i x\}_{i=1,2,\dots}$ converges weakly. Without the loss of generality, we can consider $\|x\| \leq 1$, and let

$$\bar{x} = \lim_{n \rightarrow \infty} T^n x, \tag{2.24}$$

where the limit is understood in the weak sense. We consider

$$y = \sum_{n=0}^{\infty} 2^{-n} T^n x. \tag{2.25}$$

The series that defines y is convergent in the norm of the space M_* . From the positivity of x and the properties of the operator T , it follows that

$$Ty \leq 2y, \tag{2.26}$$

and, therefore, for all $k = 1, 2, \dots$,

$$s(T^k y) \leq s(y), \tag{2.27}$$

where we denote by $s(z)$ the support of the normal functional z .

LEMMA 2.4. Let $u \in M_{*+}$ and $s(u) \leq s(y)$. Then $s(\bar{u}) \leq s(\bar{x})$, where

$$\bar{u} = \lim_{n \rightarrow \infty} T^n u. \tag{2.28}$$

Proof. In fact, We fix $\varepsilon > 0$. From the density of the set

$$\mathfrak{L}_y = \{w \in M_{*+}, w \leq \lambda y, \text{ for some } \lambda > 0\} \tag{2.29}$$

in the set

$$\mathfrak{S} = \{w \in M_{*+}, s(w) \leq s(y)\} \tag{2.30}$$

in the norm of the space M_* , it follows that there are $\lambda > 0$ and $w \in \mathfrak{L}_y$ such that

$$\|w - u\| \leq \varepsilon, \quad w \leq \lambda y. \tag{2.31}$$

Let

$$\bar{w} = \lim_{n \rightarrow \infty} T^n w. \quad (2.32)$$

Then

$$\begin{aligned} \bar{w}(1-s(\bar{x})) &= \lim_{n \rightarrow \infty} (T^n(w))(1-s(\bar{x})) \\ &\leq \lambda \cdot \lim_{n \rightarrow \infty} (T^n y)(1-s(\bar{x})) \\ &\leq \lambda \cdot \lim_{n \rightarrow \infty} \left(\sum_{k=0}^{\infty} 2^{-k} \cdot (T^{n+k} x)(1-s(\bar{x})) \right) \\ &= \lambda \cdot \sum_{k=0}^{\infty} 2^{-k} \lim_{n \rightarrow \infty} (T^{n+k} x)(1-s(\bar{x})) = 0. \end{aligned} \quad (2.33)$$

Because the operator T does not increase the norm of the functionals from M_* , we get that

$$\bar{u}(1-s(\bar{x})) = \lim_{n \rightarrow \infty} (T^n u)(1-s(\bar{x})) \leq \lim_{n \rightarrow \infty} (T^n w)(1-s(\bar{x})) + \lim_{n \rightarrow \infty} \|T^n(w - u)\| \leq \varepsilon. \quad (2.34)$$

The needed inequality follows from the arbitrariness of ε . $\square$

We introduce the following notion. For $\mu \in M_*$, we will denote by $\mu \cdot E$, where E is a projection from the algebra M , the functional

$$(\mu \cdot E)(A) = \mu(EAE), \quad (2.35)$$

where $A \in M$.

We fix $\varepsilon > 0$. We will find a number N , such that

$$(T^n x)(1-s(\bar{x})) < \varepsilon^2 \quad (2.36)$$

for $n > N$.

Then,

$$\begin{aligned} &\|T^N x \cdot s(\bar{x}) - T^N x\| \\ &= \sup_{\substack{A \in M \\ \|A\|_{\infty} \leq 1}} |(T^N x)((1-s(\bar{x}))A(1-s(\bar{x}))) \\ &\quad + (T^N x)((s(\bar{x}))A(1-s(\bar{x}))) + (T^N x)((1-s(\bar{x}))A(s(\bar{x})))| \\ &\leq \varepsilon \cdot (\varepsilon + 2\|x\|^{1/2}), \end{aligned} \quad (2.37)$$

because

$$|\mu(AB)|^2 \leq \mu(A^*A) \cdot \mu(B^*B), \quad (2.38)$$

where $\mu \in M_{*+}$ and $A, B \in M$.

Let $w \in \mathfrak{L}_{\overline{y}}$ be such that

$$w \leq \lambda \overline{x} \quad (2.39)$$

for some $\lambda > 0$ and

$$\|T^N x \cdot s(\overline{x}) - w\| \leq \varepsilon. \quad (2.40)$$

Then, for $n > N$, the following is valid:

$$\|T^n x - T^{n-N} w\| \leq \|T^{n-N}(T^N x - T^N x \cdot s(\overline{x}))\| + \|T^{n-N}(T^N x \cdot s(\overline{x}) - w)\| \leq 4 \cdot \varepsilon. \quad (2.41)$$

By taking the weak limit in the inequality (2.37) and because the unit ball of M_* is closed weakly, we will get

$$\|\overline{x} - \overline{w}\| \leq 4 \cdot \varepsilon, \quad (2.42)$$

where

$$\overline{w} = \lim_{n \rightarrow \infty} T^n w. \quad (2.43)$$

We now consider the algebra $M_{s(x)}$. The functional $\overline{x}$ is faithful on the algebra $M_{s(x)}$. We will consider the representation $\pi_{\overline{x}}$ of the algebra $M_{s(x)}$ constructed using the functional x [7]. Because the functional $\overline{x}$ is faithful, we can conclude that the representation $\pi_{\overline{x}}$ is faithful on the algebra $M_{s(\overline{x})}$, and therefore $\pi_{\overline{x}}$ is an isomorphism of the algebra $M_{s(\overline{x})}$ and some algebra $\mathfrak{A}$. The algebra $\mathfrak{A}$ is a von Neumann algebra, and its preconjugate space $\mathfrak{A}_*$ is isomorphic to the space $M_* \cdot s(\overline{x})$ ([19]). We note now that

$$TM_* \cdot s(\overline{x}) \subset M_* \cdot s(\overline{x}). \quad (2.44)$$

In fact,

$$T\mathfrak{L}_y \subset \mathfrak{L}_y, \quad (2.45)$$

and therefore, by taking the norm closure, we get

$$TS \subset S; \quad (2.46)$$

by taking now the linear span, we get

$$TM_* \cdot s(\overline{x}) \subset M_* \cdot s(\overline{x}). \quad (2.47)$$

We denote by $\overline{T}$ the isomorphic image of the operator T , acting on the space $\mathfrak{A}_*$. Let

$$u \in \mathfrak{A}_{*+}, \quad u \leq \lambda \overline{x}, \quad (2.48)$$

for some $\lambda > 0$. Then there exists the operator $B \in \mathfrak{A}'$, where $\mathfrak{A}'$ is a commutant of $\mathfrak{A}$, such that

$$(AB\Omega, \Omega) = u(A) \quad (2.49)$$

for all $A \in \mathfrak{A}$. Note, that from Lemma 2.3,

$$(\overline{T}u)(A) = u((\overline{T})^* A) = ((\overline{T})^* A)B\Omega, \Omega = (A((\overline{T}^*)' B)\Omega, \Omega). \quad (2.50)$$

Also, from

$$\overline{T}\mathfrak{A}_{*+} \subset \mathfrak{A}_{*+}, \quad \|\overline{T}u\| \leq \|u\|, \quad \overline{T}\bar{x} = \bar{x}, \quad (2.51)$$

it follows that

$$(\overline{T})^* \mathfrak{A}_+; \quad (\overline{T}^*) \mathbf{1} \leq \mathbf{1}, \quad \|(\overline{T})^* A\|_\infty \leq \|A\|_\infty, \quad (2.52)$$

for all $A \in \mathfrak{A}$. Based on the lemma, we now conclude that

$$\|(\overline{T}^* B)\|_\infty \leq \|B\|_\infty; \quad \overline{T}^{*'} \mathfrak{A}'_+ \subset \mathfrak{A}'_+; \quad \overline{T}^{*'} \mathbf{1} \leq \mathbf{1}, \quad (2.53)$$

for all $B \in \mathfrak{A}'$.

The space $\mathfrak{A}'_{\text{sa}}$ is a pre-Hilbert space of the selfadjoint operators from $\mathfrak{A}'$ with the scalar product

$$(B, C)_{\bar{x}} = (CB\Omega, \Omega), \quad (2.54)$$

and using the Kadison inequality [5], we have

$$((\overline{T}^{*'} B)(\overline{T}^{*'} B)\Omega, \Omega) \leq (\overline{T}^{*'} (B^2)\Omega, \Omega) \leq (B\Omega, B\Omega), \quad (2.55)$$

that is, the operator $\overline{T}^{*'}$ is a contraction in the pre-Hilbert space $(\mathfrak{A}'_{\text{sa}}, (\cdot, \cdot)_{\bar{x}})$.

We will identify $M_* \cdot s(\bar{x})$ and $\mathfrak{A}_*$. Because $w \in \mathfrak{L}$, that is,

$$w \leq \lambda \bar{x} \quad (2.56)$$

for some $\lambda > 0$, then

$$\overline{w} \leq \lambda \bar{x} \quad (2.57)$$

as well. Let

$$w(A) = (BA\Omega, \Omega), \quad \overline{w}(A) = (\overline{B}A\Omega, \Omega), \quad (2.58)$$

for all $A \in \mathfrak{A}$, where $B, \overline{B} \in \mathfrak{A}'$.

Let now $(a_{n,i})$ be a uniformly regular matrix. Using Lemma 2.3, we will find $k \in \mathbb{N}$ so that

$$\begin{aligned}
 & \left\| \sum_i a'_{k,i} T^i w - \bar{w} \right\| \\
 &= \sup_{\substack{A \in \mathfrak{A} \\ \|A\|_\infty = 1}} \left| \left(\sum_{i=1}^\infty a'_{k,i} (\bar{T}^{*'})^i (B - \bar{B}) A \Omega, \Omega \right) \right| \\
 &\leq \left(\sum_{i=1}^\infty a'_{k,i} (\bar{T}^{*'})^i (B - \bar{B}) \Omega, \sum_{i=1}^\infty a'_{k,i} (\bar{T}^{*'})^i (B - \bar{B}) \Omega \right)^{1/2} \cdot \sup_{\substack{A \in \mathfrak{A} \\ \|A\|_\infty \leq 1}} (A \Omega, A \Omega)^{1/2} \\
 &\leq (\bar{x}(1))^{1/2} \cdot \left\| \sum_{i=1}^\infty a'_{k,i} (\bar{T}^{*'})^i (B - \bar{B}) \right\|_{(\cdot, \cdot)_{\bar{x}}} < \varepsilon
 \end{aligned} \tag{2.59}$$

for $k > K$, where by $(a'_{n,i})$, we will denote a matrix with the elements

$$a'_{n,i} = \left(\sum_{j>N} a_{n,j} \right)^{-1} a_{n,j+N}. \tag{2.60}$$

It is easy to see that the matrix $(a'_{n,i})$ will be uniformly regular as well.

Then, for a big enough $k > K$, we will have

$$\begin{aligned}
 & \left\| \sum_i a_{k,i} T^i x - \bar{x} \right\| \leq \sum_{i \leq N} |a_{k,i}| \|T^i x - \bar{x}\| + \sum_{i > N} |a_{k,i}| \|T^i x - T^{i-N} w\| \\
 & \quad + \sum_{i > N} |a_{k,i}| \left| 1 - \left(\sum_{i > N} a_{k,i} \right)^{-1} \right| \|T^{i-N} w\| \\
 & \quad + \left\| \sum_{j=1}^\infty a_{k,j+N} \cdot \left(\sum_{i > N} a_{k,i} \right)^{-1} T^j w - \bar{w} \right\| \\
 & \quad + \left\| \left(\sum_{i \leq N} a_{k,i} \right) \cdot \bar{w} \right\| + \left\| \sum_{i > N} a_{k,i} \right\| \|\bar{w} - \bar{x}\| \\
 & \leq \sum_{i \leq N} 2 \cdot \frac{\varepsilon}{N} + \sum_{i > N} |a_{k,i}| \cdot 4\varepsilon + \sum_{i > N} |a_{k,i}| (1 - (1 + \varepsilon)^{-1}) \cdot 2 \\
 & \quad + \sum_{i \leq N} 2 \cdot \frac{\varepsilon}{N} + (1 + \varepsilon) \cdot 4\varepsilon \\
 & \leq 2\varepsilon + (1 + \varepsilon) \cdot 4\varepsilon + \varepsilon \cdot 2 \cdot (1 + \varepsilon) + \varepsilon + 2\varepsilon + (1 + \varepsilon) \cdot 4\varepsilon \leq 25\varepsilon.
 \end{aligned} \tag{2.61}$$

The arbitrariness of ε proves the needed statement. The proof of the theorem is now completed. $\square$

2.2. The case of L_1 -spaces for JBW algebras. The L_1 -spaces for semifinite JBW algebras were considered by [4] (see also [1, 12]), where it has been proven that they do coincide

with predual spaces. A semifinite JBW algebra A is always represented as

$$A = A_{\text{sp}} \dot{+} A_{\text{ex}}, \quad (2.62)$$

where A_{sp} is isometrically isomorphic to operator JW algebra, and A_{ex} is isometrically isomorphic to the space $C(X, M_3^8)$ of all continuous mappings from a Hyperstonean compact topological space X onto the exceptional Jordan algebra M_3^8 (see [11]). In this case, when A does not have direct summands of type I_2 , it is going to be a selfadjoint part of a real von Neumann algebra $R(A_{\text{sp}})$, whose complexification

$$R(A_{\text{sp}}) \dot{+} iR(A_{\text{sp}}) = M, \quad (2.63)$$

where M is the enveloping von Neumann algebra of A_{sp} , and the predual space of A , and the space

$$A_* = (A_{\text{sp}})_* \dot{+} (A_{\text{ex}})_*, \quad (2.64)$$

where $(A_{\text{sp}})_*$ is the predual space of A_{sp} , and $(A_{\text{ex}})_*$ is the predual space of A_{ex} (see, e.g., [2, 11]). The main result for the summand A_{ex} follows immediately from the result for $C(X)$, and the fact that the algebra M_3^8 is finite dimensional. So, without the loss of generality, we are interested in the operator case only. But in the operator case, the space $(A_{\text{sp}})_*$ is a selfadjoint part of $R_* = (R(A_{\text{sp}}))_*$, and

$$M_* = R_* \dot{+} iR_* \quad (2.65)$$

(see [2, 16] for details). So, the main result for R_* thus follows from the complex case by restriction of scalars, and we obtain the main result for L_1 -spaces affiliated to semifinite JBW algebras without direct type I_2 summand.

3. Main result: the case of quantum L_p -spaces ($1 < p < \infty$)

In the case of a noncommutative L_p -space for a semifinite von Neumann algebra, the main result is discussed in [25].

We will discuss here the nonassociative case.

In this section, A denotes a semifinite JBW algebra without direct summands of type I_2 , with a faithful normal trace τ . By L_p , we denote the space of operators affiliated to A , and integrated with p th power ($p > 1$, see, e.g., [1, 2, 12]). Space L_q (here $q = p/(p-1)$) is a dual as Banach space to L_p (see [1, 12]). The following theorem is valid.

THEOREM 3.1. *The following conditions for a positive contraction T in the L_p are equivalent.*

- (i) *The sequence $\{T^i x\}_{i=1,2,\dots}$ converges in $\sigma(L_p, L_q)$ topology for $x \in L_p$.*
- (ii) *For each strictly increasing sequence of natural numbers $\{k_i\}_{i=1,2,\dots}$,*

$$n^{-1} \sum_{i < n} T^{k_i} x \quad (3.1)$$

converges in norm of L_p for all $x \in L_p$.

(iii) For any uniformly regular matrix $(a_{n,i})$, the sequence $\{A_n(T)x\}_{n=1,2,\dots}$,

$$A_n(T)x = \sum_i a_{n,i} T^i x, \quad (3.2)$$

converges in norm of L_p for all $x \in L_p$.

For the sake of completeness, we give the following definitions (see, e.g., [25]) and sketch of the proof. Let ϕ be a gauge function

$$\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \quad (3.3)$$

with

$$\phi(0) = 0, \quad \lim_{t \rightarrow \infty} \phi(t) = \infty. \quad (3.4)$$

Hahn-Banach theorem implies for strictly convex Banach spaces E with conjugate E' that there exists a duality map

$$\Phi : E \longrightarrow E', \quad (3.5)$$

associated with ϕ such that

$$\langle x, \Phi(x) \rangle = \|x\| \|\Phi(x)\|, \quad \|\Phi(x)\| = \phi(x). \quad (3.6)$$

Definition 3.2. Map Φ is said to satisfy property (S) uniformly if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, such that for any $x, y \in E$,

$$|\langle x, \Phi(y) \rangle| < \delta(\epsilon) \quad (3.7)$$

implies that

$$|\langle y, \Phi(x) \rangle| < \epsilon. \quad (3.8)$$

Proof. From [12, Section 4], it follows that the duality map defined as

$$\Phi(a) = s|a|^{p-1}, \quad (3.9)$$

for

$$a = s|a| \in A \quad (3.10)$$

(where $a = s|a|$ is a polar decomposition of element a) satisfies the property (S) uniformly. Hence, the statement of the theorem follows from [25, Theorem 3.1]. $\square$

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