Using a $t$-norm $T$, we introduce the notions of $T$-fuzzy subalgebras and $T$-fuzzy $H$-ideals in BCI-algebras and investigate some of their properties.

1. Introduction

The notion of BCI-algebras was introduced by Iséki [3] which is a generalization of BCK-algebras [2]. This notion is originated from two different ways: one of the motivations is based on set theory, another motivation is from classical and nonclassical propositional calculi.

Zadeh [8] introduced the notion of fuzzy sets. Many researchers have applied this concept to mathematical branches, such as semigroup, loop, group, ring, semiring, field, near ring, vector spaces, topological spaces, functional analysis, automation. Jun et al. [4, 5] introduced the notions of fuzzy subalgebras and fuzzy ideals of BCK-algebras with respect to a $t$-norm $T$, and studied some of their properties. In this paper, we obtain some related results of $T$-fuzzy subalgebras and $T$-fuzzy $H$-ideals in BCI-algebras.

2. Preliminaries

In this section, we review some definitions that will be used in the sequel.

Definition 2.1. An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if, for all $x, y, z \in X$, the following axioms hold.

1. $((x * y) * (x * z)) * (z * y) = 0$.
2. $(x * (x * y)) * y = 0$.
3. $x * x = 0$.
4. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$.

In BCI-algebras, the following hold.

5. $(x * 0) = x$.
6. $(x * y) * z = (x * z) * y$.
7. $0 * (y * x) = (0 * y) * (0 * x)$. 

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Definition 2.2. Let $S$ be a nonempty subset of a BCI-algebra $X$, then $S$ is called a subalgebra of $X$ if $x \ast y \in S$ for all $x, y \in S$.

Definition 2.3. A subset $A$ of a BCI-algebra $(X; \ast, 0)$ is called an ideal of $X$ if for any $x, y \in X$, the following conditions hold.

(i) $0 \in A$.
(ii) $x \ast y$ and $y \in A$ imply that $x \in A$.

Definition 2.4 [6]. A subset $A$ of a BCI-algebra $(X; \ast, 0)$ is called an $H$-ideal of $X$ if for any $x, y, z \in X$, the following conditions hold.

(a) $0 \in A$.
(b) $(x \ast (y \ast z))$ and $y \in A \Rightarrow x \ast z \in A$.

Definition 2.5. A mapping $f : X \to Y$ of BCI-algebras is called a homomorphism if $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$.

Note that if $f$ is a homomorphism of BCI-algebras, then $f(0) = 0$.

Definition 2.6. Let $X$ be a nonempty set. A fuzzy (sub)set $\mu$ of the set $X$ is a mapping $\mu : X \to [0,1]$. The complement of a fuzzy set $\mu$ of a set $X$ is denoted by $\overline{\mu}$ and defined by $\overline{\mu}(x) = 1 - \mu(x)$, for all $x \in X$.

Definition 2.7 [1]. A triangular norm (t-norm) is a function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions.

(T1) $T(x,1) = x$.
(T2) $T(x,y) = T(y,x)$.
(T3) $T(x, T(y,z)) = T(T(x,y),z)$.
(T4) $T(x,y) \leq T(x,z)$ whenever $y \leq z$, for all $x, y, z \in [0,1]$.

A simple example of such defined t-norm is a function $T(x,y) = \min(x,y)$. In a general case, $T(x,y) \leq \min(x,y)$ and $T(x,0) = 0$ for all $x, y \in [0,1]$.

3. $T$-fuzzy subalgebras

In what follows, let $X$ denote a BCI-algebra unless otherwise specified.

Definition 3.1. A fuzzy set $\mu$ in $X$ is called a subalgebra of $X$ with respect to a t-norm $T$ (briefly, a $T$-fuzzy subalgebra of $X$) if $\mu(x \ast y) \geq T(\mu(x),\mu(y))$ for all $x, y \in X$.

Example 3.2. Let $X := \{0,1,2\}$ be a BCI-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
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<td>1</td>
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<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0.81$ and $\mu(x) = 0.25$ for all $x \neq 0$ and let $T_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $T_m(x,y) = \max(x+y-1,0)$ which is a t-norm for all $x, y \in [0,1]$. Then $T_m$ is a t-norm [7]. By routine calculations, it is easy
to check that \(\mu\) satisfies \(\mu(x \ast y) \geq T_m(\mu(x),\mu(y))\) for all \(x, y \in X\). Hence, \(\mu\) is a \(T_m\)-fuzzy subalgebra of \(X\).

**Theorem 3.3.** Let \(\mu\) be a \(T\)-fuzzy subalgebra of \(X\) and \(\alpha \in [0, 1]\).

(i) If \(\alpha = 1\), then \(U(\mu; \alpha)\) is either empty or a subalgebra of \(X\).

(ii) If \(T = \min\), then \(U(\mu; \alpha)\) is either empty or a subalgebra of \(X\).

(iii) \(\mu(0) \geq \mu(x)\) for all \(x \in X\).

**Proof.** (i) Assume that \(\alpha = 1\). If \(x, y \in U(\mu; 1)\), then \(\mu(x) \geq 1\) and \(\mu(y) \geq 1\). It follows from Definitions 2.7 and 3.1 that \(\mu(x \ast y) \geq T(\mu(x),\mu(y)) \geq T(1,1) = 1\) so that \(x \ast y \in U(\mu; 1)\), that is, \(U(\mu; 1)\) is a subalgebra of \(X\).

(ii) Assume that \(T = \min\) and let \(x, y \in U(\mu; \alpha)\). Then, \(\mu(x \ast y) \geq T(\mu(x),\mu(y)) = \min(\mu(x),\mu(y)) \geq \min(\alpha, \alpha) = \alpha\), and so \(x \ast y \in U(\mu; \alpha)\).

Hence \(U(\mu; \alpha)\) is a subalgebra of \(X\).

(iii) Since \(x \ast x = 0\) for all \(x \in X\), we have \(\mu(0) = \mu(x \ast x) \geq T(\mu(x),\mu(x)) = \min(\mu(x),\mu(x)) = \mu(x)\). This completes the proof. \(\square\)

**Theorem 3.4.** Let \(\mu\) be a \(T\)-fuzzy subalgebra of \(X\). If there is a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} T(\mu(x_n),\mu(x_n)) = 1\), then \(\mu(0) = 1\).

**Proof.** Let \(x \in X\), then \(\mu(0) = \mu(x \ast x) \geq T(\mu(x),\mu(x))\). Hence \(\mu(0) \geq T(\mu(x_n),\mu(x_n))\) for any \(n \in \mathbb{N}\). Since \(1 \geq \mu(0) \geq \lim_{n \to \infty} T(\mu(x_n),\mu(x_n)) = 1\), it follows that \(\mu(0) = 1\). This completes the proof. \(\square\)

**Definition 3.5.** Let \(\lambda\) and \(\mu\) be \(T\)-fuzzy subalgebras of \(X\). Then direct product of \(T\)-fuzzy subalgebras is defined by \((\lambda \times \mu)(x,y) = T(\lambda(x),\mu(y))\), for all \(x, y \in X\).

**Theorem 3.6.** If \(\mu_1\) and \(\mu_2\) are \(T\)-fuzzy subalgebras of \(X\), then \(\mu = \mu_1 \times \mu_2\) is a \(T\)-fuzzy subalgebra of \(X \times X\).

**Proof.** For any \((x_1, x_2)\) and \((y_1, y_2)\) \(\in X \times X\), we have

\[
\mu((x_1, x_2) \ast (y_1, y_2)) = \mu(x_1 \ast y_1, x_2 \ast y_2)
= (\mu_1 \times \mu_2)(x_1 \ast y_1, x_2 \ast y_2)
\geq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2))) \tag{3.1}
\geq T(T(\mu_1(x_1), \mu_2(x_2)), T(\mu_1(y_1), \mu_2(y_2)))
\geq T((\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, y_2))
= T(\mu_1(x_1, x_2), \mu_1(y_1, y_2)).
\]

Hence, \(\mu = \mu_1 \times \mu_2\) is a \(T\)-fuzzy subalgebra of \(X \times X\). \(\square\)

**Definition 3.7.** Let \(f\) be a mapping on \(X\). If \(\nu\) is a fuzzy set in \(f(X)\), then fuzzy set \(\mu = \nu \circ f\) (i.e., \((\nu \circ f)(x) = \nu(f(x))\) in \(X\) is called preimage of \(\nu\) under \(f\).

**Theorem 3.8.** An epimorphism preimage of a \(T\)-fuzzy subalgebra of \(X\) is a \(T\)-fuzzy subalgebra.
Proof. Let \( f : X \rightarrow Y \) be an epimorphism of BCI-algebras, let \( v \) be a \( T\)-fuzzy subalgebra of \( Y \), and let \( \mu \) be the preimage of \( v \) under \( f \). Then for any \( x, y \in X \), we have

\[
\mu(x \ast y) = (v \circ f)(x \ast y)
\]
\[
= v(f(x \ast y)) = v(f(x) \ast f(y))
\]
\[
\geq T(v(f(x)), v(f(y)))
\]
\[
= T((v \circ f)(x), (v \circ f)(y))
\]
\[
= T(\mu(x), \mu(y)).
\]

(3.2)

Hence, \( \mu \) is a fuzzy subalgebra of \( X \) with respect to a \( t \)-norm \( T \). \( \square \)

Definition 3.9. If \( \mu \) is a fuzzy set in a subalgebra \( X \) and \( f \) is a mapping defined on \( X \), then the fuzzy set \( \mu^f \) in \( f(X) \) defined by

\[
\mu^f(y) = \sup_{x \in f^{-1}(y)} \mu(x) \quad \forall y \in f(X)
\]

(3.3)

is called the image of \( \mu \) under \( f \).

Definition 3.10. A fuzzy set \( \mu \) in \( X \) has the Sup property if for any subset \( A \subseteq X \), there exists \( a_0 \in A \) such that \( \mu(a_0) = \sup_{a \in A} \mu(a) \).

Theorem 3.11. An epimorphism image of a fuzzy subalgebra with Sup property is a fuzzy subalgebra.

Proof. Let \( f : X \rightarrow Y \) be an epimorphism of \( X \) and let \( \mu \) be a fuzzy subalgebra of \( X \) with Sup property. Let \( f(x), f(y) \in f(X) \) and let \( x_0, y_0 \in f^{-1}(f(x)) \) be such that

\[
\mu(x_0) = \sup_{t \in f^{-1}(f(x))} \mu(t),
\]
\[
\mu(y_0) = \sup_{t \in f^{-1}(f(y))} \mu(t),
\]

(3.4)

respectively. Then,

\[
\mu^f(f(x) \ast f(y)) = \sup_{z \in f^{-1}(f(x) \ast f(y))} \mu(z)
\]
\[
\geq \min \{ \mu(x_0), \mu(y_0) \}
\]
\[
= \min \left\{ \sup_{t \in f^{-1}(f(x))} \mu(t), \sup_{t \in f^{-1}(f(y))} \mu(t) \right\}
\]
\[
= \min \{ \mu^f(f(x)), \mu^f(f(y)) \}.
\]

(3.5)

Hence \( \mu^f \) is a fuzzy subalgebra of \( Y \). \( \square \)
Definition 3.12 [7]. A $t$-norm $T$ on $[0,1]$ is called a continuous $t$-norm if $T$ is a continuous function from $[0,1] \times [0,1]$ to $[0,1]$ with respect to the usual topology. Note that the function min is a continuous $t$-norm.

Theorem 3.13. Let $T$ be a continuous $t$-norm and let $f$ be a homomorphism on $X$. If $\mu$ is a $T$-fuzzy subalgebra of $X$, then $\mu^f$ is a $T$-fuzzy subalgebra of $f(X)$.

Proof. Let $Z_1 = f^{-1}(y_1)$, $Z_2 = f^{-1}(y_2)$, and $Z_{12} = f^{-1}(y_1 \ast y_2)$, where $y_1, y_2 \in f(X)$.

Consider the set $Z_1 \ast Z_2 = \{x \in X \mid x = z_1 \ast z_2$ for some $z_1 \in Z_1$ and $z_2 \in Z_2\}$. If $x \in Z_1 \ast Z_2$, then $x = x_1 \ast x_2$ for some $x_1 \in Z_1$ and $x_2 \in Z_2$.

Thus $f(x) = f(x_1 \ast x_2) = f(x_1) \ast f(x_2) = y_1 \ast y_2$, that is, $x \in f^{-1}(y_1 \ast y_2) = Z_{12}$.

Hence $Z_1 \ast Z_2 \subseteq Z_{12}$. It follows that

$$
\mu^f(y_1 \ast y_2) = \sup_{x \in f^{-1}(y_1 \ast y_2)} \mu(x) \\
= \sup_{x \in Z_{12}} \mu(x) \geq \sup_{x \in Z_1 \ast Z_2} \mu(x) \\
\geq \sup_{x_1 \in Z_1, x_2 \in Z_2} \mu(x_1 \ast x_2) \\
\geq \sup_{x_1 \in Z_1, x_2 \in Z_2} T(\mu(x_1), \mu(x_2)).
$$

\(3.6\)

Since $T$ is continuous, for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that if $\sup_{x_1 \in Z_1} \mu(x_1) - x_1^* \leq \delta$ and $\sup_{x_2 \in Z_2} \mu(x_2) - x_2^* \leq \delta$, then

$$
T\left(\sup_{x_1 \in Z_1} \mu(x_1), \sup_{x_2 \in Z_2} \mu(x_2)\right) - T(x_1^*, x_2^*) \leq \varepsilon.
$$

\(3.7\)

Choose $z_1 \in Z_1$ and $z_2 \in Z_2$ such that $\sup_{x_1 \in Z_1} \mu(x_1) - \mu(z_1) \leq \delta$ and $\sup_{x_2 \in Z_2} \mu(x_2) - \mu(z_2) \leq \delta$, then $T(\sup_{x_1 \in Z_1} \mu(x_1), \sup_{x_2 \in Z_2} \mu(x_2)) - T(\mu(z_1), \mu(z_2)) \leq \varepsilon$. Consequently, $\mu^f(y_1 \ast y_2) \geq \sup_{x_1 \in Z_1, x_2 \in Z_2} T(\mu(x_1), \mu(x_2)) \geq T(\sup_{x_1 \in Z_1} \mu(x_1), \sup_{x_2 \in Z_2} \mu(x_2)) = T(\mu^f(y_1), \mu^f(y_2))$, which shows that $\mu^f$ is a $T$-fuzzy subalgebra of $f(X)$. \[\square\]

4. $T$-fuzzy $H$-ideals

Definition 4.1. A fuzzy set $\mu$ in $X$ is called $T$-fuzzy ideals of $X$ if

1. $\mu(0) \geq \mu(x)$ for all $x \in X$,
2. $\mu(x) \geq T(\mu(x \ast y), \mu(y))$ for all $x, y \in X$.

Definition 4.2. A fuzzy set $\mu$ in $X$ is called $T$-fuzzy $H$-ideals of $X$ if

1. (TF1) $\mu(0) \geq \mu(x)$ for all $x \in X$,
2. (TF2) $\mu(x \ast z) \geq T(\mu(x \ast (y \ast z)), \mu(y))$ for all $x, y, z \in X$. 


Example 4.3. Let $X := \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>c</td>
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<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a fuzzy set $\mu : X \rightarrow [0,1]$ by $\mu(0) = t_1$ and $\mu(x) = t_2$ for all $x \neq 0$, where $t_1 > t_2$ and let $T_m : [0,1] \times [0,1] \rightarrow [0,1]$ be a function defined by $T_m(x, y) = \max(x + y - 1, 0)$ which is a $t$-norm for all $x, y \in [0,1]$. By routine calculations, it is easy to check that $\mu$ is a $T_m$-fuzzy $H$-ideal of $X$.

Proposition 4.4. Every $T$-fuzzy $H$-ideal in a BCI-algebra $X$ is a $T$-fuzzy ideal of $X$.

Proof. For $x, y, z \in X$, we have

$$
\mu_A(x \ast z) \geq T(\mu_A(x \ast (y \ast z)), \mu_A(y)),
\mu_A(x \ast 0) \geq T(\mu_A(x \ast (y \ast 0)), \mu_A(y)) \\ (\text{putting } z = 0) \quad (4.1)
\mu_A(x) \geq T(\mu_A(x \ast y), \mu_A(y)) \\ (\text{using (5)}),
$$

which completes the proof. \hfill \Box

Proposition 4.5. Every $T$-fuzzy $H$-ideal of a BCI-algebra $X$ is a $T$-fuzzy subalgebra of $X$.

Proof. For $x, y, z \in X$, we have

$$
\mu_A(x \ast z) \geq T(\mu_A(x \ast (y \ast z)), \mu_A(y)),
\mu_A(x \ast y) \geq T(\mu_A(x \ast (y \ast y)), \mu_A(y)) \\ (\text{replacing } z \text{ by } y) \quad (4.2)
\mu_A(x \ast y) \geq T(\mu_A(x \ast 0), \mu_A(y)) \\ (\text{using (3)})
\mu_A(x \ast y) \geq T(\mu_A(x), \mu_A(y)) \\ (\text{using (5)}).
$$

This ends the proof. \hfill \Box

Theorem 4.6. If $\mu$ is a $T$-fuzzy $H$-ideal of $X$, then each nonempty level subset $U(\mu; 1)$ is $H$-ideal of $X$.

Proof. Suppose that $\mu$ is a $T$-fuzzy $H$-ideal of $X$. Since $U(\mu, 1)$ is nonempty, there exists $x \in U(\mu; 1)$. It follows from (TF1) that $\mu(0) \geq \mu(x) \geq 1$, that is, $0 \in U(\mu; \alpha)$. Let $x, y, z \in X$ be such that $x \ast (y \ast z) \in U(\mu; 1)$ and $y \in U(\mu; 1)$. Then $\mu(x \ast z) \geq T(\mu(x \ast (y \ast z)), \mu(y)) \geq T(1,1) = 1$ so that $x \ast z \in U(\mu; 1)$.

Hence $U(\mu; 1)$ is a $H$-ideal of $X$. \hfill \Box
Theorem 4.7. If $\lambda$ and $\mu$ are $T$-fuzzy $H$-ideals of a BCI-algebra $X$, then $\lambda \times \mu$ is a $T$-fuzzy $H$-ideal of $X \times X$.

Proof. For any $(x, y) \in X \times X$, we have $(\lambda \times \mu)(0, 0) = T(\lambda(0), \mu(0)) \geq T(\lambda(x), \mu(y)) = (\lambda \times \mu)(x, y)$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2) \in X \times X$. Then

\[
(\lambda \times \mu)(x \ast z) = (\lambda \times \mu)((x_1, x_2) \ast (z_1, z_2))
= (\lambda \times \mu)(x_1 \ast z_1, x_2 \ast z_2)
= T(\lambda(x_1 \ast z_1), \mu(x_2 \ast z_2))
\geq T(T(\lambda(x_1 \ast (y_1 \ast z_1)), \lambda(y_1)), T(\mu(x_2 \ast (y_2 \ast z_2)), \mu(y_2)))
= T(T(\lambda(x_1 \ast (y_1 \ast z_1)), \lambda(y_1)), T(\mu(x_2 \ast (y_2 \ast z_2)), \mu(y_2)))
= T(T(\lambda \times \mu)((x_1 \ast (y_1 \ast z_1), x_2 \ast (y_2 \ast z_2))), (\lambda \times \mu)((y_1, y_2)))
= T(\mu((x_1, x_2) \ast ((y_1, y_2) \ast (z_1, z_2)), (\lambda \times \mu)((y_1, y_2)))
= T(\mu((x \ast (y \ast z)), (\lambda \times \mu)(y)).
\]

Hence $\lambda \times \mu$ is a $T$-fuzzy $H$-ideal of $X \times X$. \hfill \Box

Theorem 4.8. Let $f : X \to Y$ be a homomorphism of BCI-algebras. If $\mu$ is a $T$-fuzzy $H$-ideal of $Y$, then $\mu^f$ is a $T$-fuzzy $H$-ideal of $X$.

Proof. For any $x \in X$, we have $\mu^f(x) = \mu(f(x)) \leq \mu((0)) = \mu(f(0)) = \mu(0)$. Thus, $\mu^f(x) \leq \mu^f(0)$, for all $x \in X$. Let $x, y, z \in X$. Then

\[
T(\mu^f(x \ast (y \ast z)), \mu^f(y)) = T(\mu(f(x \ast (y \ast z))), \mu(f(y)))
= T(\mu(f(x) \ast (f(y) \ast f(z))), \mu(f(y)))
\leq \mu(f(x) \ast f(z)) = \mu(f(x \ast z)) = \mu^f(x \ast z).
\]

Hence $\mu^f$ is a $T$-fuzzy $H$-ideal of $X$. \hfill \Box

Theorem 4.9. Let $f : X \to Y$ be an epimorphism of BCI-algebras. If $\mu^f$ is a $T$-fuzzy $H$-ideal of $X$, then $\mu$ is a $T$-fuzzy $H$-ideal of $Y$.

Proof. Let $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Then $\mu(y) = \mu(f(x)) = \mu^f(x) \leq \mu^f(0) = \mu(f(0)) = \mu(0)$. Let $x, y, z \in Y$. Then there exist $a, b, c \in X$ such that $f(a) = x$, $f(b) = y$, and $f(c) = z$. It follows that

\[
\mu(x \ast z) = \mu(f(a) \ast f(c)) = \mu(f(a \ast c)) = \mu^f(a \ast c)
\geq T(\mu^f(a \ast (b \ast c)), \mu^f(b))
= T(\mu(f(a \ast (b \ast c))), \mu(f(b)))
= T(\mu(f(a) \ast (f(b) \ast f(c))), \mu(f(b)))
= T(\mu(x \ast (y \ast z)), \mu(y)).
\]

Hence $\mu$ is a $T$-fuzzy $H$-ideal of $Y$. \hfill \Box
Theorem 4.13. Let $T$ be a $T$-norm and let $\lambda$ and $\mu$ be two fuzzy sets in $X$. Then the $T$-product of $\lambda$ and $\mu$ is denoted by $[\lambda \cdot \mu]_T$ and defined by $[\lambda \cdot \mu]_T(x) = T(\lambda(x), \mu(x))$, for all $x \in X$.

Note that
(i) $[\lambda \cdot \mu]_T = [\mu \cdot \lambda]_T$,
(ii) $[\lambda \cdot \mu]_T$ is a fuzzy set in $X$.

Theorem 4.11. Let $\lambda$ and $\mu$ be two $T$-fuzzy $H$-ideals of $X$. If a $T$-norm $T^*$ dominates $T$, that is, if $T^*(T(a, y), T(b, \delta)) \geq T(T^*(a, b), T^*(y, \delta))$ for all $a, b, y, \delta \in [0, 1]$, then $T^*$-product $[\lambda \cdot \mu]^+_T$ is a $T$-fuzzy $H$-ideal of $X$.

Proof. For any $x \in X$, we have $[\lambda \cdot \mu]^+_T(0) = T^*(\lambda(0), \mu(0)) \geq T^*(\lambda(x), \mu(x)) = [\lambda \cdot \mu]^+_T(x)$. Let $x, y, z \in X$. Then

$$
[\lambda \cdot \mu]^+_T(x \ast z) = T^*(\lambda(x \ast z), \mu(x \ast z)) \\
\geq T^*(T(\lambda(x \ast (y \ast z)), \lambda(y)), T(\mu(x \ast (y \ast z)), \mu(y))) \\
\geq T(T^*(\lambda(x \ast (y \ast z)), \mu(x \ast (y \ast z))), T^*(\lambda(y), \mu(y))) \\
= T([\lambda \cdot \mu]^+_T(x \ast (y \ast z)), [\lambda \cdot \mu]^+_T(y)).
$$

This proves that $[\lambda \cdot \mu]^+_T$ is a $T$-fuzzy $H$-ideal of $X$. □


Theorem 4.13. Let $T$ and $T^*$ be $T$-norms in which $T^*$ dominates $T$. Let $f : X \rightarrow Y$ be an epimorphism of BCI-algebras. If $\lambda$ and $\mu$ are $T$-fuzzy $H$-ideals of $X$, then $f^{-1}([\lambda \cdot \mu]^+_T) = [f^{-1}(\lambda), f^{-1}(\mu)]^+_T$.

Proof. For any $x \in X$, we have

$$
[f^{-1}([\lambda \cdot \mu]^+_T)](x) = [\lambda \cdot \mu]^+_T(f(x)) \\
= T^*(\lambda(f(x)), \mu(f(x))) \\
= T^*(f^{-1}(\lambda)(x), [f^{-1}(\mu)](x)) \\
= [f^{-1}(\lambda), f^{-1}(\mu)]^+_T(x).
$$

This completes the proof. □

Corollary 4.14. If $f : X \rightarrow Y$ is an epimorphism of BCI-algebras, then $f^{-1}([\lambda \cdot \mu]_T) = [f^{-1}(\lambda), f^{-1}(\mu)]_T$ for any $T$-fuzzy $H$-ideals $\lambda$ and $\mu$ of $Y$.

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