A NOTE ON SELF-EXTREMAL SETS IN $L_p(\Omega)$ SPACES

VIET NGUYEN-KHAC AND KHIEM NGUYEN-VAN

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We give a necessary condition for a set in $L_p(\Omega)$ spaces ($1 < p < \infty$) to be self-extremal that partially extends our previous results to the case of L_p spaces. Examples of self-extremal sets in $L_p(\Omega)$ ($1 < p < \infty$) are also given.

In [4, 5], we introduced the notion of (self-) extremal sets of a Banach space $(X, \|\cdot\|)$. For a nonempty bounded subset A of X, we denote by $d(A)$ its diameter and by $r(A)$ the relative Chebyshev radius of A with respect to the closed convex hull $\overline{\text{co}}A$ of A, that is, $r(A) := \inf_{y\in\overline{\text{co}}A} \sup_{x\in A} \|x - y\|$. The self-Jung constant of X is defined by $J_s(X) := \sup\{r(A) : A \subset X, \text{ with } d(A) = 1\}$. If in this definition we replace $r(A)$ by the relative Chebyshev radius $r_X(A)$ of A with respect to the whole X, we get the Jung constant $J(X)$ of X. Recall that a bounded subset A of X consisting of at least two points is said to be extremal (resp., self-extremal) if $r_X(A) = J(X)d(A)$ (resp., $r(A) = J_s(X)d(A)$).

Throughout the note, unless otherwise mentioned, we will work with the following assumption: (Ω,μ) *is a σ-finite measure space such that $L_p(\Omega)$ is infinite-dimensional*. The Jung and self-Jung constants of $L_p(\Omega)$ ($1 \le p < \infty$) were determined in [1, 3, 6, 7]:

$$J(L_p(\Omega)) = J_s(L_p(\Omega)) = \max\{2^{1/p-1}, 2^{-1/p}\}. \tag{1}$$

Theorem 1. *If $1 < p < \infty$ and A is self-extremal in $L_p(\Omega)$, then $\kappa(A) = d(A)$.*

Here $\kappa(A) := \inf\{\varepsilon > 0 : A$ can be covered by finitely many sets of diameter $\le \varepsilon\}$—the Kuratowski measure of noncompactness of A (for our convenience we use the notation $\kappa(A)$ in this note).

Before proving our theorem, we need the following results which for convenience we reformulate in the form of Lemmas 2 and 3.

Lemma 2 (see [1], Theorem 1.1). *Let X be a reflexive strictly convex Banach space and A a finite subset of X. Then there exists a subset $B \subset A$ such that*

(i) *$r(B) \ge r(A)$;*

(ii) *$\|x - b\| = r(B)$ for every $x \in B$, where b is the relative Chebyshev center of B, that is, $b \in \overline{\text{co}}B$ and $\sup_{x\in B} \|x - b\| = r(B)$.*

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Lemma 3 (see [8], Theorem 15.1). *Let (Ω,μ) be a σ-finite measure space, $1 < p < \infty$, $x_1,\dots,x_n$ vectors in $L_p(\Omega)$, and $t_1,\dots,t_n$ nonnegative numbers such that $\sum_{i=1}^n t_i = 1$. The following inequality holds:*

$$2\sum_{i=1}^{n} t_i \left\| x_i - \sum_{j=1}^{n} t_j x_j \right\|^{\alpha} \le \sum_{i,j=1}^{n} t_i t_j \|x_i - x_j\|^{\alpha}, \tag{2}$$

where

$$\alpha = \begin{cases} \dfrac{p}{p-1} & \text{if } 1 < p < 2, \\ p & \text{if } p \ge 2. \end{cases} \tag{3}$$

Proof of Theorem 1. Since $r(A)$ and $d(A)$ remain the same with replacing A by $\overline{\text{co}}A$, we may assume that A is closed convex and $r(A) = 1$. For each integer $n \ge 2$, we have

$$\bigcap_{x\in A} B\left(x, 1 - \frac{1}{n}\right) \cap A = \varnothing, \tag{4}$$

where $B(x,r)$ denotes the closed ball centered at x with radius r which is weakly compact since $L_p(\Omega)$ is reflexive. Hence there exist $x_{q_{n-1}+1}, x_{q_{n-1}+2},\dots,x_{q_n}$ in A (with convention $q_1 = 0$) such that

$$\bigcap_{i=q_{n-1}+1}^{q_n} B\left(x_i, 1 - \frac{1}{n}\right) \cap A = \varnothing. \tag{5}$$

Set $A_n := \{x_{q_{n-1}+1}, x_{q_{n-1}+2},\dots,x_{q_n}\}$. By Lemma 2, there exists a subset $B_n = \{y_{s_{n-1}+1}, y_{s_{n-1}+2},\dots,y_{s_n}\}$ of A_n satisfying properties (i)-(ii) of the lemma. Let us denote the relative Chebyshev center of B_n by b_n, and let $r_n := r(B_n)$. By what we said above, we have $r_n > 1 - 1/n$ and $\|y_i - b_n\| = r_n$ for every $i \in I_n := \{s_{n-1}+1, s_{n-1}+2,\dots,s_n\}$. Since B_n is a finite set, there exist non-negative numbers $t_{s_{n-1}+1}, t_{s_{n-1}+2},\dots,t_{s_n}$ with $\sum_{i\in I_n} t_i = 1$ such that $b_n = \sum_{i\in I_n} t_i y_i$. Applying Lemma 3, one gets

$$2r_n^{\alpha} = 2\sum_{i\in I_n} t_i \left\| y_i - \sum_{j\in I_n} t_j y_j \right\|^{\alpha} \le \sum_{i,j\in I_n} t_i t_j \|y_i - y_j\|^{\alpha}, \tag{6}$$

where α is as in (3).

Setting $B_\infty := \{y_{s_{n-1}+1}, y_{s_{n-1}+2},\dots,y_{s_n}\}_{n=2}^{\infty}$, we claim that $\kappa(B_\infty) = d(A)$. Evidently $\kappa(B_\infty) \le d(A)$ by definition. If $\kappa(A_\infty) < d(A)$, so there exist $\varepsilon_0 \in (0, d(A))$ satisfying $\kappa(B_\infty) \le d(A) - \varepsilon_0$, and subsets $D_1, D_2,\dots,D_m$ of $L_p(\Omega)$ with $d(D_i) \le d(A) - \varepsilon_0$ for every $i = 1,2,\dots,m$

such that $B_\infty \subset \bigcup_{i=1}^m D_i$. Then one can find at least one set among $D_1, D_2, \dots, D_m$, say D_1, with the property that there are infinitely many n satisfying

$$\sum_{i \in J_n} t_i \geq \frac{1}{m}, \tag{7}$$

where

$$J_n := \{i \in I_n : y_i \in D_1\}. \tag{8}$$

From (1), it follows that $(d(A))^\alpha = (1/J_s(L_p(\Omega)))^\alpha = 2$. In view of (6), we have, for all n satisfying (7),

$$\begin{aligned} 2 \cdot r_n^\alpha &\leq \sum_{i,j \in I_n} t_i t_j \|y_i - y_j\|^\alpha \\ &\leq (d(A) - \varepsilon_0)^\alpha \cdot \left(\sum_{i,j \in J_n} t_i t_j \right) + (d(A))^\alpha \cdot \left(1 - \sum_{i,j \in J_n} t_i t_j \right) \\ &\leq 2 - \left[(d(A))^\alpha - (d(A) - \varepsilon_0)^\alpha \right] \cdot \frac{1}{m^2}. \end{aligned} \tag{9}$$

On the other hand, obviously $1 - 1/n < r_n \leq 1$, therefore $\lim_{n \to \infty} r_n = 1$. We get a contradiction with (9) since there are infinitely many n satisfying (7).

One concludes that $\kappa(B_\infty) = d(A)$, and hence $\kappa(A) = d(A)$.

The proof of Theorem 1 is complete. □

Observe that no relatively compact set A in $L_p(\Omega)$ $(1 < p < \infty)$ is self-extremal by Theorem 1. Hence we obtain an immediate extension of Gulevich's result for $L_p(\Omega)$ spaces.

Corollary 4 (cf. [2]). *Suppose that $1 < p < \infty$ and that A is a relatively compact set in $L_p(\Omega)$ with $d(A) > 0$. Then $r(A) < (1/\sqrt[\alpha]{2})d(A)$, where α is as in (3).*

The following theorem gives a necessary condition for a set in $L_p(\Omega)$ $(1 < p < \infty)$ to be self-extremal.

Theorem 5. *Under the assumptions of Theorem 1, for every $\varepsilon \in (0, d(A))$, every positive integer m, there exists an m-simplex $\Delta(\varepsilon, m)$ with vertices in A such that each edge of $\Delta(\varepsilon, m)$ has length not less than $d(A) - \varepsilon$.*

Proof. We will assume A is closed convex and $r(A) = 1$. From the proof of Theorem 1, we derived a sequence $\{y_{s_{n-1}+1}, y_{s_{n-1}+2}, \dots, y_{s_n}\}_{n=2}^\infty$ in A and a sequence of positive numbers $\{t_{s_{n-1}+1}, t_{s_{n1}+2}, \dots, t_{s_n}\}_{n=2}^\infty$ (with convention $s_1 = 0$) such that

$$2 \cdot r_n^\alpha \leq \sum_{i,j \in I_n} t_i t_j \|y_i - y_j\|^\alpha, \quad \sum_{i \in I_n} t_i = 1, \tag{10}$$

where $r_n \in (1 - 1/n, 1]$, α is as in (3), and $I_n := \{s_{n-1} + 1, s_{n-1} + 2, \dots, s_n\}$.

We denote

$$
\begin{aligned}
T_{nj} &:= \sum_{i\in I_n} t_i\|y_i - y_j\|^\alpha,\\
S_n &:= \left\{ j\in I_n : T_{nj} \geq 2\cdot r_n^\alpha \cdot \left(1-\sqrt{1-r_n^\alpha}\right)\right\},\\
S_n(y_j) &:= \left\{ i\in I_n : \|y_i - y_j\|^\alpha \geq 2\cdot\left(1-\frac{1}{\sqrt[4]{n}}\right)\right\}, \quad j\in S_n,\\
\widehat{S}_n(y_j) &:= \{y_i : i\in S_n(y_j)\}, \quad j\in S_n,\\
\lambda_n &:= \sum_{i\in I_n\setminus S_n} t_i = 1-\sum_{i\in S_n} t_i.
\end{aligned}
\tag{11}
$$

One can proceed furthermore as follows. We have

$$
\begin{aligned}
2r_n^\alpha &\leq \sum_{i,j\in I_n} t_i t_j \|y_i - y_j\|^\alpha\\
&= \sum_{j\in S_n} t_j \sum_{i\in I_n} t_i \|y_i - y_j\|^\alpha + \sum_{j\in I_n\setminus S_n} t_j \sum_{i\in I_n} t_i\|y_i - y_j\|^\alpha\\
&\leq 2\sum_{j\in S_n} t_j + 2r_n^\alpha\left(1-\sqrt{1-r_n^\alpha}\right)\sum_{j\in I_n\setminus S_n} t_j\\
&= 2-2\lambda_n\left(1-r_n^\alpha + r_n^\alpha\sqrt{1-r_n^\alpha}\right)\\
&\leq 2-2\lambda_n\sqrt{1-r_n^\alpha}.
\end{aligned}
\tag{12}
$$

Hence $\lambda_n \leq \sqrt{1-r_n^\alpha} \to 0$, as $n\to\infty$. Thus $\lim_{n\to\infty}(\sum_{i\in S_n} t_i) = \lim_{n\to\infty}(1-\lambda_n) = 1$. On the other hand,

$$
2r_n^\alpha \leq \sum_{i,j\in I_n} t_i t_j\|y_i - y_j\|^\alpha \leq 2\left(1-\left(\sum_{i\in I_n} t_i^2\right)\right) \leq 2(1-t_i^2) \tag{13}
$$

for every $i\in I_n$. Therefore $t_i \leq \sqrt{1-r_n^\alpha} \to 0$ as $n\to\infty$. One concludes that the cardinality $|S_n|$ of S_n tends to ∞ as $n\to\infty$. In a similar manner (cf. [5, the proof of Theorem 3.4]), for every $\varepsilon\in(0,d(A))$ and a given positive integer m, we choose n sufficiently large satisfying

$$
|S_n| > m, \quad \frac{2\alpha m}{\sqrt[4]{n}} < 1, \quad 2\left(1-\frac{1}{\sqrt[4]{n}}\right) \geq (d(A)-\varepsilon)^\alpha \tag{14}
$$

such that for every $1\leq k\leq m$ and every choice of $i_1, i_2, \dots, i_k \in S_n$, we have

$$
\bigcap_{\nu=1}^{k} \widehat{S}_n(y_{i_\nu}) \neq \varnothing. \tag{15}
$$

With m and n as above and a fixed $j \in S_n$, setting $z_1 := y_j$, we take consecutively $z_2 \in \widehat{S}_n(z_1), z_3 \in \widehat{S}_n(z_1) \cap \widehat{S}_n(z_2), \ldots, z_{m+1} \in \bigcap_{k=1}^{m} \widehat{S}_n(z_k)$. One sees that

$$||z_i - z_j||^{\alpha} \geq 2\left(1 - \frac{1}{\sqrt[4]{n}}\right) \geq (d(A) - \varepsilon)^{\alpha} \tag{16}$$

for all $i \neq j$ in $\{1, 2, \ldots, m+1\}$, with n sufficiently large. We obtain an m-simplex formed by $z_1, z_2, \ldots, z_{m+1}$, whose edges have length not less than $d(A) - \varepsilon$, as claimed.

The proof of Theorem 5 is complete. □

Remark 6. (i) Since for $L_p(\Omega)$ spaces $J_s = J$, the extremal sets in $L_p(\Omega)$ are also self-extremal. Thus we obtain a similar result for extremal sets in $L_p(\Omega)$ via Theorem 5 above.

(ii) In particular, $\Omega = \mathbb{N}, \mu(A) := \text{card}(A), A \subset \mathbb{N}$ leads to the ℓ_p space case [5, Theorem 3.4].

Example 7. (i) Let $p \geq 2$, consider a sequence $\{\Omega_n\}_{i=1}^{\infty}$ consisting of measurable subsets of Ω such that

$$0 < \mu(\Omega_i) < \infty, \quad i = 1, 2, \ldots; \qquad \Omega_i \cap \Omega_j = \varnothing \quad \forall i \neq j; \qquad \bigcup_{i=1}^{\infty} \Omega_i = \Omega. \tag{17}$$

Let χ_{Ω_i} denote the characteristic function of Ω_i, and set

$$A := \{f_i\}_{i=1}^{\infty}, \quad f_i := \frac{\chi_{\Omega_i}}{[\mu(\Omega_i)]^{1/p}}. \tag{18}$$

One can check easily that $r(A) = 1$, $d(A) = 2^{1/p}$, hence A is a self-extremal set in $L_p(\Omega)$.

(ii) In the case $1 < p < 2$, we set $B := \{r_i\}_{i=0}^{\infty}$, where $\{r_i\}_{i=0}^{\infty}$ is the sequence of Rademacher functions in $L_p[0,1]$. If $r \in \text{co}\{r_0, r_1, \ldots, r_n\}$ and $k \geq n+1$, then it is easy to see that $d(B) = 2^{1-1/p}$ and

$$||r - r_k||_p := \left(\int_0^1 |r - r_k|^p d\mu\right)^{1/p} \geq \left|\int_0^1 (r - r_k) r_k d\mu\right| = 1, \tag{19}$$

hence $r(B) = 1$. Thus B is a self-extremal set in $L_p[0,1]$ with $1 < p < 2$. This is in contrast to the ℓ_p case [5], where we conjectured that there are no (self)-extremal sets in ℓ_p spaces with $1 < p < 2$.

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Viet Nguyen-Khac: Institute of Mathematics, Vietnamese Academy of Science and Technology, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam

E-mail address: nkviet@math.ac.vn

Khiem Nguyen-Van: Department of Mathematics & Informatics, Hanoi University of Education, Cau Giay District, Hanoi, Vietnam

E-mail address: nguyenvankhiem77@yahoo.com