CONTINUITY FOR MAXIMAL COMMUTATOR OF BOCHNER-RIESZ OPERATORS ON SOME WEIGHTED HARDY SPACES

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We show the boundedness for the commutator of Bochner-Riesz operator on some weighted $H^1$ space.

1. Introduction

Let $b$ be a locally integrable function. The maximal operator $B^\delta_{r,b}$ associated with the commutator generated by the Bochner-Riesz operator is defined by

$$B^\delta_{r,b}(f)(x) = \sup_{r>0} |B^\delta_{r,b}(f)(x)|,$$  \hspace{1cm} (1.1)

where

$$B^\delta_{r,b}(f)(x) = \int_{\mathbb{R}^n} B^\delta_r(x-y)f(y)(b(x)-b(y))dy$$ \hspace{1cm} (1.2)

and $(B^\delta_r(\hat{f}))(\xi) = (1-r^2|\xi|^2)^{\delta/2}\hat{f}(\xi)$. We also define that

$$B^\delta_{*,b}(f)(x) = \sup_{r>0} |B^\delta_r(f)(x)|,$$  \hspace{1cm} (1.3)

which is the Bochner-Riesz operator (see [8]). Let $E$ be the space $E = \{h : \|h\| = \sup_{r>0} |h(r)| < \infty\}$, then, for each fixed $x \in \mathbb{R}^n$, $B^\delta_r(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to $E$, and it is clear that $B^\delta_{*,b}(f)(x) = \|B^\delta_r(f)(x)\|$ and $B^\delta_{*,b}(f)(x) = \|b(x)B^\delta_r(f)(x) - B^\delta_r(bf)(x)\|$.

As well known, a classical result of Coifman et al. [4] proved that the commutator $[b,T]$ generated by $\text{BMO}(\mathbb{R}^n)$ functions and the Calderón-Zygmund operator is bounded on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$). However, it was observed that $[b,T]$ is not bounded, in general, from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ for $p \leq 1$. But, if $H^p(\mathbb{R}^n)$ is replaced by some suitable atomic space $H^p_{\text{B}}(\mathbb{R}^n)$ and $H^1_{\text{B}}(\mathbb{R}^n)$ (see [1, 6, 7, 9]), then $[b,T]$ maps continuously $H^p_{\text{B}}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ and $H^1_{\text{B}}(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$ for $p \in (n/(n+1), 1]$. The main purpose of this paper is to establish the weighted boundedness of the commutators.
related to Bochner-Riesz operator and BMO($R^n$) function on some weighted $H^1$ space. We first introduce some definitions (see [1, 6, 7, 9]).

**Definition 1.1.** Let $b$, $w$ be locally integrable functions and $w \in A_1$ (i.e., $Mw(x) \leq cw(x)$ a.e.). A bounded measurable function $a$ on $R^n$ is said to be $(w, b)$-atom if

(i) $\text{supp } a \subset B = B(x_0, r)$,
(ii) $\|a\|_{L^\infty} \leq w(B)^{-1}$,
(iii) $\int a(y)dy = \int a(y)b(y)dy = 0$.

A temperate distribution $f$ is said to belong to $H^1_b(w)$ if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x), \quad (1.4)$$

where $a_j$’s are $(w, b)$-atoms, $\lambda_j \in \mathbb{C}$, and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. Moreover, $\|f\|_{H^1_b(w)} \sim \sum_{j=1}^{\infty} |\lambda_j|$.

**Definition 1.2.** Let $w \in A_1$. A function $f$ is said to belong to weighted Block $H^1$ space $H^1_b(w)$ if $f$ can be written as (1.4), where $a_j$’s are $w$-atoms (i.e., $a_j$’s satisfy Definition 1.1(i), (ii), and (iii)’ $\int a(y)dy = 0$) and $\lambda_j \in \mathbb{C}$ with

$$\sum_{j=1}^{\infty} |\lambda_j| \left(1 + \log^+ \frac{1}{|\lambda_j|}\right) < \infty. \quad (1.5)$$

Moreover, $\|f\|_{H^1_b(w)} \sim \sum_{j=1}^{\infty} |\lambda_j| \left(1 + \log^+ \left(\frac{\sum_i |\lambda_i|}{|\lambda_j|}\right)\right)$.

Now, we formulate our results as follows.

**Theorem 1.3.** Let $b \in \text{BMO}(R^n)$ and $w \in A_1$. Then the maximal commutator $B^\delta_{*b}$ is bounded from $H^1_b(w)$ to $L^1_w(R^n)$ when $\delta > (n - 1)/2$.

**Theorem 1.4.** Let $b \in \text{BMO}(R^n)$ and $w \in A_1$. Then the maximal commutator $B^\delta_{*b}$ is bounded from $H^1_b(w)$ to $L^{1,\infty}_w(R^n)$ when $\delta > (n - 1)/2$.

**Theorem 1.5.** Let $b \in \text{BMO}(R^n)$ and $w \in A_1$. Then the maximal commutator $B^\delta_{*b}$ is bounded from $H^1_b(w)$ to $L^{1,\infty}_w(R^n)$ when $\delta > (n - 1)/2$.

### 2. Proof of theorems

**Proof of Theorem 1.3.** It suffices to show that there exists a constant $C > 0$ such that for every $(w, b)$-atom $a$,

$$\|B^\delta_{*b} (a)\|_{L^1_w} \leq C. \quad (2.1)$$

Let $a$ be a $(w, b)$-atom supported on a ball $B = B(x_0, R)$. We write

$$\int_{R^n} [B^\delta_{*b}(a)(x)] w(x)dx$$

$$\begin{equation}
= \int_{|x-x_0| \leq 2R} [B^\delta_{*b}(a)(x)] w(x)dx + \int_{|x-x_0| > 2R} [B^\delta_{*b}(a)(x)] w(x)dx \equiv I + II. \quad (2.2)
\end{equation}$$
For $I$, taking $q > 1$, by Hölder’s inequality and the $L^q$-boundedness of $B_{\ast, b}^\delta$ (see [2]), we see that

$$I \leq C \left\| B_{\ast, b}^\delta (a) \right\|_{L^q} \cdot w(2B)^{1-1/q} \leq C \left\| a \right\|_{L^q} w(2B)^{1-1/q} \leq C. \quad (2.3)$$

For $II$, let $b_0 = |B(x_0, R)|^{-1} \int_{B(x_0, R)} b(y) dy$, then

$$II \leq \sum_{k=1}^\infty \int_{2^{k+1}R \geq |x-x_0| > 2^k R} |b(x) - b_0| B_{\ast}^\delta (a)(x) w(x) dx \quad (2.4)$$

and

$$+ \sum_{k=1}^\infty \int_{2^{k+1}R \geq |x-x_0| > 2^k R} B_{\ast}^\delta ((b - b_0) a)(x) w(x) dx = II_1 + II_2. \quad (2.5)$$

For $II_1$, we choose $\delta_0$ such that

$$n - \frac{1}{2} < \delta_0 \leq \frac{n + 1}{2} \left( \delta, \min \left( \delta, \frac{n+1}{2} \right) \right) \quad (2.6)$$

and consider the following two cases.

Case 1 ($0 < r \leq R$). In this case, note that (see [8])

$$|B^\delta (z)| \leq C (1 + |z|)^{-\delta (n+1)/2}, \quad (2.7)$$

we have, for $|x-x_0| > 2|y-x_0|$, \n

$$|B^\delta_r (a)(x)| \leq C r^{-n} \int_{B(x_0, R)} \frac{|a(y)|}{1 + |x-y|/r} \delta^{(n+1)/2} d \gamma \quad (2.8)$$

and

$$\leq C |B|^{(\delta_{n+1}+n)/2} |z|^{\delta_{n+1}+n/2} w(B)^{-1}. \quad (2.9)$$

Case 2 ($r > R$). In this case, note that

$$\nabla^\beta B^\delta (z) \leq C (1 + |z|)^{-\delta (n+1)/2}$$

for any $\beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$ and $|x-x_0| > 2|y-x_0|$, where

$$\nabla^\beta = \left( \frac{\partial}{\partial x_1} \right)^{\beta_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\beta_n}, \quad (2.10)$$

by the vanishing condition of $a$, we gain

$$\left| B^\delta_r (a)(x) \right| \leq C r^{-(n+1)} \int_{B(x_0, R)} \frac{|a(y)| |y-x_0|}{1 + |x-y|/r} \delta^{(n+1)/2} d \gamma \quad (2.11)$$

\n
$$\leq C |B|^{(\delta_{n+1}+n)/2} |z|^{\delta_{n+1}+n/2} w(B)^{-1}. \quad (2.12)$$
Combining Case 1 with Case 2, we obtain

\[
II_1 \leq C \sum_{k=1}^{\infty} 2^{-k(\delta_0+(n+1)/2)} w(B)^{-1} |2^{k+1}B| \left( \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |b(x) - b_0|^p dx \right)^{1/p} \\
\times \frac{1}{|B|} \int_B w(x)^p dx 
\]

for any ball \( B \) and some \( 1 < p < \infty \) (see [10]). Using the properties of BMO(\( R^n \)) functions (see [10]), and noting \( w \in A_1 \), then

\[
\frac{w(B_2)}{|B_2|} \cdot \frac{|B_1|}{w(B_1)} \leq C \tag{2.13}
\]

for all balls \( B_1, B_2 \) with \( B_1 \subset B_2 \). Thus, by Hölder’s and reverse of Hölder’s inequalities for \( w \in A_1 \), we get, for \( 1/p + 1/p' = 1 \),

\[
II_1 \leq C \sum_{k=1}^{\infty} 2^{-k(\delta_0+(n+1)/2)} w(B)^{-1} |2^{k+1}B| \left( \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |b(x) - b_0|^p dx \right)^{1/p} \\
\times \left( \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} w(x)^p dx \right)^{1/p} 
\]

\[
\leq C \|b\|_{\text{BMO}} \sum_{k=1}^{\infty} k^2 \left( 2^{-k(\delta_0-(n-1)/2)} \left( \frac{w(2kB)}{|2kB|} \right)^{1/p} \right) \leq C. \tag{2.14}
\]

For \( II_2 \), similar to the estimate of \( II_1 \), we obtain

\[
B^*_{(b-b_0)}(x) \leq C \|b\|_{\text{BMO}} w(B)^{-1} |B|^{(\delta_0+(n+1)/2)/n} |x-x_0|^{-(\delta_0+(n+1)/2)}, \tag{2.15}
\]

thus

\[
II_2 \leq C \|b\|_{\text{BMO}} \sum_{k=1}^{\infty} w(B)^{-1} |B|^{(\delta_0+(n+1)/2)/n} |2^kB|^{-(\delta_0+(n+1)/2)/n} w(2kB) \\
\leq C \|b\|_{\text{BMO}} \sum_{k=1}^{\infty} 2^{-k(\delta_0-(n-1)/2)} \left( \frac{w(2kB)}{|2kB|} \right) \leq C. \tag{2.16}
\]

This finishes the proof of Theorem 1.3. \( \square \)

To prove Theorem 1.4, we recall the following lemma (see [5, 10]).
Lemma 2.1. Let $w \geq 0$ and $\{g_k\}$ be a sequence of measurable functions satisfying
\[
\|g_k\|_{L_w^{1-\alpha}} \leq 1. \tag{2.17}
\]
Then, for every numerical sequence $\{\lambda_k\}$,
\[
\left\| \sum_k \lambda_k g_k \right\|_{L_w^{1-\alpha}} \leq C \sum_k |\lambda_k| \left(1 + \log \left(\sum_j |\lambda_j| / |\lambda_k|\right)\right). \tag{2.18}
\]

Proof of Theorem 1.4. By Lemma 2.1, it is enough to show that there exists a constant $C$ such that
\[
\|B^\delta_{*\cdot,b}(a)\|_{L_w^{1-\alpha}} \leq C \quad \text{for each } w\text{-atom } a. \tag{2.19}
\]
Let $a$ be a $w$-atom supported on a ball $B = B(x_0, r)$. We write
\[
w(\{x \in \mathbb{R}^n : B^\delta_{*\cdot,b}(a)(x) > \lambda\}) \\
\leq w(\{x \in 2B : B^\delta_{*\cdot,b}(a)(x) > \lambda\}) + w(\{x \in (2B)^c : B^\delta_{*\cdot,b}(a)(x) > \lambda\}) = I + II. \tag{2.20}
\]
For $I$, by the $L^q$-boundedness of $B^\delta_{*,b}$ for $q > 1$, we gain
\[
I \leq \lambda^{-1} \|B^\delta_{*,b}(a)\chi_{2B}\|_{L_w^1} \leq C\lambda^{-1} \|B^\delta_{*,b}(a)\|_{L_w^1} \cdot w(B)^{1-1/q} \\
\leq C\lambda^{-1} \|a\|_{L_w^1} \cdot w(B)^{1-1/q} \leq C\lambda^{-1}. \tag{2.21}
\]
For $II$, let $b_0 = |B|^{-1} \int_B b(x)dx$, notice that
\[
B^\delta_{*,b}(a)(x) = \|b(x)B^\delta_{*,b}(a)(x) - B^\delta_{*,b}(ba)(x)\| \\
= \|b(x) - b_0\| B^\delta_{*,b}(a)(x) - B^\delta_{*,b}((b, b_0) a)(x)\| \\
\leq |b(x) - b_0| B^\delta_{*,b}(a)(x) + B^\delta_{*,b}((b, b_0) a)(x) \\
\leq b(x) - b_0 |B^\delta_{*,b}(a)(x) + B^\delta_{*,b}((b, b_0) a)(x),
\]
we have
\[
II \leq w\left(\left\{x \in (2B)^c : |b(x) - b_0| g^\delta_{*,a}(a)(x) > \frac{\lambda}{2}\right\}\right) \\
+ w\left(\left\{x \in (2B)^c : g^\delta_{*,a}((b, b_0) a)(x) > \frac{\lambda}{2}\right\}\right) = II_1 + II_2. \tag{2.23}
\]
Similar to the proof of Theorem 1.3, we get
\[
II_1 \leq C\lambda^{-1} \int_{(2B)^c} |b(x) - b_0| B^\delta_{*,b}(a)(x)w(x)dx \\
= C\lambda^{-1} \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^kB} |b(x) - b_0| B^\delta_{*,b}(a)(x)w(x)dx \leq C\lambda^{-1} \|b\|_{\text{BMO}}, \tag{2.24}
\]
\[
II_2 \leq C\lambda^{-1} \int_{(2B)^c} B^\delta_{*,b}((b, b_0) a)(x)w(x)dx \leq C\lambda^{-1} \|b\|_{\text{BMO}}.
\]
Combining the estimate of $I, II_1,$ and $II_2,$ we gain

$$w(\{x \in \mathbb{R}^n : B_{*,b}^\delta(a)(x) > \lambda\}) \leq C\lambda^{-1} \|b\|_{\text{BMO}}. \quad (2.25)$$

This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. Given $f \in H^1(w),$ let $f = \sum_j \lambda_j a_j$ be the atomic decomposition for $f.$ By a limiting argument, it suffices to show Theorem 1.5 for a finite sum of $f = \sum_Q \lambda_Q a_Q$ with $\sum_Q |\lambda_Q| \leq C \|f\|_{H^1(w)}.$ We may assume that each $Q$ (the supporting cube of $a_Q$) is dyadic. For $\lambda > 0$ by [3, Lemma 4.1], there exists a collection of pairwise disjoint dyadic cubes $\{S\}$ such that

$$\sum_Q \sum_{Q \subset S} |\lambda_Q| \leq C\lambda |S|, \quad \forall S,$$

$$\sum_S |S| \leq \lambda^{-1} \sum_Q |\lambda_Q|, \quad \left\| \sum_{Q \notin S} \lambda_Q |Q|^{-1} \chi_Q \right\|_{L^\infty} \leq C\lambda. \quad (2.26)$$

Let $E = \bigcup_S \overline{S},$ where for a fixed cube $Q,$ $\overline{Q}$ denotes the cube with the same center as $Q$ but with the side-length $4\sqrt{n}$ times that of $Q.$ Then, $|E| \leq C\lambda^{-1} \|f\|_{H^1}.$ Set $M(x) = \sum_S \sum_{Q \subset S} \lambda_Q a_Q, N(x) = f(x) - M(x).$ By the $L^2$ boundedness of $B_{*,b}^\delta$ and the well-known argument, it suffices to show that

$$w(\{x \in E^c : B_{*,b}^\delta(M)(x) > \lambda\}) \leq C\lambda^{-1} \|f\|_{H^1(w)}. \quad (2.27)$$

Because $B_{*,b}^\delta(M)(x) \leq \sum_S \sum_{Q \subset S} |\lambda_Q| B_{*,b}^\delta(a_Q)(x),$ we have

$$w(\{x \in E^c : B_{*,b}^\delta(M)(x) > \lambda\})$$

$$\leq C\lambda^{-1} \int_{E^c} B_{*,b}^\delta(M)(x)w(x)dx$$

$$\leq C\lambda^{-1} \sum_S \sum_Q |\lambda_Q| \sum_{k=1}^\infty \int_{2^k Q} B_{*,b}^\delta(a_Q)(x)w(x)dx, \quad (2.28)$$

similar to the estimate of Theorem 1.3, we get, when $x \in E^c,$

$$B_{*,b}^\delta(a_Q)(x) \leq C\|b\|_{\text{BMO}} w(B)^{-1} |Q|^{(\delta_0+(n+1)/2)/n} |x-x_0|^{-(\delta_0+(n+1)/2)}$$

$$+ C |b(x) - b_0| w(B)^{-1} 2^{-k(\delta_0+(n+1)/2)} \quad (2.29)$$
thus, by Hölder’s and reverse of Hölder’s inequalities for \( w \in A_1 \), we obtain

\[
\begin{align*}
  & w(\{ x \in E^c : B^\delta_{*;\eta}(M)(x) > \lambda \}) \\
  & \leq C \lambda^{-1} w(B)^{-1} \sum_S \sum_{Q \subset S} |\lambda_Q| \sum_{k=1}^{\infty} k 2^{-k(\delta_0+(n+1)/2)} w(2^k Q) \\
  & \leq C \lambda^{-1} \sum_S \sum_{Q \subset S} |\lambda_Q| \sum_{k=1}^{\infty} k 2^{-k(\delta_0-(n-1)/2)} \\
  & \leq C \lambda^{-1} \sum_S \sum_{Q \subset S} |\lambda_Q| \leq C \lambda^{-1} \| f \|_{H^1(w)}. 
\end{align*}
\]

(2.30)

This finishes the proof of Theorem 1.5. \( \square \)

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