SEMICOMPACTNESS IN L -TOPOLOGICAL SPACES

FU-GUI SHI

Received 8 October 2004 and in revised form 8 June 2005

The concepts of semicompactness, countable semicompactness, and the semi-Lindelöf property are introduced in L -topological spaces, where L is a complete de Morgan algebra. They are defined by means of semiopen L -sets and their inequalities. They do not rely on the structure of basis lattice L and no distributivity in L is required. They can also be characterized by semiclosed L -sets and their inequalities. When L is a completely distributive de Morgan algebra, their many characterizations are presented.

1. Introduction

The notion of semicompactness [3] was introduced in L -topological spaces by Kudri. In Kudri's work [6], he followed the lines of his definition of compactness which is equivalent to the notion of strong fuzzy compactness in [7, 8, 13]. However, Kudri's semicompactness relies on the structure of L and L is required to be completely distributive.

In [10, 12], a new definition of fuzzy compactness is presented in L -topological spaces by means of an inequality, which does not depend on the structure of L and no distributivity is required in L . When L is a completely distributive de Morgan algebra, it is equivalent to the notion of fuzzy compactness in [7, 8, 13].

Following the lines of [10, 12], we will introduce a new definition of semicompactness in L -topological spaces by means of semiopen L -sets and their inequality, where L is a complete de Morgan algebra. This definition does not rely on the structure of basis lattice L and no distributivity in L is required. It can also be characterized by semiclosed L -sets and their inequality. When L is a completely distributive de Morgan algebra, its many characterizations are presented. Moreover, we also will introduce the notions of countable semicompactness and the semi-Lindelöf property and research their properties.

2. Preliminaries

Throughout this paper, $(L, \vee, \wedge, ')$ is a complete de Morgan algebra, X a nonempty set. L^X is the set of all L -fuzzy sets (or L -sets for short) on X . The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$.

An element a in L is called prime element if $a \geq b \wedge c$ implies that $a \geq b$ or $a \geq c$. a in L is called a coprime element if a' is a prime element [5]. The set of nonunit prime elements in L is denoted by $P(L)$. The set of nonzero coprime elements in L is denoted by $M(L)$.

The binary relation $<$ in L is defined as follows: for $a, b \in L$, $a < b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [4]. In a completely distributive de Morgan algebra L , each element b is a sup of $\{a \in L \mid a < b\}$. $\{a \in L \mid a < b\}$ is called the greatest minimal family of b in the sense of [7, 13], in symbol $\beta(b)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' < b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations in [9]:

$$A^{(a)} = \{x \in X \mid A(x) \not\leq a\}, \quad A_{(a)} = \{x \in X \mid a \in \beta(A(x))\}. \quad (2.1)$$

An L -topological space (or L -space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\underline{0}$, $\underline{1}$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L -topology on X . Each member of $\mathcal{T}$ is called an open L -set and its quasicomplementation is called a closed L -set.

Definition 2.1 (see [7, 13]). For a topological space (X, τ) , let $\omega_L(\tau)$ denote the family of all the lower semicontinuous maps from (X, τ) to L , that is, $\omega_L(\tau) = \{A \in L^X \mid A^{(a)} \in \tau, a \in L\}$. Then $\omega_L(\tau)$ is an L -topology on X , in this case, $(X, \omega_L(\tau))$ is topologically generated by (X, τ) .

Definition 2.2 (see [7, 13]). An L -space $(X, \mathcal{T})$ is weak induced if for all $a \in L$, for all $A \in \mathcal{T}$, it follows that $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in $\mathcal{T}$.

It is obvious that $(X, \omega_L(\tau))$ is weak induced.

LEMMA 2.3 (see [11]). *Let $(X, \mathcal{T})$ be a weakly induced L -space, $a \in L$, $A \in \mathcal{T}$. Then $A_{(a)}$ is an open set in $[\mathcal{T}]$.*

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ . $2^{[\Phi]}$ denotes the set of all countable subfamilies of Φ .

Definition 2.4 (see [10, 12]). Let $(X, \mathcal{T})$ be an L -space, $G \in L^X$ is called (countably) compact if for every (countably) family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right). \quad (2.2)$$

Definition 2.5 (see [10]). Let $(X, \mathcal{T})$ be an L -space, $G \in L^X$ is said to have the Lindelöf property if for every family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right). \quad (2.3)$$

LEMMA 2.6 (see [10]). Let L be a complete Heyting algebra, let $f : X \rightarrow Y$ be a map, $f_L^- : L^X \rightarrow L^Y$ is the extension of f , then for any family $\mathcal{P} \subseteq L^Y$,

$$\bigvee_{y \in Y} \left(f_L^-(G)(y) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^-(B)(x) \right). \quad (2.4)$$

Definition 2.7 (see [1]). An L -set G in an L -space $(X, \mathcal{T})$ is called semiopen if there exists $A \in \mathcal{T}$ such that $A \leq G \leq \text{cl}(A)$. G is called semiclosed if G' is semiopen.

Definition 2.8. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces. A map $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ is called

- (1) semicontinuous [1] if $f_L^-(G)$ is semiopen in $(X, \mathcal{T}_1)$ for every open L -set G in $(Y, \mathcal{T}_2)$;
- (2) irresolute [2] if $f_L^-(G)$ is semiopen in $(X, !\mathcal{T}_1)$ for every semiopen L -set G in $(Y, \mathcal{T}_2)$.

3. Definition and characterizations of semicompactness

Definition 3.1. Let $(X, \mathcal{T})$ be an L -space. $G \in L^X$ is called (countably) semicompact if for every (countable) family $\mathcal{U}$ of semiopen L -sets, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right). \quad (3.1)$$

Definition 3.2. Let $(X, \mathcal{T})$ be an L -space. $G \in L^X$ is said to have the semi-Lindelöf property (or be a semi-Lindelöf L -set) if for every family $\mathcal{U}$ of semiopen L -sets, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right). \quad (3.2)$$

Example 3.3. Let X be any nonempty set and let A be a $[0, 1]$ -set on X defined as $A(x) = 0.5$, for all $x \in X$. Let $\mathcal{T} = \{\emptyset, X, A\}$. Then the set of all semiopen $[0, 1]$ -sets in $(X, \mathcal{T})$ is $\mathcal{T}$. In this case, any $[0, 1]$ -set in $(X, \mathcal{T})$ is semicompact, hence it is countably semicompact and has the semi-Lindelöf property.

Obviously, we have the following theorem.

THEOREM 3.4. Semicompactness implies countably semicompactness and the semi-Lindelöf property. Moreover, an L -set having the semi-Lindelöf property is semicompact if and only if it is countably semicompact.

Since an open L -set must be semiopen, we have the following theorem.

THEOREM 3.5. Semicompactness implies compactness, countably semicompactness implies countably compactness, and the semi-Lindelöf property implies the Lindelöf property.

From Definitions 3.1 and 3.2, we can obtain the following two theorems by using quasicomplementation.

THEOREM 3.6. *Let $(X, \mathcal{T})$ be an L -space. $G \in L^X$ is (countably) semicompact if and only if for every (countable) family $\mathcal{B}$ of semiclosed L -sets, it follows that*

$$\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{B}} B(x) \right) \geq \bigwedge_{\mathcal{F} \in 2^{[\mathcal{B}]}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right). \quad (3.3)$$

THEOREM 3.7. *Let $(X, \mathcal{T})$ be an L -space. $G \in L^X$ has the semi-Lindelöf property if and only if for every family $\mathcal{B}$ of semiclosed L -sets, it follows that*

$$\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{B}} B(x) \right) \geq \bigwedge_{\mathcal{F} \in 2^{[\mathcal{B}]}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right). \quad (3.4)$$

In order to present characterizations of semicompactness, countable semicompactness and the semi-Lindelöf property, we generalize the notions of a -shading and a - R -neighborhood family in [10, 12] as follows.

Definition 3.8. Let $(X, \mathcal{T})$ be an L -space, $a \in L \setminus \{1\}$, and $G \in L^X$. A family $\mathcal{A} \subseteq L^X$ is said to be

- (1) an a -shading of G if for any $x \in X$, $(G'(x) \vee \bigvee_{A \in \mathcal{A}} A(x)) \not\leq a$;
- (2) a strong a -shading of G if $\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{A}} A(x)) \not\leq a$;
- (3) an a - R -neighborhood family of G if for any $x \in X$, $(G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x)) \not\leq a$;
- (4) a strong a - R -neighborhood family of G if $\bigvee_{x \in X} (G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x)) \not\leq a$.

It is obvious that a strong a -shading of G is an a -shading of G , a strong a - R -neighborhood family of G is an a - R -neighborhood family of G , and $\mathcal{P}$ is a strong a - R -neighborhood family of G if and only if $\mathcal{P}'$ is a strong a -shading of G .

Definition 3.9. Let $a \in L \setminus \{0\}$ and $G \in L^X$. A subfamily $\mathcal{A}$ of L^X is said to have weak a -nonempty intersection in G if $\bigvee_{x \in X} (G(x) \wedge \bigwedge_{A \in \mathcal{A}} A(x)) \geq a$. $\mathcal{A}$ is said to have the finite (countable) weak a -intersection property in G if every finite (countable) subfamily $\mathcal{F}$ of $\mathcal{A}$ has weak a -nonempty intersection in G .

From Definitions 3.1, 3.2, Theorems 3.5 and 3.6, we immediately obtain the following two results.

THEOREM 3.10. *Let $(X, \mathcal{T})$ be an L -space and $G \in L^X$. Then the following conditions are equivalent.*

- (1) G is (countably) semicompact.
- (2) For any $a \in L \setminus \{1\}$, each (countable) semiopen strong a -shading $\mathcal{U}$ of G has a finite subfamily which is a strong a -shading of G .
- (3) For any $a \in L \setminus \{0\}$, each (countable) semiclosed strong a - R -neighborhood family $\mathcal{P}$ of G has a finite subfamily which is a strong a - R -neighborhood family of G .
- (4) For any $a \in L \setminus \{0\}$, each (countable) family of semiclosed L -sets which has the finite weak a -intersection property in G has weak a -nonempty intersection in G .

THEOREM 3.11. *Let $(X, \mathcal{T})$ be an L -space and $G \in L^X$. Then the following conditions are equivalent.*

- (1) G has the semi-Lindelöf property.
- (2) For any $a \in L \setminus \{1\}$, each semiopen strong a -shading $\mathcal{U}$ of G has a countable subfamily which is a strong a -shading of G .
- (3) For any $a \in L \setminus \{0\}$, each semiclosed strong a - R -neighborhood family $\mathcal{P}$ of G has a countable subfamily which is a strong a - R -neighborhood family of G .
- (4) For any $a \in L \setminus \{0\}$, each family of semiclosed L -sets which has the countable weak a -intersection property in G has weak a -nonempty intersection in G .

4. Properties of (countable) semicompactness

THEOREM 4.1. *Let L be a complete Heyting algebra. If both G and H are (countably) semicompact, then $G \vee H$ is (countably) semicompact.*

Proof. For any (countable) family $\mathcal{P}$ of semiclosed L -sets, by Theorem 3.5 we have that

$$\begin{aligned}
 & \bigvee_{x \in X} \left((G \vee H)(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \\
 &= \left\{ \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \right\} \vee \left\{ \bigvee_{x \in X} \left(H(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \right\} \\
 &\geq \left\{ \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \vee \left\{ \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left(H(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \\
 &= \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left((G \vee H)(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right).
 \end{aligned} \tag{4.1}$$

This shows that $G \vee H$ is (countably) semicompact. □

Analogously, we have the following result.

THEOREM 4.2. *Let L be a complete Heyting algebra. If both G and H have the semi-Lindelöf property, then $G \vee H$ has the semi-Lindelöf property.*

THEOREM 4.3. *If G is (countably) semicompact and H is semiclosed, then $G \wedge H$ is (countably) semicompact.*

Proof. For any (countable) family $\mathcal{P}$ of semiclosed L -sets, by Theorem 3.5 we have that

$$\begin{aligned}
 & \bigvee_{x \in X} \left((G \wedge H)(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \\
 &= \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P} \cup \{H\}} B(x) \right) \\
 &\geq \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P} \cup \{H\})}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right)
 \end{aligned}$$

$$\begin{aligned}
&= \left\{ \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \wedge \left\{ \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left(G(x) \wedge H(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \\
&= \left\{ \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left(G(x) \wedge H(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right) \right\} \\
&= \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left((G \wedge H)(x) \wedge \bigwedge_{B \in \mathcal{F}} B(x) \right).
\end{aligned} \tag{4.2}$$

This shows that $G \wedge H$ is (countably) semicompact. $\square$

Analogously, we have the following result.

THEOREM 4.4. *If G has the semi-Lindelöf property and H is semiclosed, then $G \wedge H$ has the semi-Lindelöf property.*

THEOREM 4.5. *Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ be an irresolute map. If G is a semicompact (resp., countably semicompact, semi-Lindelöf) L -set in $(X, \mathcal{T}_1)$, then so is $f_L^-(G)$ in $(Y, \mathcal{T}_2)$.*

Proof. We only prove that the theorem is true for semicompactness. Suppose that $\mathcal{P}$ is a family of semiclosed L -sets in $(Y, \mathcal{T}_2)$, by Lemma 2.6 and semicompactness of G , we have that

$$\begin{aligned}
&\bigvee_{y \in Y} \left(f_L^-(G)(y) \wedge \bigwedge_{B \in \mathcal{P}} B(y) \right) \\
&= \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} f_L^-(B)(x) \right) \\
&\geq \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{F}} f_L^-(B)(x) \right) \\
&= \bigwedge_{\mathcal{F} \in 2^{(\mathcal{P})}} \bigvee_{y \in Y} \left(f_L^-(G)(y) \wedge \bigwedge_{B \in \mathcal{F}} B(y) \right).
\end{aligned} \tag{4.3}$$

Therefore $f_L^-(G)$ is semicompact. $\square$

Analogously, we have the following result.

THEOREM 4.6. *Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ be a semi-continuous map. If G is a semicompact (resp., countably semicompact, semi-Lindelöf) L -set in $(X, \mathcal{T}_1)$, then $f_L^-(G)$ is a compact (countably compact, Lindelöf) L -set in $(Y, \mathcal{T}_2)$.*

Definition 4.7. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L -spaces. A map $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ is called strongly irresolute if $f_L^-(G)$ is open in $(X, \mathcal{T}_1)$ for every semiopen L -set G in $(Y, \mathcal{T}_2)$.

It is obvious that a strongly irresolute map is irresolute.

Analogously, we have the following result.

THEOREM 4.8. *Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ be a strongly irresolute map. If G is a compact (resp., countably compact, Lindelöf) L -set in $(X, \mathcal{T}_1)$, then $f_L^-(G)$ is a semicompact (countably semicompact, semi-Lindelöf) L -set in $(Y, \mathcal{T}_2)$.*

5. Further characterizations of semicompactness and goodness

In this section, we assume that L is a completely distributive de Morgan algebra.

Now we generalize the notions of β_a -open cover and Q_a -open cover [10] as follows.

Definition 5.1. Let $(X, \mathcal{T})$ be an L -space, $a \in L \setminus \{0\}$, and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a β_a -cover of G if for any $x \in X$, it follows that $a \in \beta(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x))$. $\mathcal{U}$ is called a strong β_a -cover of G if $a \in \beta(\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)))$.

It is obvious that a strong β_a -cover of G must be a β_a -cover of G .

Definition 5.2. Let $(X, \mathcal{T})$ be an L -space, $a \in L \setminus \{0\}$, and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a Q_a -cover of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \geq a$.

It is obvious that a β_a -cover of G must be a Q_a -cover of G .

Analogous to the proof of [10, Theorem 2.9], we can obtain the following theorem.

THEOREM 5.3. *Let $(X, \mathcal{T})$ be an L -space and $G \in L^X$. Then the following conditions are equivalent.*

- (1) G is (countably) semicompact.
- (2) For any $a \in L \setminus \{0\}$, each (countable) semiclosed strong a - R -neighborhood family $\mathcal{P}$ of G has a finite subfamily which is a strong a - R -neighborhood family of G .
- (3) For any $a \in L \setminus \{0\}$, each (countable) semiclosed strong a - R -neighborhood family $\mathcal{P}$ of G has a finite subfamily which is an a - R -neighborhood family of G .
- (4) For any $a \in L \setminus \{0\}$ and any (countable) semiclosed strong a - R -neighborhood family $\mathcal{P}$ of G , there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ such that $\mathcal{F}$ is a strong b - R -neighborhood family of G .
- (5) For any $a \in L \setminus \{0\}$ and any (countable) semiclosed strong a - R -neighborhood family $\mathcal{P}$ of G , there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ such that $\mathcal{F}$ is a b - R -neighborhood family of G .
- (6) For any $a \in M(L)$, each (countable) semiclosed strong a - R -neighborhood family $\mathcal{P}$ of G has a finite subfamily which is a strong a - R -neighborhood family of G .
- (7) For any $a \in M(L)$, each (countable) semiclosed strong a - R -neighborhood family $\mathcal{P}$ of G has a finite subfamily which is an a - R -neighborhood family of G .
- (8) For any $a \in M(L)$ and any (countable) semiclosed strong a - R -neighborhood family $\mathcal{P}$ of G , there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta^*(a)$ such that $\mathcal{F}$ is a strong b - R -neighborhood family of G .
- (9) For any $a \in M(L)$ and any (countable) semiclosed strong a - R -neighborhood family $\mathcal{P}$ of G , there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta^*(a)$ such that $\mathcal{F}$ is a b - R -neighborhood family of G .
- (10) For any $a \in L \setminus \{1\}$, each (countable) semiopen strong a -shading $\mathcal{U}$ of G has a finite subfamily which is a strong a -shading of G .

(11) For any $a \in L \setminus \{1\}$, each (countable) semiopen strong a -shading $\mathcal{U}$ of G has a finite subfamily which is an a -shading of G .

(12) For any $a \in L \setminus \{1\}$ and any (countable) semiopen strong a -shading $\mathcal{U}$ of G , there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha(a)$ such that $\mathcal{V}$ is a strong b -shading of G .

(13) For any $a \in L \setminus \{1\}$ and any (countable) semiopen strong a -shading $\mathcal{U}$ of G , there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha(a)$ such that $\mathcal{V}$ is a b -shading of G .

(14) For any $a \in P(L)$, each (countable) semiopen strong a -shading $\mathcal{U}$ of G has a finite subfamily which is a strong a -shading of G .

(15) For any $a \in P(L)$, each (countable) semiopen strong a -shading $\mathcal{U}$ of G has a finite subfamily which is an a -shading of G .

(16) For any $a \in P(L)$ and any (countable) semiopen strong a -shading $\mathcal{U}$ of G , there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha^*(a)$ such that $\mathcal{V}$ is a strong b -shading of G .

(17) For any $a \in P(L)$ and any (countable) semiopen strong a -shading $\mathcal{U}$ of G , there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha^*(a)$ such that $\mathcal{V}$ is a b -shading of G .

(18) For any $a \in L \setminus \{0\}$, each (countable) semiopen strong β_a -cover $\mathcal{U}$ of G has a finite subfamily which is a strong β_a -cover of G .

(19) For any $a \in L \setminus \{0\}$, each (countable) semiopen strong β_a -cover $\mathcal{U}$ of G has a finite subfamily which is a β_a -cover of G .

(20) For any $a \in L \setminus \{0\}$ and any (countable) semiopen strong β_a -cover $\mathcal{U}$ of G , there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ with $a \in \beta(b)$ such that $\mathcal{V}$ is a strong β_b -cover of G .

(21) For any $a \in L \setminus \{0\}$ and any (countable) semiopen strong β_a -cover $\mathcal{U}$ of G , there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ with $a \in \beta(b)$ such that $\mathcal{V}$ is a β_b -cover of G .

(22) For any $a \in M(L)$, each (countable) semiopen strong β_a -cover $\mathcal{U}$ of G has a finite subfamily which is a strong β_a -cover of G .

(23) For any $a \in M(L)$, each (countable) semiopen strong β_a -cover $\mathcal{U}$ of G has a finite subfamily which is a β_a -cover of G .

(24) For any $a \in M(L)$ and any (countable) semiopen strong β_a -cover $\mathcal{U}$ of G , there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a strong β_b -cover of G .

(25) For any $a \in M(L)$ and any (countable) semiopen strong β_a -cover $\mathcal{U}$ of G , there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a β_b -cover of G .

(26) For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each (countable) semiopen Q_a -cover of G has a finite subfamily which is a Q_b -cover of G .

(27) For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each (countable) semiopen Q_a -cover of G has a finite subfamily which is a β_b -cover of G .

(28) For any $a \in L \setminus \{0\}$ and any $b \in \beta(a) \setminus \{0\}$, each (countable) semiopen Q_a -cover of G has a finite subfamily which is a strong β_b -cover of G .

(29) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each (countable) semiopen Q_a -cover of G has a finite subfamily which is a Q_b -cover of G .

(30) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each semiopen Q_a -cover of G has a finite subfamily which is a β_b -cover of G .

(31) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each (countable) semiopen Q_a -cover of G has a finite subfamily which is a strong β_b -cover of G .

Analogously, we also can present characterizations of the semi-Lindelöf property.

LEMMA 5.4. Let $(X, \omega(\tau))$ be generated topologically by (X, τ) . If A is a semiopen L -set in (X, τ) , then χ_A is a semiopen set in $(X, \omega(\tau))$. If B is a semiopen L -set in $(X, \omega(\tau))$, then $B_{(a)}$ is a semiopen set in (X, τ) for every $a \in L$.

Proof. If A is a semiopen set in (X, τ) , then there exists $D \in \tau$ such that $D \subseteq A \subseteq \text{cl}(D)$. Thus we have that

$$\chi_D \leq \chi_A \leq \chi_{\text{cl}(D)} = \text{cl}(\chi_D). \quad (5.1)$$

This shows that χ_A is semiopen.

If B is a semiopen L -set in $(X, \omega(\tau))$, then there exists $E \in \omega(\tau)$ such that $E \leq B \leq \text{cl}(E)$. Thus we have that $E_{(a)} \subseteq B_{(a)} \subseteq \text{cl}(E)_{(a)}$. From [9], we can obtain that $\text{cl}(E)_{(a)} \subseteq \text{cl}(E_{(a)})$. Hence by Lemma 2.3, we know that $B_{(a)}$ is a semiopen set in (X, τ) . $\square$

The following two theorems show that semicompactness, countable semicompactness and the semi-Lindelöf property are good extensions.

THEOREM 5.5. Let (X, τ) be a topological space and let $(X, \omega(\tau))$ be generated topologically by (X, τ) . Then $(X, \omega(\tau))$ is (countably) semicompact if and only if (X, τ) is (countably) semicompact.

Proof. Necessity. Let $\mathcal{A}$ be a (countable) semiopen cover of (X, τ) . Then $\{\chi_A \mid A \in \mathcal{A}\}$ is a family of semiopen L -sets in $(X, \omega(\tau))$ with $\bigwedge_{x \in X} (\bigvee_{A \in \mathcal{A}} \chi_A(x)) = 1$. From (countable) semicompactness of $(X, \omega(\tau))$, we know that

$$\bigvee_{\mathcal{V} \in 2^{(\mathcal{A})}} \bigwedge_{x \in X} \left(\bigvee_{A \in \mathcal{V}} \chi_A(x) \right) = \bigvee_{\mathcal{V} \in 2^{(\mathcal{A})}} \bigwedge_{x \in X} \left(\bigvee_{A \in \mathcal{V}} \chi_A(x) \right) = 1. \quad (5.2)$$

This implies that there exists $\mathcal{V} \in 2^{(\mathcal{A})}$ such that $\bigwedge_{x \in X} (\bigvee_{A \in \mathcal{V}} \chi_A(x)) = 1$. Hence, $\mathcal{V}$ is a cover of (X, τ) . Therefore (X, τ) is (countably) semicompact.

Sufficiency. Let $\mathcal{U}$ be a (countable) family of semiopen L -sets in $(X, \omega(\tau))$ and let $\bigwedge_{x \in X} (\bigvee_{B \in \mathcal{U}} B(x)) = a$. If $a = 0$, then obviously we have that

$$\bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{U}} B(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{V}} B(x) \right). \quad (5.3)$$

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a) \setminus \{0\}$, we have that

$$b \in \beta \left(\bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{U}} B(x) \right) \right) \subseteq \bigcap_{x \in X} \beta \left(\bigvee_{B \in \mathcal{U}} B(x) \right) = \bigcap_{x \in X} \bigcup_{B \in \mathcal{U}} \beta(B(x)). \quad (5.4)$$

From Lemma 5.4, this implies that $\{B_{(b)} \mid B \in \mathcal{U}\}$ is a semiopen cover of (X, τ) . From (countable) semicompactness of (X, τ) , we know that there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $\{B_{(b)} \mid B \in \mathcal{V}\}$ is a cover of (X, τ) . Hence $b \leq \bigwedge_{x \in X} (\bigvee_{B \in \mathcal{V}} B(x))$. Further, we have that

$$b \leq \bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{V}} B(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{V}} B(x) \right). \quad (5.5)$$

This implies that

$$\bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{U}} B(x) \right) = a = \bigvee \{b \mid b \in \beta(a)\} \leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{V}} B(x) \right). \quad (5.6)$$

Therefore, $(X, \omega(\tau))$ is (countably) semicompact. $\square$

Analogously, we have the following result.

THEOREM 5.6. *Let (X, τ) be a topological space and let $(X, \omega(\tau))$ be generated topologically by (X, τ) . Then $(X, \omega(\tau))$ has the semi-Lindelöf property if and only if (X, τ) has the semi-Lindelöf property.*

Acknowledgment

This work is supported by the National Natural Science Foundation of China (10371079) and the Base Research Foundation of Beijing Institute of Technology.

References

- [1] K. K. Azad, *On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl. **82** (1981), no. 1, 14–32.
- [2] D. Çoker and A. H. Eş, *On fuzzy S -closed spaces*, Doğa Mat. **11** (1987), no. 3, 145–152.
- [3] Ch. Dorsett, *Semicompact R_1 and product spaces*, Bull. Malaysian Math. Soc. (2) **3** (1980), no. 1, 15–19.
- [4] Ph. Dwinger, *Characterization of the complete homomorphic images of a completely distributive complete lattice. I*, Nederl. Akad. Wetensch. Indag. Math. **44** (1982), no. 4, 403–414.
- [5] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott, *A Compendium of Continuous Lattices*, Springer, Berlin, 1980.
- [6] S. R. T. Kudri, *Semicompactness and S^* -closedness in L -fuzzy topological spaces*, Fuzzy Sets and Systems **109** (2000), no. 2, 223–231.
- [7] Y.-M. Liu and M.-K. Luo, *Fuzzy Topology*, Advances in Fuzzy Systems—Applications and Theory, vol. 9, World Scientific, New Jersey, 1997.
- [8] R. Lowen, *A comparison of different compactness notions in fuzzy topological spaces*, J. Math. Anal. Appl. **64** (1978), no. 2, 446–454.
- [9] F.-G. Shi, *The theory and applications of L_β -nested sets and L_α -nested sets*, Mohu Xitong yu Shuxue **9** (1995), no. 4, 65–72 (Chinese).
- [10] ———, *Countable compactness and the Lindelöf property of L -fuzzy sets*, Iranian Journal of Fuzzy Systems **1** (2004), no. 1, 79–88.
- [11] ———, *A new notion of fuzzy compactness in L -topological spaces*, Inform. Sci. **173** (2005), 35–48.
- [12] ———, *Fuzzy compactness in L -topological spaces*, Acta Math. Sinica, submitted.
- [13] G. J. Wang, *Theory of L -Fuzzy Topological Space*, Shanxi Normal University Press, Xian, 1988 (Chinese).

Fu-Gui Shi: Department of Mathematics, School of Science, Beijing Institute of Technology, Beijing 100081, China

E-mail address: fuguishi@bit.edu.cn