EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A SEMILINEAR ELLIPTIC SYSTEM

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We consider the existence, the nonexistence, and the uniqueness of solutions of some special systems of nonlinear elliptic equations with boundary conditions. In a particular case, the system reduces to the homogeneous Dirichlet problem for the biharmonic equation $\Delta^2 u = |u|^p$ in a ball with $p > 0$.

1. Introduction

In this paper, we are interested in the existence, the nonexistence, and the uniqueness question for the following problem:

$$\begin{aligned}\Delta u &= |v|^{q-1}v \quad \text{in } B_R, \\ \Delta v &= |u|^p \quad \text{in } B_R, \\ u &= \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_R,\end{aligned}\tag{1.1}$$

where B_R denotes the open ball of radius R centered at the origin in $\mathbb{R}^n$ ($n \geq 1$), $\partial/\partial \nu$ is the outward normal derivative, and $p, q > 0$.

Concerning uniqueness, we have the following theorem.

THEOREM 1.1. (i) *Let $p > 0$, $q \geq 1$ with $pq \neq 1$. Then (1.1) has at most one nontrivial radial solution $(u, v) \in (C^2(\overline{B}_R))^2$.*

(ii) *Let $p > 0$, $q \geq 1$ with $pq = 1$. Assume that (1.1) has a nontrivial radial solution $(u, v) \in (C^2(\overline{B}_R))^2$. Then all nontrivial radial solutions are given by $(\theta^q u, \theta v)$, where $\theta > 0$ is an arbitrary constant.*

When $q = 1$ and $p \in (0, 1) \cup (1, \infty)$, Theorem 1.1 was established in [4] (see also the references therein). When $n = 1$, $q = 1$, and $p > 1$, the uniqueness of a nontrivial solution follows from a general result given in [5].

When $q = 1$, $p > 1$, and

$$p < \frac{n+4}{n-4} \quad \text{if } n \geq 5,\tag{1.2}$$

the existence of a nontrivial solution was proved in [2, 5, 11]. The case $q = 1$ and $0 < p < 1$ is well known: see, for instance, [4, 6]. Moreover, when $q = 1$, any nontrivial solution of (1.1) is positive in B_R because the Green function of Δ^2 with Dirichlet boundary conditions is positive in B_R [1, 8]. Then it was proved in [2, 11, 12] that problem (1.1) has no nontrivial solutions, whether radial or not, if

$$p \geq \frac{n+4}{n-4} \quad (n \geq 5). \quad (1.3)$$

We will prove a nonexistence result and an existence result.

THEOREM 1.2. *Suppose $n \geq 3$. Let $p, q > 0$ satisfy*

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n}. \quad (1.4)$$

(i) *Let $(u, v) \in (C^2(\overline{B}_R))^2$ be a solution of problem (1.1) such that $u \geq 0$ in B_R . Then $u = v = 0$.*

(ii) *If $(u, v) \in (C^2(\overline{B}_R))^2$ is a radial solution of problem (1.1), then $u = v = 0$.*

THEOREM 1.3. (i) *Let $p > 0$, $q \geq 1$ with $pq \neq 1$ satisfy*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n} \quad \text{if } n \geq 3. \quad (1.5)$$

Then (1.1) has a nontrivial radial solution $(u, v) \in (C^2(\overline{B}_R))^2$.

(ii) *Let $p > 0$, $q \geq 1$ with $pq = 1$. Then there exists $R > 0$ such that (1.1) has a nontrivial radial solution $(u, v) \in (C^2(\overline{B}_R))^2$.*

Remark 1.4. Notice that when $pq \leq 1$, (1.5) holds.

In the sequel, Δ denotes equally the Cartesian and the polar form of the Laplacian.

In Section 2, we give some preliminary results. Theorem 1.1 is proved in Section 3 using the same approach as in [4, 7]. In Section 4, we prove Theorem 1.2. We prove Theorem 1.3 in Section 5: the proof is based on a two-dimensional shooting argument for the ordinary differential equations associated to radial solutions of (1.1) [3, 5, 7, 15, 16]. The fact that $q \geq 1$ is crucial in the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

In this section, we first examine the structure of nontrivial radial solutions of (1.1).

LEMMA 2.1. *Let $(u, v) \in (C^2(\overline{B}_R))^2$ be a nontrivial radial solution of (1.1). Then $u' < 0$ on $(0, R)$, $\Delta u(R) = u''(R) > 0$ and $v' > 0$ on $(0, R]$, $v(0) < 0 < v(R)$.*

Proof. Clearly $u = 0$ if and only if $v = 0$. We have

$$r^{n-1}v'(r) = \int_0^r s^{n-1} |u(s)|^p ds \geq 0, \quad 0 \leq r \leq R. \quad (2.1)$$

Assume that $v(0) \geq 0$. Then (2.1) implies that $v \geq 0$ on $[0, R]$, hence $\Delta u \geq 0$ on $[0, R]$. Therefore $r^{n-1}u'(r)$ is nondecreasing in $[0, R]$. Since $u'(0) = u'(R) = u(R) = 0$, we deduce

that $u = 0$ and we reach a contradiction. The case where $v(R) \leq 0$ can be handled in the same way. Therefore we have $v(0) < 0 < v(R)$. We claim that $u(0) \neq 0$. Indeed assume that $u(0) = 0$. Using (2.1) and the first equation in (1.1), we deduce that there exists $R' \in (0, R)$ such that $r^{n-1}u'(r)$ is nonincreasing in $[0, R']$ and nondecreasing in $[R', R]$. Since $u'(0) = u'(R) = 0$, we obtain that $u' \leq 0$ in $[0, R]$. Using the fact that $u(0) = u(R) = 0$, we deduce that $u = 0$ in $[0, R]$ and we get a contradiction. Now (2.1) implies that $v' > 0$ in $(0, R]$. Let $R' \in (0, R)$ be such that $v(R') = 0$. Using the first equation in (1.1), we deduce that $r^{n-1}u'(r)$ is decreasing in $[0, R']$ and increasing in $[R', R]$. Since $u'(0) = u'(R) = 0$, we obtain $u' < 0$ in $(0, R)$. $\square$

LEMMA 2.2. Assume that $n \geq 1$ and $p, q > 0$. Let $\alpha, \beta > 0$ be fixed. If $(u, v) \in (C^2(\mathbb{R}^n))^2$ is a radial solution of

$$\begin{aligned} \Delta u &= |v|^{q-1}v, & r > 0, \\ \Delta v &= |u|^p, & r > 0, \\ u(0) &= \alpha, & v(0) = -\beta, & u'(0) = v'(0) = 0 \end{aligned} \quad (2.2)$$

such that $uu' < 0$ on $(0, \infty)$, then $v < 0$ on $(0, \infty)$.

Proof. We have $0 < u \leq \alpha$ on $[0, \infty)$. Therefore

$$r^{n-1}v'(r) = \int_0^r s^{n-1}u(s)^p ds > 0 \quad \text{for } r > 0. \quad (2.3)$$

Assume that the conclusion of the lemma is false. Then (2.3) implies that there exist $a, b > 0$ such that

$$v(r) \geq a \quad \text{for } r \geq b. \quad (2.4)$$

We deduce that

$$(r^{n-1}u'(r))' \geq a^q r^{n-1} \quad \text{for } r \geq b, \quad (2.5)$$

hence

$$r^{n-1}u'(r) \geq a^q \frac{r^n - b^n}{n} + b^{n-1}u'(b) \quad \text{for } r \geq b, \quad (2.6)$$

which implies that $u'(r) > 0$ for r large and we reach a contradiction. $\square$

Now we give a lemma which is needed in the proof of Theorem 1.3.

LEMMA 2.3. Assume that $n \geq 1$ and $p, q > 0$. Let $\alpha, \beta > 0$ be fixed. Assume that for some $a > 0$, $(u, v) \in (C^2(\overline{B}_a))^2$ is a radial solution of

$$\begin{aligned} \Delta u &= |v|^{q-1}v & \text{in } [0, a], \\ \Delta v &= |u|^p & \text{in } [0, a], \\ u(0) &= \alpha, & v(0) = -\beta, & u'(0) = v'(0) = 0 \end{aligned} \quad (2.7)$$

such that $uu' < 0$ on $(0, a)$. Then

$$|v(r)| \leq d \max(\beta, \alpha^{(p+1)/(q+1)}), \quad 0 \leq r \leq a, \quad (2.8)$$

where

$$d = \left(1 + \frac{q+1}{p+1}\right)^{1/(q+1)}. \quad (2.9)$$

Proof. We have $0 < u \leq \alpha$ on $[0, a]$. As in Lemma 2.2 we deduce that $v' > 0$ on $(0, a]$. We have

$$\int_0^r (v' \Delta u + u' \Delta v) ds = \int_0^r (|v|^{q-1} v v' + u^p u') ds \quad (2.10)$$

for $r \in [0, a]$. Since

$$\begin{aligned} \int_0^r (v' \Delta u + u' \Delta v) ds &= \int_0^r (u' v')' ds + 2(n-1) \int_0^r \frac{u'(s) v'(s)}{s} ds \\ &= u'(r) v'(r) + 2(n-1) \int_0^r \frac{u'(s) v'(s)}{s} ds, \end{aligned} \quad (2.11)$$

$$\int_0^r (|v|^{q-1} v v' + u^p u') ds = \frac{|v(r)|^{q+1}}{q+1} + \frac{u(r)^{p+1}}{p+1} - \frac{\beta^{q+1}}{q+1} - \frac{\alpha^{p+1}}{p+1}, \quad (2.12)$$

we obtain

$$\frac{|v(r)|^{q+1}}{q+1} + \frac{u(r)^{p+1}}{p+1} = \frac{\beta^{q+1}}{q+1} + \frac{\alpha^{p+1}}{p+1} + u'(r) v'(r) + 2(n-1) \int_0^r \frac{u'(s) v'(s)}{s} ds \quad (2.13)$$

for $r \in [0, a]$, which implies that

$$|v(r)|^{q+1} \leq \beta^{q+1} + \frac{q+1}{p+1} \alpha^{p+1}, \quad 0 \leq r \leq a, \quad (2.14)$$

and the lemma follows. $\square$

3. Proof of Theorem 1.1

(i) Let (u, v) and (w, z) be two nontrivial radial solutions of (1.1). Let s and t be defined by

$$s = 2 \frac{q+1}{pq-1}, \quad t = 2 \frac{p+1}{pq-1}. \quad (3.1)$$

For $\lambda > 0$ we set

$$\tilde{w}(r) = \lambda^s w(\lambda r), \quad \tilde{z}(r) = \lambda^t z(\lambda r), \quad 0 \leq r \leq \frac{R}{\lambda}. \quad (3.2)$$

By Lemma 2.1, $\tilde{w} > 0$ on $[0, R/\lambda)$ and then we have

$$\begin{aligned}\Delta \tilde{w}(r) &= |\tilde{z}(r)|^{q-1} z(r), \quad 0 \leq r \leq \frac{R}{\lambda}, \\ \Delta \tilde{z}(r) &= \tilde{w}(r)^p, \quad 0 \leq r \leq \frac{R}{\lambda}, \\ \tilde{w}\left(\frac{R}{\lambda}\right) &= \tilde{w}'\left(\frac{R}{\lambda}\right) = 0.\end{aligned}\tag{3.3}$$

Choose λ such that $\lambda^s w(0) = u(0)$. Then we have

$$\tilde{w}(0) = u(0).\tag{3.4}$$

We want to show that

$$\tilde{z}(0) = v(0).\tag{3.5}$$

Suppose that $\tilde{z}(0) < v(0)$. If there exists $a \in (0, \min(R, R/\lambda)]$ such that $\tilde{z} - v < 0$ on $[0, a)$ and $(\tilde{z} - v)(a) = 0$, then $\Delta(\tilde{w} - u) < 0$ on $[0, a)$. Equation (3.4) and the maximum principle imply that $\tilde{w} - u < 0$ on $(0, a]$. Therefore $\Delta(\tilde{z} - v) < 0$ on $(0, a]$ and the maximum principle implies that $\tilde{z} - v > (\tilde{z} - v)(a) = 0$ on $[0, a)$, a contradiction. Thus $\tilde{z} - v < 0$ on $[0, \min(R, R/\lambda)]$. Then, as before, we show that $\tilde{w} - u < 0$ on $(0, \min(R, R/\lambda)]$. Since

$$(\tilde{w} - u)\left(\min\left(R, \frac{R}{\lambda}\right)\right) = \begin{cases} -u\left(\frac{R}{\lambda}\right) & \text{if } \lambda > 1, \\ 0 & \text{if } \lambda = 1, \\ \tilde{w}(R) & \text{if } \lambda < 1, \end{cases}\tag{3.6}$$

we deduce that $\lambda > 1$ with the help of Lemma 2.1. Now using the fact that $r^{n-1}(\tilde{w} - u)'(r)$ is decreasing in $[0, R/\lambda]$, we get $(\tilde{w} - u)'(R/\lambda) < 0$. Since $(\tilde{w} - u)'(R/\lambda) = -u'(R/\lambda) > 0$ by Lemma 2.1, we again obtain a contradiction. The case $\tilde{z}(0) > v(0)$ can be handled in the same way. Thus (3.5) is proved.

Now we define the functions U , W , F , and G_n by

$$\begin{aligned}U(r) &= (u(r), v(r)), \quad 0 \leq r \leq R, \\ W(r) &= (\tilde{w}(r), \tilde{z}(r)), \quad 0 \leq r \leq \frac{R}{\lambda}, \\ F(x, y) &= (|y|^{q-1}y, x^p), \quad x \geq 0, y \in \mathbb{R},\end{aligned}\tag{3.7}$$

$$G_n(r, s) = \begin{cases} r - s & \text{if } n = 1, \\ s \ln\left(\frac{r}{s}\right) & \text{if } n = 2, \\ \frac{s}{n-2} \left(1 - \left(\frac{s}{r}\right)^{n-2}\right) & \text{if } n \geq 3 \end{cases}\tag{3.8}$$

for $0 \leq s \leq r$. Using (3.4), (3.5), and the fact that $u'(0) = \tilde{w}'(0) = v'(0) = \tilde{z}'(0) = 0$, we easily obtain

$$U(r) - W(r) = \int_0^r G_n(r, s)(F(U(s)) - F(W(s)))ds \quad (3.9)$$

for $r \in [0, \min(R, R/\lambda)]$. When $p \geq 1$, F is locally Lipschitz continuous, and using Gronwall's lemma we obtain $U = W$ on $[0, \min(R, R/\lambda)]$. When $p \in (0, 1)$, let $a \in (0, \min(R, R/\lambda))$ be fixed. Then $u(0) \geq u(r) \geq u(a) > 0$, $\tilde{w}(0) = u(0) \geq \tilde{w}(r) \geq \tilde{w}(a) > 0$ for $r \in [0, a]$. Since F is locally Lipschitz continuous on $(0, +\infty) \times \mathbb{R}$, as before we obtain $U = W$ on $[0, a]$. By continuity we get $U = W$ on $[0, \min(R, R/\lambda)]$. Now we deduce that $\lambda = 1$ and thus $(u, v) = (w, z)$ on $[0, R]$.

(ii) Let (u, v) be a nontrivial radial solution of problem (1.1). Then, for any $\theta > 0$, $(w, z) = (\theta^q u, \theta v)$ is a nontrivial radial solution of problem (1.1). Now let (w, z) be a nontrivial radial solution of (1.1). Choose $\theta > 0$ such that $\theta^q u(0) = w(0)$ and define $\tilde{w} = \theta^q u$, $\tilde{z} = \theta v$. Then $(\tilde{w}, \tilde{z})$ is a nontrivial radial solution of (1.1) such that $\tilde{w}(0) = w(0)$. Arguing as in part (i), we show that $\tilde{z}(0) = z(0)$ and that $(\tilde{w}, \tilde{z}) = (w, z)$.

Remark 3.1. Our technique also applies when there is a homogeneous dependence on the radius $|x|$. More precisely, for $p > 0$, $q \geq 1$, and $pq \neq 1$, the following system

$$\begin{aligned} \Delta u &= |x|^\mu |v|^{q-1} v \quad \text{in } B_R, \\ \Delta v &= |x|^\nu |u|^p \quad \text{in } B_R, \\ u &= \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_R, \end{aligned} \quad (3.10)$$

where $\mu, \nu \geq 0$, has at most one nontrivial radial solution (u, v) . Indeed, the arguments are the same with s and t in (2.1) replaced by

$$s = \frac{2(q+1) + \nu + q\mu}{pq - 1}, \quad t = \frac{2(p+1) + \mu + p\nu}{pq - 1}. \quad (3.11)$$

Now let $p > 0$, $q \geq 1$ with $pq = 1$. Assume that problem (3.10) has a nontrivial radial solution (u, v) . Then all nontrivial radial solutions are given by $(\theta^q u, \theta v)$, where $\theta > 0$ is an arbitrary constant.

4. Proof of Theorem 1.2

(i) Let $(u, v) \in (C^2(\overline{B_R}))^2$ be a solution of problem (1.1) such that $u \geq 0$ in B_R . We have $x \cdot \nu(x) = R$ for all $x \in \partial B_R$. Multiplying the first equation in (1.1) by $x \cdot \nabla v$ and integrating over B_R , we get

$$\int_{B_R} (x \cdot \nabla v) \Delta u \, dx = \int_{B_R} (x \cdot \nabla v) |v|^{q-1} v \, dx. \quad (4.1)$$

Integrating by parts, we obtain

$$\int_{B_R} (x \cdot \nabla v) |v|^{q-1} v \, dx = -\frac{n}{q+1} \int_{B_R} |v|^{q+1} \, dx + \frac{R}{q+1} \int_{\partial B_R} |v|^{q+1} \, d\sigma. \quad (4.2)$$

Similarly we get

$$\int_{B_R} (x \cdot \nabla u) \Delta v \, dx = \int_{B_R} (x \cdot \nabla u) u^p \, dx = -\frac{n}{p+1} \int_{B_R} u^{p+1} \, dx. \quad (4.3)$$

Now we have

$$\int_{B_R} ((x \cdot \nabla v) \Delta u + (x \cdot \nabla u) \Delta v) \, dx = (n-2) \int_{B_R} \nabla u \cdot \nabla v \, dx. \quad (4.4)$$

Then we deduce that

$$\frac{R}{q+1} \int_{\partial B_R} |v|^{q+1} \, d\sigma = \frac{n}{q+1} \int_{B_R} |v|^{q+1} \, dx + \frac{n}{p+1} \int_{B_R} u^{p+1} \, dx + (n-2) \int_{B_R} \nabla u \cdot \nabla v \, dx. \quad (4.5)$$

Since

$$\begin{aligned} \int_{B_R} \nabla u \cdot \nabla v \, dx &= - \int_{B_R} v \Delta u \, dx = - \int_{B_R} |v|^{q+1} \, dx, \\ \int_{B_R} \nabla u \cdot \nabla v \, dx &= - \int_{B_R} u \Delta v \, dx = - \int_{B_R} u^{p+1} \, dx, \end{aligned} \quad (4.6)$$

we can write

$$\frac{R}{q+1} \int_{\partial B_R} |v|^{q+1} \, d\sigma = n \left(\frac{1}{p+1} + \frac{1}{q+1} - \frac{n-2}{n} \right) \int_{B_R} |v|^{q+1} \, dx. \quad (4.7)$$

Using (1.4) we deduce that $v = 0$ on ∂B_R . The maximum principle implies that $v \leq 0$ in B_R . Therefore $\Delta u \leq 0$ in B_R . The Hopf boundary point lemma implies that $u = 0$ in B_R and (i) is proved.

(ii) follows from (i) and Lemma 2.1.

Remark 4.1. Clearly Theorem 1.2(i) can be extended to more general domains and more general nonlinearities as in [2, 11, 12] and Theorem 1.2(ii) can be extended to more general nonlinearities.

5. Proof of Theorem 1.3

We will use a two-dimensional shooting argument for the ordinary differential equations associated to radial solutions of (1.1) [3, 5, 7, 15, 16]. We consider the one-dimensional (singular if $n \geq 2$) initial value problem (2.2) where $\alpha > 0$, $\beta > 0$.

We will need a series of lemmas. We begin with a standard local existence and uniqueness result.

LEMMA 5.1. *For any $\alpha > 0$, $\beta > 0$ there exists $T = T(\alpha, \beta) > 0$ such that problem (2.2) on $[0, T]$ has a unique solution $(u, v) \in (C^2[0, T])^2$.*

Proof. Let $\alpha, \beta > 0$ be given. Choose $T = T(\alpha, \beta) > 0$ such that

$$T = \min \left(\left(\frac{n\alpha}{\beta^q} \right)^{1/2}, \left(\frac{n\beta}{\alpha^p} \right)^{1/2} \right), \quad (5.1)$$

and consider the set of functions

$$Z = \left\{ (u, v) \in (C[0, T])^2; \frac{\alpha}{2} \leq u(r) \leq \alpha, -\beta \leq v(r) \leq -\frac{\beta}{2} \text{ for } 0 \leq r \leq T \right\}. \quad (5.2)$$

Clearly Z is a bounded closed convex subset of the Banach space $(C[0, T])^2$ endowed with the norm $\|(u, v)\| = \max(\|u\|_\infty, \|v\|_\infty)$. Define

$$L(u, v)(r) = \left(\alpha + \int_0^r G_n(r, s) |v(s)|^{q-1} v(s) ds, -\beta + \int_0^r G_n(r, s) |u(s)|^p ds \right) \quad (5.3)$$

for $r \in [0, T]$ and $(u, v) \in (C[0, T])^2$, where G_n is defined in (3.8). It is easily verified that L is a compact operator mapping Z into itself, and so there exists $(u, v) \in Z$ such that $(u, v) = L(u, v)$ by the Schauder fixed point theorem. Clearly $(u, v) \in (C^2[0, T])^2$ and (u, v) is a solution of (2.2) on $[0, T]$. Since the right-hand side in (2.2) is Lipschitz continuous in $(u, v) \in [\alpha/2, \alpha] \times [-\beta, -\beta/2]$, the uniqueness follows. $\square$

Remark 5.2. Notice that $u(r) > 0$ and $v(r) < 0$ for $r \in [0, T]$. Then direct integration of the system (2.2) implies that $u' < 0$ and $v' > 0$ in $(0, T]$.

In view of Lemma 5.1, for any $\alpha, \beta > 0$ problem (2.2) has a unique local solution: let $[0, R_{\alpha, \beta})$ denote the maximum interval of existence of that solution ($R_{\alpha, \beta} = +\infty$ possibly). If $0 < p < 1$, the uniqueness of the solution could fail at any point r where $u(r) = 0$. In this case, $R_{\alpha, \beta}$ could also depend on the particular solution itself. Define

$$P_{\alpha, \beta} = \{s \in (0, R_{\alpha, \beta}); u(\alpha, \beta, r)u'(\alpha, \beta, r) < 0 \ \forall r \in (0, s]\}, \quad (5.4)$$

where $(u(\alpha, \beta, \cdot), v(\alpha, \beta, \cdot))$ is a solution of (2.2) in $[0, R_{\alpha, \beta})$. $P_{\alpha, \beta} \neq \emptyset$ by Remark 5.2. Set

$$r_{\alpha, \beta} = \sup P_{\alpha, \beta}. \quad (5.5)$$

Notice that the solution is unique on $[0, r_{\alpha, \beta}]$, so $r_{\alpha, \beta}$ depends only on α, β .

LEMMA 5.3. $u'(\alpha, \beta, r) < 0$ for $r \in (0, r_{\alpha, \beta})$ and $v'(\alpha, \beta, r) > 0$ for $r \in (0, R_{\alpha, \beta})$.

Proof. The first assertion follows from the definition of $r_{\alpha, \beta}$. Since $u(\alpha, \beta, r) > 0$ for $r \in [0, r_{\alpha, \beta})$, integrating the second equation in (2.2) from 0 to $r \in (0, R_{\alpha, \beta})$ we obtain $v'(\alpha, \beta, r) > 0$ for $r \in (0, R_{\alpha, \beta})$. $\square$

LEMMA 5.4. For any $\alpha, \beta > 0$, $r_{\alpha, \beta} < \infty$.

Proof. Assume that $r_{\alpha, \beta} = \infty$. We easily get a contradiction when $n = 1$ or 2 . Now if $n \geq 3$, we set $z = -v$. By Lemma 2.2, $z > 0$ on $[0, \infty)$ and we have

$$\begin{aligned} -\Delta u &= z^q, & r > 0, \\ -\Delta z &= u^p, & r > 0. \end{aligned} \quad (5.6)$$

Since p, q satisfy (1.5), we obtain a contradiction with the help of the nonexistence results established in [9, 10, 13, 14]. $\square$

LEMMA 5.5. *For any $a \in [T(\alpha, \beta), r_{\alpha, \beta})$, there exists $b = b(\alpha, \beta, a) > 0$ such that the maximal extension of (u, v) includes the interval $[0, a + b]$. Moreover,*

$$b(\alpha, \beta, a) = \frac{m(\alpha, \beta)}{a + \sqrt{a^2 + m(\alpha, \beta)}}, \quad (5.7)$$

where

$$m(\alpha, \beta) = \min \left(\frac{n\beta}{2^{p-1}\alpha^p}, \frac{n\alpha}{2^{q-1}d^q (\max(\beta, \alpha^{(p+1)/(q+1)}))^q} \right), \quad (5.8)$$

with d given in Lemma 2.3.

Proof. Lemma 5.5 is essentially a local existence result, with initial data $u(a), v(a), u'(a), v'(a)$ at $r = a$. Let

$$W = \left\{ (u, v) \in (C[a, a + b])^2; |u(r) - u(a)| \leq \alpha, 0 \leq v(r) - v(a) \leq \beta \text{ for } a \leq r \leq a + b \right\}, \quad (5.9)$$

where $b = b(\alpha, \beta, a)$ is given in the lemma. W is a bounded closed convex subset of the Banach space $(C[a, a + b])^2$ equipped with the norm $\|(u, v)\| = \max(\|u\|_\infty, \|v\|_\infty)$. Consider the mapping $S(u, v) = (S_1(u, v), S_2(u, v))$ on $(C[a, a + b])^2$ given by

$$\begin{aligned} S_1(u, v)(r) &= u(a) + \int_a^r \frac{dt}{t^{n-1}} \int_0^t s^{n-1} |v(s)|^{q-1} v(s) ds, \\ S_2(u, v)(r) &= v(a) + \int_a^r \frac{dt}{t^{n-1}} \int_0^t s^{n-1} |u(s)|^p ds \end{aligned} \quad (5.10)$$

for $a \leq r \leq a + b$, where we also denote by u, v the unique solution of (2.2) on $[0, a]$. Let $(u, v) \in W$. Using Lemma 5.3, we have

$$|u(s)| \leq u(a) + \alpha \leq 2\alpha, \quad s \in [a, a + b]. \quad (5.11)$$

Therefore we get

$$0 \leq S_2(u, v)(r) - v(a) \leq 2^{p-1}\alpha^p \frac{r^2 - a^2}{n} \leq \beta, \quad r \in [a, a + b]. \quad (5.12)$$

By Lemma 2.3 we have

$$|v(s)| \leq |v(a)| + \beta \leq 2d \max(\beta, \alpha^{(p+1)/(q+1)}), \quad s \in [a, a + b]. \quad (5.13)$$

Therefore for $a \leq r \leq a + b$, we obtain

$$|S_1(u, v)(r) - u(a)| \leq 2^{q-1}d^q \left(\max(\beta, \alpha^{(p+1)/(q+1)}) \right)^q \frac{r^2 - a^2}{n} \leq \alpha. \quad (5.14)$$

We have thus proved that $S(W) \subset W$. Since S is a compact operator, there exists $(u, v) \in W$ such that $(u, v) = S(u, v)$ by the Schauder fixed point theorem. Clearly $(u, v) \in (C^2[a, a+b])^2$ and (u, v) is a solution of (2.2) on $[a, a+b]$ which extends the solution (u, v) on $[0, a]$. $\square$

LEMMA 5.6. For any $\alpha, \beta > 0$,

$$R_{\alpha, \beta} \geq r_{\alpha, \beta} + \frac{m(\alpha, \beta)}{r_{\alpha, \beta} + \sqrt{r_{\alpha, \beta}^2 + m(\alpha, \beta)}}. \quad (5.15)$$

Proof. By Lemma 5.5, for any $a \in (T(\alpha, \beta), r_{\alpha, \beta})$ we have

$$R_{\alpha, \beta} > a + \frac{m(\alpha, \beta)}{a + \sqrt{a^2 + m(\alpha, \beta)}}. \quad (5.16)$$

The lemma follows by letting $a \rightarrow r_{\alpha, \beta}$. $\square$

PROPOSITION 5.7. For any $\alpha > 0$, there exists a unique $\beta > 0$ such that $u(\alpha, \beta, r_{\alpha, \beta}) = u'(\alpha, \beta, r_{\alpha, \beta}) = 0$.

Proof. We first prove the uniqueness. Let $\alpha > 0$ be fixed. Suppose that there exist $\beta > \gamma > 0$ such that $u(\alpha, \beta, r_{\alpha, \beta}) = u'(\alpha, \beta, r_{\alpha, \beta}) = u(\alpha, \gamma, r_{\alpha, \gamma}) = u'(\alpha, \gamma, r_{\alpha, \gamma}) = 0$. Using the same arguments as in the proof of (3.5) we obtain a contradiction.

Now we prove the existence. Suppose that there exists $\alpha > 0$ such that for any $\beta > 0$ $u(\alpha, \beta, r_{\alpha, \beta}) > 0$ or $u'(\alpha, \beta, r_{\alpha, \beta}) < 0$. Define the sets

$$\begin{aligned} B &= \{\beta > 0; u(\alpha, \beta, r_{\alpha, \beta}) = 0, u'(\alpha, \beta, r_{\alpha, \beta}) < 0\}, \\ C &= \{\beta > 0; u(\alpha, \beta, r_{\alpha, \beta}) > 0, u'(\alpha, \beta, r_{\alpha, \beta}) = 0\}. \end{aligned} \quad (5.17)$$

$\square$

The proof of the proposition is completed by using the next two lemmas which contradict the fact that

$$(0, +\infty) = B \cup C. \quad (5.18)$$

LEMMA 5.8. (i) Suppose $B \neq \emptyset$. Then there exists $m > 0$ such that $m \leq \inf B$.

(ii) Suppose $C \neq \emptyset$. Then there exists $M > 0$ such that $M \geq \sup C$.

LEMMA 5.9. B and C are open.

Proof of Lemma 5.8. We have

$$u(\alpha, \beta, r) = \alpha + \int_0^r G_n(r, s) |v(\alpha, \beta, s)|^{q-1} v(\alpha, \beta, s) ds, \quad 0 \leq r < R_{\alpha, \beta}, \quad (5.19)$$

$$v(\alpha, \beta, r) = -\beta + \int_0^r G_n(r, s) |u(\alpha, \beta, s)|^p ds, \quad 0 \leq r < R_{\alpha, \beta}. \quad (5.20)$$

(i) Let $\beta \in B$. Assume first that $v(\alpha, \beta, \cdot) < 0$ on $[0, r_{\alpha, \beta}]$. Then Lemma 5.3 and (5.19) imply

$$r_{\alpha, \beta} \geq \left(\frac{2n\alpha}{\beta^q} \right)^{1/2}. \quad (5.21)$$

Now, if there exists $s_{\alpha, \beta} \in [0, r_{\alpha, \beta}]$ such that $v(\alpha, \beta, s_{\alpha, \beta}) = 0$, Lemma 5.3 implies that $-\beta \leq v(\alpha, \beta, \cdot) < 0$ in $[0, s_{\alpha, \beta}]$ and $v(\alpha, \beta, \cdot) > 0$ in $(s_{\alpha, \beta}, r_{\alpha, \beta}]$. Then from (5.19) we get

$$\begin{aligned} \alpha &= - \int_0^{r_{\alpha, \beta}} G_n(r_{\alpha, \beta}, s) |v(\alpha, \beta, s)|^{q-1} v(\alpha, \beta, s) ds \\ &\leq \int_0^{s_{\alpha, \beta}} G_n(r_{\alpha, \beta}, s) |v(\alpha, \beta, s)|^q ds \leq \beta^q \int_0^{s_{\alpha, \beta}} G_n(r_{\alpha, \beta}, s) ds \leq \beta^q \frac{r_{\alpha, \beta}^2}{2n}, \end{aligned} \quad (5.22)$$

and (5.21) still holds.

Suppose that $\inf B = 0$ and let (β_j) be a sequence in B decreasing to zero. Then $r_{\alpha, \beta_j} \rightarrow +\infty$ by (5.21). Let $r > 0$ be fixed. We can assume that $r_{\alpha, \beta_j} > r$ for all j . If $v(\alpha, \beta_j, s) < 0$ for $s \in [0, r]$, we have

$$u(\alpha, \beta_j, r) = \alpha - \int_0^r G_n(r, s) |v(\alpha, \beta_j, s)|^q ds \geq \alpha - \frac{r^2 \beta_j^q}{2n}. \quad (5.23)$$

If $s_{\alpha, \beta_j} < r$, we have

$$\begin{aligned} u(\alpha, \beta_j, r) &= \alpha - \int_0^{s_{\alpha, \beta_j}} G_n(r, s) |v(\alpha, \beta_j, s)|^q ds + \int_{s_{\alpha, \beta_j}}^r G_n(r, s) v(\alpha, \beta_j, s)^q ds \\ &\geq \alpha - \int_0^{s_{\alpha, \beta_j}} G_n(r, s) |v(\alpha, \beta_j, s)|^q ds \\ &\geq \alpha - \beta_j^q \int_0^{s_{\alpha, \beta_j}} G_n(r, s) ds \geq \alpha - \frac{r^2 \beta_j^q}{2n}. \end{aligned} \quad (5.24)$$

Therefore using Lemma 5.3 we obtain

$$u(\alpha, \beta_j, s) \geq \alpha - \frac{r^2 \beta_j^q}{2n} \quad \text{for } s \in [0, r], \quad (5.25)$$

from which we deduce that

$$u(\alpha, \beta_j, s) \geq \frac{\alpha}{2} \quad (5.26)$$

for $s \in [0, r]$ and j large. From (5.20) we get

$$v(\alpha, \beta_j, r) \geq -\beta_j + \frac{r^2 \alpha^p}{2^{p+1} n} \quad (5.27)$$

for j large. Thus if we choose r such that

$$-\beta_j + \frac{r^2 \alpha^p}{2^{p+1} n} \geq 1, \quad (5.28)$$

using Lemma 5.3 we get

$$v(\alpha, \beta_j, s) \geq 1 \quad (5.29)$$

for $r \leq s \leq r_{\alpha, \beta_j}$ and j large. We also have

$$-\beta_j \leq v(\alpha, \beta_j, s) \leq -\beta_j + \frac{r^2 \alpha^p}{2n} \quad (5.30)$$

for $s \in [0, r]$. Therefore there exists $c > 0$ such that

$$|v(\alpha, \beta_j, s)| \leq c \quad (5.31)$$

for $s \in [0, r]$ and all j . There exists $k > 0$ such that

$$\int_r^{r_{\alpha, \beta_j}} G_n(r_{\alpha, \beta_j}, s) ds \geq k r_{\alpha, \beta_j}^2 \quad (5.32)$$

for j large. Now we write

$$\begin{aligned} \alpha &= - \int_0^{r_{\alpha, \beta_j}} G_n(r_{\alpha, \beta_j}, s) |v(\alpha, \beta_j, s)|^{q-1} v(\alpha, \beta_j, s) ds \\ &= - \int_0^r G_n(r_{\alpha, \beta_j}, s) |v(\alpha, \beta_j, s)|^{q-1} v(\alpha, \beta_j, s) ds \\ &\quad - \int_r^{r_{\alpha, \beta_j}} G_n(r_{\alpha, \beta_j}, s) v(\alpha, \beta_j, s)^q ds \\ &\leq c^q \int_0^r G_n(r_{\alpha, \beta_j}, s) ds - \int_r^{r_{\alpha, \beta_j}} G_n(r_{\alpha, \beta_j}, s) ds \\ &\leq c^q r r_{\alpha, \beta_j} - k r_{\alpha, \beta_j}^2 \end{aligned} \quad (5.33)$$

for j large, where we have used the fact that $G_n(r_{\alpha, \beta_j}, s) \leq r_{\alpha, \beta_j} - s$ for $0 \leq s \leq r_{\alpha, \beta_j}$. Since the last term above tends to $-\infty$, we get a contradiction.

(ii) Let $\beta \in C$. We claim that $v(\alpha, \beta, r_{\alpha, \beta}) > 0$. If not, by Lemma 5.3 we have $\Delta u(\alpha, \beta, \cdot) < 0$ on $[0, r_{\alpha, \beta}]$ for some $\beta \in C$. Since $u'(\alpha, \beta, 0) = 0$, we obtain $u'(\alpha, \beta, r_{\alpha, \beta}) < 0$, a contradiction. Therefore (5.20) implies

$$\beta < \int_0^{r_{\alpha, \beta}} G_n(r_{\alpha, \beta}, s) u(\alpha, \beta, s)^p ds \quad (5.34)$$

for $\beta \in C$. Suppose that $\sup C = +\infty$ and let (β_j) be a sequence in C increasing to $+\infty$. Since $0 < u(\alpha, \beta_j, r) \leq \alpha$ for $r \in [0, r_{\alpha, \beta_j}]$, (5.34) implies that $r_{\alpha, \beta_j} \rightarrow +\infty$ as $j \rightarrow +\infty$. Then we can assume that $r_{\alpha, \beta_j} \geq 1$ and that $\alpha^p \leq \beta_j$ for all j . From (5.20) we get

$$-\beta_j \leq v(\alpha, \beta_j, r) \leq -\frac{2n-1}{2n} \beta_j \leq -\frac{\beta_j}{2} \quad \text{for } r \in [0, 1], \quad (5.35)$$

and using (5.19) we deduce that $u(\alpha, \beta_j, 1) \leq \alpha - \beta_j^q / n 2^{q+1}$. But then $u(\alpha, \beta_j, 1) < 0$ for j large and we reach a contradiction. $\square$

Remark 5.10. The proof above shows that, when $\beta \in C$, there exists $s_{\alpha,\beta} \in (0, r_{\alpha,\beta})$ such that $v(\alpha, \beta, \cdot) < 0$ on $[0, s_{\alpha,\beta})$ and $v(\alpha, \beta, \cdot) > 0$ on $(s_{\alpha,\beta}, r_{\alpha,\beta}]$. When $\beta \in B$, $s_{\alpha,\beta}$ may not exist.

Proof of Lemma 5.9

Case 1 ($p \geq 1$). Then the right-hand side of (2.2) is Lipschitz continuous. Let $\beta \in B$. We have $u(\alpha, \beta, r_{\alpha,\beta}) = 0$ and $u'(\alpha, \beta, r_{\alpha,\beta}) < 0$. Therefore we can find $\varepsilon > 0$ such that

$$u(\alpha, \beta, r_{\alpha,\beta} + \varepsilon) < 0, \quad u'(\alpha, \beta, r_{\alpha,\beta} + \varepsilon) < 0. \quad (5.36)$$

But then by continuous dependence on initial data, there exists $\eta > 0$ such that

$$u(\alpha, \gamma, r_{\alpha,\beta} + \varepsilon) < 0, \quad u'(\alpha, \gamma, r_{\alpha,\beta} + \varepsilon) < 0 \quad (5.37)$$

for $|\gamma - \beta| < \eta$. The first inequality in (5.37) implies that there exists $x \in (0, r_{\alpha,\beta} + \varepsilon)$ such that $u(\alpha, \gamma, x) = 0$ and $u(\alpha, \gamma, r) > 0$ for $r \in [0, x)$. $\Delta v(\alpha, \gamma, r) > 0$ for $r \in [0, x)$ and $\Delta v(\alpha, \gamma, r) \geq 0$ for $r \in [x, r_{\alpha,\beta} + \varepsilon]$. Then $v'(\alpha, \gamma, r) > 0$ for $r \in (0, r_{\alpha,\beta} + \varepsilon]$ and $v(\alpha, \gamma, \cdot)$ is increasing on $[0, r_{\alpha,\beta} + \varepsilon]$. We deduce that $\Delta u(\alpha, \gamma, \cdot)$ is increasing on $[0, r_{\alpha,\beta} + \varepsilon]$. If $\Delta u(\alpha, \gamma, r_{\alpha,\beta} + \varepsilon) \leq 0$, then $u'(\alpha, \gamma, r) < 0$ for $r \in (0, r_{\alpha,\beta} + \varepsilon]$. If $\Delta u(\alpha, \gamma, r_{\alpha,\beta} + \varepsilon) > 0$, then there exists $s_{\alpha,\gamma} \in (0, r_{\alpha,\beta} + \varepsilon)$ such that $\Delta u(\alpha, \gamma, \cdot) < 0$ in $[0, s_{\alpha,\gamma})$ and $\Delta u(\alpha, \gamma, \cdot) > 0$ in $(s_{\alpha,\gamma}, r_{\alpha,\beta} + \varepsilon]$. We deduce that $u'(\alpha, \gamma, \cdot)$ is decreasing (resp., increasing) in $[0, s_{\alpha,\gamma}]$ (resp., $[s_{\alpha,\gamma}, r_{\alpha,\beta} + \varepsilon]$). Since $u'(\alpha, \gamma, 0) = 0$, the second inequality in (5.37) implies that $u'(\alpha, \gamma, r) < 0$ for $r \in (0, r_{\alpha,\beta} + \varepsilon]$. Therefore $x = r_{\alpha,\gamma}$ for $|\gamma - \beta| < \eta$ and $(\beta - \eta, \beta + \eta) \subset B$. Thus B is open. Now let $\beta \in C$. We have $u(\alpha, \beta, r_{\alpha,\beta}) > 0$ and $u'(\alpha, \beta, r_{\alpha,\beta}) = 0$. By Remark 5.10, we have $v(\alpha, \beta, r_{\alpha,\beta}) > 0$, hence $\Delta u(\alpha, \beta, r_{\alpha,\beta}) = u''(\alpha, \beta, r_{\alpha,\beta}) > 0$. Therefore we can find $\varepsilon > 0$ such that

$$u(\alpha, \beta, r) > 0, \quad r \in [0, r_{\alpha,\beta} + \varepsilon], \quad u'(\alpha, \beta, r_{\alpha,\beta} + \varepsilon) > 0. \quad (5.38)$$

Then by continuous dependence on initial data, there exists $\eta > 0$ such that

$$u(\alpha, \gamma, r) > 0, \quad r \in [0, r_{\alpha,\beta} + \varepsilon], \quad u'(\alpha, \gamma, r_{\alpha,\beta} + \varepsilon) > 0 \quad (5.39)$$

for $|\gamma - \beta| < \eta$. The second inequality in (5.39) implies that there exists $x \in (0, r_{\alpha,\beta} + \varepsilon)$ such that $u'(\alpha, \gamma, x) = 0$ and $u'(\alpha, \gamma, r) < 0$ for $r \in (0, x)$. Therefore $x = r_{\alpha,\gamma}$ for $|\gamma - \beta| < \eta$ and $(\beta - \eta, \beta + \eta) \subset C$. Thus C is open.

Case 2 ($0 < p < 1$). We first show that C is open. Indeed let $\beta \in C$. Since $u(\alpha, \beta, r) > 0$ for $r \in [0, r_{\alpha,\beta}]$, the system (2.2) is Lipschitz continuous in u and v when u is in a neighborhood of the interval $[u(\alpha, \beta, r_{\alpha,\beta}), \alpha]$ in $(0, \infty)$, and the solution $u(\alpha, \beta, \cdot)$, $v(\alpha, \beta, \cdot)$ can be uniquely extended to $[0, r_{\alpha,\beta} + t]$ for some $t > 0$, with $u(\alpha, \beta, r) > 0$ for $r \in [0, r_{\alpha,\beta} + t]$. Then we can argue as in Case 1. Now we show that B is open. As in [15], this case is much more difficult. We begin with the following two steps. Let $\beta \in B$.

Step 1. There exists $c > 0$ and $\eta > 0$ such that when $|\beta - \gamma| < \eta$, the solutions $u(\alpha, \gamma, \cdot)$, $v(\alpha, \gamma, \cdot)$, and $u(\alpha, \beta, \cdot)$, $v(\alpha, \beta, \cdot)$ are defined on $[0, r_{\alpha,\beta} + c]$.

By Lemma 5.6, $u(\alpha, \beta, \cdot)$, $v(\alpha, \beta, \cdot)$ can be extended to the interval $[0, r_{\alpha, \beta} + b(\alpha, \beta, r_{\alpha, \beta})]$ where

$$b(\alpha, \beta, r_{\alpha, \beta}) = \frac{m(\alpha, \beta)}{r_{\alpha, \beta} + \sqrt{r_{\alpha, \beta}^2 + m(\alpha, \beta)}}. \quad (5.40)$$

Fix $\omega \in (0, r_{\alpha, \beta} - T(\alpha, \beta))$ and $\mu = r_{\alpha, \beta} - \omega$. Then $T(\alpha, \beta) < \mu < r_{\alpha, \beta}$ and by Lemma 5.3

$$0 < u(\alpha, \beta, \mu) \leq u(\alpha, \beta, r) \leq \alpha, \quad 0 \leq r \leq \mu. \quad (5.41)$$

Since the system (2.2) is Lipschitz continuous in u and v when u is in a neighborhood of the interval $[u(\alpha, \beta, \mu), \alpha]$ in $(0, \infty)$, the continuous dependence on initial data implies that there exists $\eta > 0$ such that when $|\gamma - \beta| < \eta$ the solution $u(\alpha, \gamma, \cdot)$, $v(\alpha, \gamma, \cdot)$ is defined on $[0, \mu]$ and $u(\alpha, \gamma, r) > 0$ for $r \in [0, \mu]$, $u'(\alpha, \gamma, r) < 0$ for $r \in (0, \mu]$, hence $r_{\alpha, \gamma} > \mu$. By taking η smaller if necessary, we can assume that $T(\alpha, \gamma) < \mu$, hence $T(\alpha, \gamma) < \mu < r_{\alpha, \gamma}$. By Lemma 5.5 we can extend $u(\alpha, \gamma, \cdot)$, $v(\alpha, \gamma, \cdot)$ to $[0, \mu + b(\alpha, \gamma, \mu)]$. By taking η smaller if necessary, we can assume that

$$b(\alpha, \gamma, \mu) > \frac{b(\alpha, \beta, \mu)}{2} > \frac{b(\alpha, \beta, r_{\alpha, \beta})}{2} = 2c. \quad (5.42)$$

Thus if we choose ω to satisfy also $\omega \leq c$, we get

$$\mu + b(\alpha, \gamma, \mu) = r_{\alpha, \beta} - \omega + b(\alpha, \gamma, \mu) \geq r_{\alpha, \beta} + c. \quad (5.43)$$

Thus $u(\alpha, \gamma, \cdot)$, $v(\alpha, \gamma, \cdot)$ extend to the interval $[0, r_{\alpha, \beta} + c]$ and $c < b(\alpha, \beta, r_{\alpha, \beta})$ so that $u(\alpha, \beta, \cdot)$, $v(\alpha, \beta, \cdot)$ also exist on $[0, r_{\alpha, \beta} + c]$.

Step 2. We claim that there exist $\varepsilon \in (0, c)$ and $\delta \in (0, \eta)$ such that

$$|u'(\alpha, \gamma, r) - u'(\alpha, \beta, r_{\alpha, \beta})| \leq \frac{1}{2} |u'(\alpha, \beta, r_{\alpha, \beta})| \quad (5.44)$$

(recall that $u'(\alpha, \beta, r_{\alpha, \beta}) < 0$) when $|\gamma - \beta| < \delta$ and $|r - r_{\alpha, \beta}| \leq \varepsilon$. Let $\varepsilon \in (0, c)$, $|\gamma - \beta| < \eta$, and $r \in [r_{\alpha, \beta} - \varepsilon, r_{\alpha, \beta} + \varepsilon]$. By Step 1 and integration of (2.2) we have

$$\begin{aligned} & u'(\alpha, \gamma, r) - u'(\alpha, \beta, r_{\alpha, \beta}) \\ &= u'(\alpha, \gamma, r) - u'(\alpha, \beta, r) + u'(\alpha, \beta, r) - u'(\alpha, \beta, r_{\alpha, \beta}) \\ &= (u'(\alpha, \gamma, r_{\alpha, \beta} - \varepsilon) - u'(\alpha, \beta, r_{\alpha, \beta} - \varepsilon)) \frac{(r_{\alpha, \beta} - \varepsilon)^{n-1}}{r^{n-1}} \\ & \quad + \int_{r_{\alpha, \beta} - \varepsilon}^r \frac{s^{n-1}}{r^{n-1}} \left(|v(\alpha, \gamma, s)|^{q-1} v(\alpha, \gamma, s) - |v(\alpha, \beta, s)|^{q-1} v(\alpha, \beta, s) \right) ds \\ & \quad + u'(\alpha, \beta, r_{\alpha, \beta}) \left(\frac{r_{\alpha, \beta}^{n-1}}{r^{n-1}} - 1 \right) + \int_{r_{\alpha, \beta}}^r \frac{s^{n-1}}{r^{n-1}} |v(\alpha, \beta, s)|^{q-1} v(\alpha, \beta, s) ds. \end{aligned} \quad (5.45)$$

We deduce that

$$\begin{aligned}
 & |u'(\alpha, \gamma, r) - u'(\alpha, \beta, r_{\alpha, \beta})| \\
 & \leq |u'(\alpha, \gamma, r_{\alpha, \beta} - \varepsilon) - u'(\alpha, \beta, r_{\alpha, \beta} - \varepsilon)| + |u'(\alpha, \beta, r_{\alpha, \beta})| \left| \frac{r_{\alpha, \beta}^{n-1}}{r^{n-1}} - 1 \right| \\
 & \quad + \int_{r_{\alpha, \beta} - \varepsilon}^r \frac{s^{n-1}}{r^{n-1}} |v(\alpha, \gamma, s)|^q ds + \int_{r_{\alpha, \beta} - \varepsilon}^{r_{\alpha, \beta}} \frac{s^{n-1}}{r^{n-1}} |v(\alpha, \beta, s)|^q ds.
 \end{aligned} \tag{5.46}$$

The proof of Lemma 5.5 gives the following estimate for $|\gamma - \beta| < \eta$:

$$|v(\alpha, \gamma, r)| \leq 2d \max(\gamma, \alpha^{(p+1)/(q+1)}), \quad r_{\alpha, \beta} - \varepsilon \leq r \leq r_{\alpha, \beta} + \varepsilon. \tag{5.47}$$

By making ε smaller if necessary we have

$$\begin{aligned}
 & \int_{r_{\alpha, \beta} - \varepsilon}^r \frac{s^{n-1}}{r^{n-1}} |v(\alpha, \gamma, s)|^q ds + \int_{r_{\alpha, \beta} - \varepsilon}^{r_{\alpha, \beta}} \frac{s^{n-1}}{r^{n-1}} |v(\alpha, \beta, s)|^q ds \leq \frac{1}{4} |u'(\alpha, \beta, r_{\alpha, \beta})|, \\
 & \left| \frac{r_{\alpha, \beta}^{n-1}}{r^{n-1}} - 1 \right| \leq \frac{1}{8}
 \end{aligned} \tag{5.48}$$

for $r_{\alpha, \beta} - \varepsilon \leq r \leq r_{\alpha, \beta} + \varepsilon$. Then from (5.46) we obtain

$$|u'(\alpha, \gamma, r) - u'(\alpha, \beta, r_{\alpha, \beta})| \leq |u'(\alpha, \gamma, r_{\alpha, \beta} - \varepsilon) - u'(\alpha, \beta, r_{\alpha, \beta} - \varepsilon)| + \frac{3}{8} |u'(\alpha, \beta, r_{\alpha, \beta})| \tag{5.49}$$

for $|\gamma - \beta| < \eta$ and $|r - r_{\alpha, \beta}| \leq \varepsilon$. Now let ε be fixed. By continuous dependence on initial data and the fact that $u(\alpha, \beta, r) > u(\alpha, \beta, r_{\alpha, \beta} - \varepsilon)$ for $r \in [0, r_{\alpha, \beta} - \varepsilon)$, we can choose $\delta \in (0, \eta)$ such that

$$|u'(\alpha, \gamma, r_{\alpha, \beta} - \varepsilon) - u'(\alpha, \beta, r_{\alpha, \beta} - \varepsilon)| \leq \frac{1}{8} |u'(\alpha, \beta, r_{\alpha, \beta})| \tag{5.50}$$

for $|\gamma - \beta| < \delta$ and our claim follows.

Now assume that B is not open. Equation (5.18) implies that there exist $\beta \in B$ and a sequence (β_j) in C such that $\beta_j \rightarrow \beta$ and $r_{\alpha, \beta_j} \rightarrow T \in [0, \infty]$. Assume first that $T > r_{\alpha, \beta}$. Then we can assume that there exists $c' \in (0, c)$ such that $r_{\alpha, \beta_j} \geq r_{\alpha, \beta} + c'$ for all j . We can also assume that ε in Step 2 is such that $0 < \varepsilon < c'$. Since $u(\alpha, \beta, r_{\alpha, \beta}) = 0$ and $u'(\alpha, \beta, r_{\alpha, \beta}) < 0$, there exists $0 < \varepsilon' \leq \varepsilon$ such that

$$0 < u(\alpha, \beta, r_{\alpha, \beta} - \varepsilon') < \frac{1}{4} |u'(\alpha, \beta, r_{\alpha, \beta})| \varepsilon. \tag{5.51}$$

By continuous dependence on initial data, there exists $\delta' \in (0, \delta)$ such that

$$u(\alpha, \gamma, r_{\alpha, \beta} - \varepsilon') < 2u(\alpha, \beta, r_{\alpha, \beta} - \varepsilon') \tag{5.52}$$

when $|\gamma - \beta| < \delta'$. Now let j_0 be such that $|\beta_j - \beta| < \delta'$ for $j \geq j_0$. By Step 2, for $|r - r_{\alpha,\beta}| \leq \varepsilon$ and $j \geq j_0$ we have

$$|u'(\alpha, \beta_j, r)| = |u'(\alpha, \beta, r_{\alpha,\beta})| + u'(\alpha, \beta, r_{\alpha,\beta}) - u'(\alpha, \beta_j, r) \geq \frac{1}{2} |u'(\alpha, \beta, r_{\alpha,\beta})|. \quad (5.53)$$

Therefore for $j \geq j_0$,

$$\begin{aligned} u(\alpha, \beta_j, r_{\alpha,\beta} + \varepsilon) &\leq u(\alpha, \beta_j, r_{\alpha,\beta} - \varepsilon') - \min_{|r - r_{\alpha,\beta}| \leq \varepsilon} |u'(\alpha, \beta_j, r)| (\varepsilon + \varepsilon') \\ &< 2u(\alpha, \beta, r_{\alpha,\beta} - \varepsilon') - \frac{1}{2} |u'(\alpha, \beta, r_{\alpha,\beta})| \varepsilon < 0. \end{aligned} \quad (5.54)$$

Then we obtain a contradiction since $\beta_j \in C$. Now assume that $T \leq r_{\alpha,\beta}$. By Step 2 we have

$$|u'(\alpha, \beta_j, r_{\alpha,\beta_j}) - u'(\alpha, \beta, r_{\alpha,\beta})| = |u'(\alpha, \beta, r_{\alpha,\beta})| \leq \frac{1}{2} |u'(\alpha, \beta, r_{\alpha,\beta})| \quad (5.55)$$

for $j \geq j_0$ and we get a contradiction.

Now we can complete the proof of Theorem 1.3.

(i) Let $\alpha > 0$ be fixed. By Proposition 5.7, there exists a unique $\beta > 0$ such that $u(\alpha, \beta, r_{\alpha,\beta}) = u'(\alpha, \beta, r_{\alpha,\beta}) = 0$. With s and t defined in (2.1), we set

$$w(r) = \left(\frac{r_{\alpha,\beta}}{R}\right)^s u\left(\alpha, \beta, \frac{r_{\alpha,\beta}}{R} r\right), \quad z(r) = \left(\frac{r_{\alpha,\beta}}{R}\right)^t v\left(\alpha, \beta, \frac{r_{\alpha,\beta}}{R} r\right), \quad 0 \leq r \leq R. \quad (5.56)$$

Then (w, z) is a nontrivial radial solution of problem (1.1).

(ii) follows from Proposition 5.7. □

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